FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

$$G(x, y, y') = 0$$

$\lozenge$ in normal form:

$$y' = F(x, y)$$

$\lozenge$ in differential form:

$$M(x, y)dx + N(x, y)dy = 0$$

- Last time we discussed first-order linear ODE: $y' + q(x)y = h(x)$.
  We next consider first-order nonlinear equations.

NONLINEAR FIRST-ORDER ODEs

- No general method of solution for 1st-order ODEs beyond linear case; rather, a variety of techniques that work on a case-by-case basis.

Examples:

i) Bring equation to separated-variables form, that is, $y' = \alpha(x)/\beta(y)$;
then equation can be integrated.
Cases covered by this include $y' = \varphi(ax + by)$; $y' = \varphi(y/x)$.

ii) Reduce to linear equation by transformation of variables.
Examples of this include Bernoulli's equation.

iii) Bring equation to exact-differential form, that is
$M(x, y)dx + N(x, y)dy = 0$ such that $M = \partial\phi/\partial x$, $N = \partial\phi/\partial y$.
Then solution determined from $\phi(x, y) =$ const.

• Useful reference for the ODE part of this course (worked problems and examples)

Schaum's Outline Series

Differential Equations

R. Bronson and G. Costa
McGraw-Hill (Third Edition, 2006)

◊ Chapters 1 to 7: First-order ODE.

First order nonlinear equations

Although no general method for solution is available, there are several cases of physically relevant nonlinear equations which can be solved analytically :

Separable equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x)}{g(y)}$$

Solution :

$$\boxed{\int g(y)dy = \int f(x)dx}$$

Ex 1

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \mathrm{e}^x \quad \Longrightarrow \quad \int \frac{dy}{y^2} = \int \mathrm{e}^x \mathrm{d}x$$

i.e $\frac{-1}{y} = \mathrm{e}^x + c$ or $\boxed{y = \frac{-1}{(\mathrm{e}^x + c)}}$

Almost separable equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(ax+by)$$

Change variables : $z = ax + by$ $\qquad \frac{\mathrm{d}z}{\mathrm{d}x} = a + b\frac{\mathrm{d}y}{\mathrm{d}x}$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = a + bf(z) \quad \Rightarrow \quad \boxed{x = \int \frac{1}{(a+bf(z))}\,\mathrm{d}z.}$$

Ex 2 $\qquad \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = (-4x+y)^2}$ $\qquad z = y - 4x \quad \Rightarrow \quad \frac{\mathrm{d}z}{\mathrm{d}x} = -4 + \frac{\mathrm{d}y}{\mathrm{d}x} = z^2 - 4$

$$x = \tfrac{1}{4}\ln\left(\tfrac{z-2}{z+2}\right) + \mathrm{C} \qquad \boxed{\Rightarrow \quad y = 4x + 2\,\frac{(1+ke^{4x})}{(1-ke^{4x})}}$$

k a constant

Homogeneous equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y/x).$$

The equation is invariant under $x \to sx, \quad y \to sy$ .... homogeneous

Solution $$y = \mathrm{v}x \quad \Rightarrow \quad y' = \mathrm{v}'x + \mathrm{v}.$$

$$i.e. \quad \mathrm{v}' = \frac{1}{x}(f(\mathrm{v}) - \mathrm{v})$$

$$\boxed{\int \frac{\mathrm{d}v}{f(\mathrm{v}) - \mathrm{v}} = \int \frac{\mathrm{d}x}{x} = \ln x + \text{constant.}}$$

Ex 3 $$\boxed{xy\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = (x+y)^2 \mathrm{e}^{-y/x}}$$ Homogeneous

Change variables $y = \mathrm{v}x \quad \Rightarrow \quad y' = \mathrm{v}'x + \mathrm{v}.$

$$(\mathrm{v}'x + \mathrm{v}) - \mathrm{v} = \frac{(1+\mathrm{v})^2}{\mathrm{v}}\mathrm{e}^{-\mathrm{v}} \quad \Rightarrow \quad \ln x = \int \frac{\mathrm{e}^{\mathrm{v}}\mathrm{v}\mathrm{d}\mathrm{v}}{(1+\mathrm{v})^2}.$$

To evaluate integral change variables $u \equiv 1 + \mathrm{v}$

$$\mathrm{e}^{-1}\int\left(\frac{1}{u} - \frac{1}{u^2}\right)\mathrm{e}^{u}\mathrm{d}u = \mathrm{e}^{-1}\left[\frac{\mathrm{e}^{u}}{u}\right].$$

$$\boxed{i.e. \quad \ln x = \frac{e^{\frac{y}{x}}}{1+\frac{y}{x}}}$$

Homogeneous but for constants

$$\frac{dy}{dx} = \frac{x+2y+1}{x+y+2}$$

$$x = x'+a, \quad y = y'+b \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{dy'}{dx} = \frac{dy'}{dx'}.\frac{dx'}{dx} = \frac{dy'}{dx'}$$

$1+a+2b=0$

$$\frac{dy'}{dx'} = \frac{x'+2y'+1+a+2b}{x'+y'+2+a+b}$$

$2+a+b=0$

$a=-3, \quad b=1$

$$\boxed{\frac{dy'}{dx'} = \frac{x'+2y'}{x'+y'}}$$

Homogeneous

The Bernoulli equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n, \qquad n \neq 1$$

To solve, change variable to $z = y^{1-n}$ $\Rightarrow$ $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

Gives the equation $\boxed{\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)}$ 1st order Linear

Ex 4 $\boxed{y' + y = y^{2/3}}$ $z = y^{1-n} = y^{1/3}$ $\Rightarrow$ $z' + \frac{z}{3} = \frac{1}{3}$

1st order linear

Integrating factor $\mathrm{e}^{x/3}$ $\Rightarrow$ $ze^{x/3} = \int e^{x/3}dx/3$

$$\boxed{z = y^{1/3} = 1 + ce^{-x/3}}$$

Exercise:

Solve the equation $2\, y\,' = y/x + x^2/y$
with initial condition $y(1) = 2$.

- This equation is Bernoulli with $n = -1$.
  - Set $z = y^2$. Then $z' - z/x = x^2$.
    - Integrating factor $I(x) = 1/x$

$$\Rightarrow \;\; z(x) = x\, [\int dx\; x^2/x + \mathrm{const.}] = x^3/2 + \mathrm{const.}\; x$$

$$\text{Thus } \; y = z^{1/2} = \pm\sqrt{x^3/2 + \mathrm{const.}\; x}$$

- Initial condition $y(1) = 2$

$$\Rightarrow \;\; y(x) = \sqrt{\frac{x^3 + 7x}{2}}$$

Homework

1. Solve the differential equation

$$2\,\frac{dy}{dx} = \frac{y(x+y)}{x^2} \qquad \text{“homogeneous”}$$

with $y(1) = -1$.

[Answ.: $y = x/(1 - 2\sqrt{x})$]

2. Solve the differential equation

$$\frac{dy}{dx} + xy = xy^2 \qquad \text{“Bernoulli”}$$

with $y(0) = 1/2$.

[Answ.: $y = 1/(1 + e^{x^2/2})$]

Exact equations

- A first-order ODE

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if there exists a function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = M \quad , \qquad \frac{\partial \phi}{\partial y} = N \ .$$

- In this case the differential equation can be recast as

$$d\phi = M(x, y)dx + N(x, y)dy = 0$$

so that the solution to it is determined by

$$\phi(x, y) = \text{constant} \ .$$

Example: Solve the equation $xy' = -2\tan y$.

• This equation can be rewritten as

$$2x\sin y\ dx + x^2\cos y\ dy = 0\ ,$$

$$\text{i.e.,}\quad M(x,y) = 2x\sin y\ \ ,\qquad N(x,y) = x^2\cos y\ ,$$

which is exact because

$$\frac{\partial\phi}{\partial x} = 2x\sin y\ \Rightarrow\ \phi(x,y) = x^2\sin y + \alpha(y)$$

$$\frac{\partial\phi}{\partial y} = x^2\cos y\ \Rightarrow\ x^2\cos y + \alpha'(y) = x^2\cos y\ \Rightarrow\ \alpha = \text{constant}$$

• Therefore $\phi(x,y) = x^2\sin y + \mathrm{c}$,
and the general solution is determined by $x^2\sin y = \text{const.}$:

$$\Rightarrow\ y(x) = \arcsin\left(\text{const.}/x^2\right)$$

DIFFERENTIAL EQUATIONS AND FAMILIES OF CURVES

- General solution of a first-order ODE $y' = f(x, y)$
  contains an arbitrary constant: $y = (x, c)$
  ▷ one curve in x, y plane for each value of c
  ▷ general solution can be thought of as one-parameter family of curves

Example: $y' = -x/y$.

$$\text{separable equation} \quad \Rightarrow \int y \; dy = - \int x \; dx \;\Rightarrow\; y^2/2 = -x^2/2 + c$$

$$\text{i.e., } \; x^2 + y^2 = \text{constant} : \quad \text{family of circles centered at origin}$$

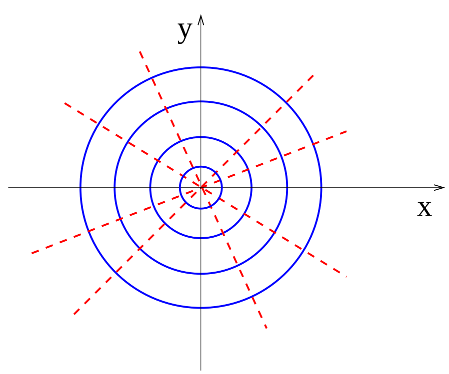


Fig.1

Orthogonal trajectories

• Given the family of curves representing solutions of ODE $y' = f(x, y)$, orthogonal trajectories are given by a second family of curves which are solutions of

$$y' = -1/f(x, y).$$

◊ Then each curve in either family is perpendicular to every curve in the other family.

Example:

Find the orthogonal trajectories to the family of circles $y' = -x/y$.

• Solve $y' = y/x$.

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln y = \ln x + \text{constant}$$

i.e., $y = cx$: family of straight lines through the origin

Homework

a) Find the family of curves corresponding to solutions of the ODE

$$y' = (y^2 - x^2)/(2xy).$$

b) Find the orthogonal trajectories to the above family of curves.

• homogeneous equation $y' = f(y/x)$ with $f(y/x) = (y/x - x/y)/2$

solvable by $y \to v = y/x$ and separation of variables

$\Rightarrow$ $x^2 + y^2 = cx$: family of circles tangent to $y-$axis at 0

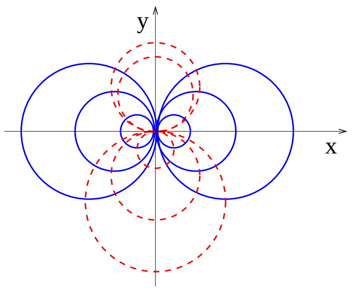


Fig.2

• orthogonal trajectories found by solving $y' = -2xy/(y^2 - x^2)$

$\Rightarrow$ $x^2 + y^2 = ky$: family of circles tangent to $x-$axis at 0

EXPLOITING FIRST-ORDER METHODS TO TREAT EQUATIONS OF HIGHER ORDER IN SPECIAL CASES

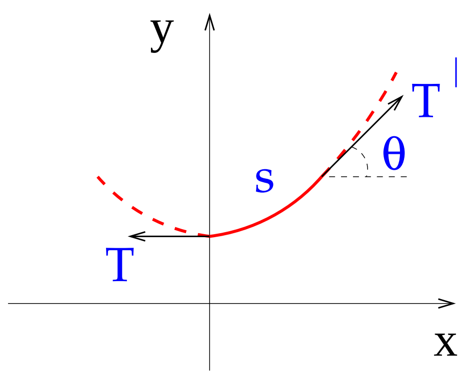
♣ y not present in 2nd-order equation $F(x, y, y', y'') = 0$

$\Rightarrow$ setting $y' = q$ yields 1st-order equation for $q(x)$.

♣ x not present in 2nd-order equation $F(x, y, y', y'') = 0$

$\Rightarrow$ setting $y' = q$, $y'' = dq/dx = q(dq/dy)$ yields $G(y, q, dq/dy) = 0$.

Example: homogeneous, flexible chain hanging under its own weight



ρ = *linear mass density*

Using Newton's law, the shape y(x) of the chain obeys the 2nd−order nonlinear differential equation

$$y'' = a\sqrt{1 + (y')^2} \quad , \quad a \equiv \rho g / T$$

Setting $y' = q \implies q' = a\sqrt{1 + q^2}$

- Separation of variables $\Rightarrow \displaystyle\int \frac{1}{\sqrt{1+q^2}}\, dq = a \int dx$

- Using $q = dy/dx = 0$ at $x = 0 \;\Rightarrow\; \ln(q + \sqrt{1+q^2}) = ax$

- Solving for $q \;\Rightarrow\; q = dy/dx = (e^{ax} - e^{-ax})/2$

$$\text{Thus } y(x) = \frac{1}{a}\,\frac{e^{ax} + e^{-ax}}{2} + \text{constant} = \frac{1}{a}\,\cosh ax + \text{constant}$$

This curve is called a *catenary*.

Historical note. The problem of the catenary was the subject of a challenge posed by Jakob Bernoulli in 1690, in response to which the problem was solved the following year indipendently by Johann Bernoulli, Leibniz and Huygens.

Homework

1. Find the function $y(x)$ obeying the differential equation

$$y'^2 = x^2 y''$$

and the conditions $y(0) = 2$, $y'(1) = 2$.
[Hint: set $y' = q$ and apply separation of variables.]

[Answ.: $y(x) = 2(1 - x) - 4\ln(1 - x/2)$]

2. Find the function $y(x)$ obeying the differential equation

$$y'' = y' e^y$$

and the conditions $y(0) = 0$, $y'(0) = 1$.
[Hint: $y' = q$; $y'' = dq/dx = q(dq/dy)$; solve equation for $q(y)$.]

[Answ.: $y(x) = -\ln(1 - x)$]

Summary

$\Diamond$ No general method of solution for 1st-order ODEs beyond linear case; rather, a variety of techniques that work on a case-by-case basis.

Main guiding criteria:

- methods to bring equation to separated-variables form
- methods to bring equation to exact differential form
- transformations that linearize the equation

$\Diamond$ 1st-order ODEs correspond to families of curves in x, y plane
$\Rightarrow$ geometric interpretation of solutions

$\Diamond$ Equations of higher order may be reduceable to first-order problems in special cases — e.g. when y or x variables are missing from 2nd order equations