Lecture 4

- Roots of complex numbers

- Characterization of a polynomial by its roots

- Techniques for solving polynomial equations

ROOTS OF COMPLEX NUMBERS

Def.:

- A number u is said to be an n-th root of complex number z if $u^n = z$, and we write $u = z^{1/n}$.

Th.:

- Every complex number has exactly n distinct n-th roots.

Let $z = r(\cos\theta + i\sin\theta)$; $u = \rho(\cos\alpha + i\sin\alpha)$. Then

$$r(\cos\theta + i\sin\theta) = \rho^n(\cos\alpha + i\sin\alpha)^n = \rho^n(\cos n\alpha + i\sin n\alpha)$$

$$\Rightarrow \ \rho^n = r \ , \quad n\alpha = \theta + 2\pi k \qquad (k \text{ integer})$$

$$\text{Thus } \ \rho = r^{1/n} \ , \quad \alpha = \theta/n + 2\pi k/n \ .$$

n distinct values for k from 0 to $n-1$. $(z \neq 0)$

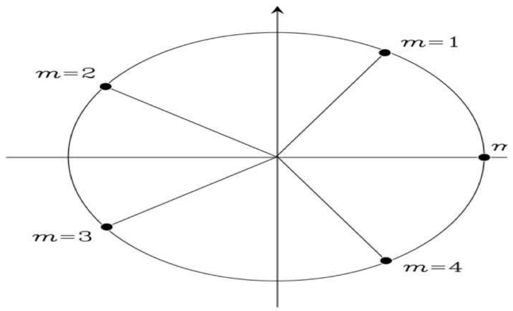
$$\text{So } \ u = z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right] \ , \quad k = 0, 1, \ldots, n-1$$

Note. $f(z) = z^{1/n}$ is a "multi-valued" function.

Example 1 : nth roots of unity :

$$x^n = 1 \quad (i.e.\, x^n - 1 = 0)$$

$$\Rightarrow \quad x = 1^{1/n}$$



$$1 = e^{2m\pi\imath} \quad \Rightarrow \quad 1^{1/n} = e^{2m\pi\imath/n}$$

$$= \cos(\frac{2m\pi}{n}) + \imath \sin(\frac{2m\pi}{n})$$

$$1^{1/5} = \cos(\frac{2m\pi}{5}) + \imath \sin(\frac{2m\pi}{5}) \quad (m = 0, 1, 2, 3, 4).$$

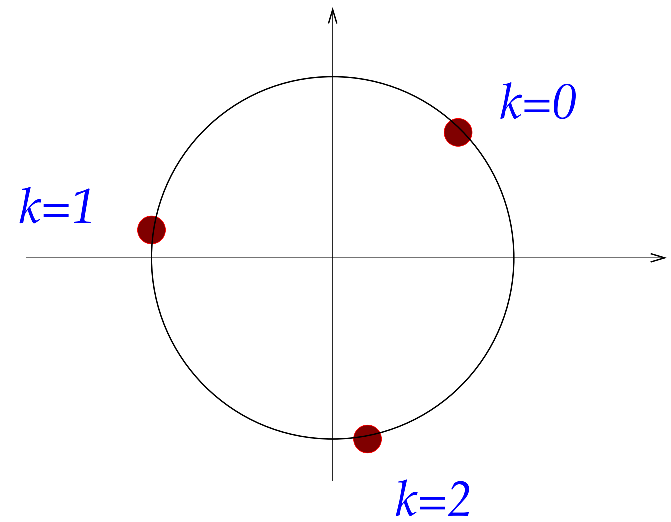
Example 2. Find all cubic roots of $z = -1 + i$:

$$u = (-1+i)^{1/3}$$

$$u = (\sqrt{2})^{1/3}\left[\cos\left(\frac{3\pi}{4}\frac{1}{3} + \frac{2\pi k}{3}\right) + i\sin\left(\frac{3\pi}{4}\frac{1}{3} + \frac{2\pi k}{3}\right)\right] \quad , \quad k = 0, 1, 2$$

that is,

$$k = 0 : \quad 2^{1/6}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right)$$
$$k = 1 : \quad 2^{1/6}\left(\cos\tfrac{11\pi}{12} + i\sin\tfrac{11\pi}{12}\right)$$
$$k = 2 : \quad 2^{1/6}\left(\cos\tfrac{19\pi}{12} + i\sin\tfrac{19\pi}{12}\right)$$



- Equivalently:

$$u = (-1+i)^{1/3} = e^{(1/3)\ln(-1+i)} = e^{(1/3)[\ln\sqrt{2} + i(3\pi/4 + 2k\pi)]}$$

$$= (\sqrt{2})^{1/3} e^{i(\pi/4 + 2k\pi/3)}$$

Roots of polynomials

$$P(z) \equiv a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 .$$

$$P(z = z_i) = 0 \qquad \Rightarrow z_i \text{ is a root}$$

Characterising a polynomial by its roots ..the "fundamental theorem of algebra"

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = a_n (z - z_1)(z - z_2)\cdots(z - z_n)$$

In mathematics, the fundamental theorem of algebra states that every non-zero single-variable polynomial, with complex coefficients, has exactly as many complex roots as its degree, if repeated roots are counted up to their multiplicity.

[Gauss, 1799]

- proof of fundamental theorem of algebra is given in the course "Functions of a complex variable", Short Option S1

Roots of polynomials

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Characterising a polynomial by its roots

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = a_n (z - z_1)(z - z_2) \cdots (z - z_n)$$

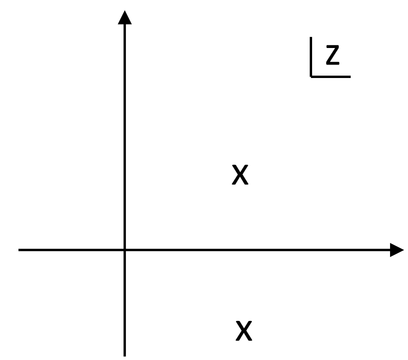
$$= a_n \left(z^n - z^{n-1} \sum_{j=1}^{n} z_j + \cdots + (-1)^n \prod_{j=1}^{n} z_j\right).$$

Comparing coefficients of z^{n-1} and z^0

$$\frac{a_{n-1}}{a_n} = -\sum_j z_j \quad ; \quad \frac{a_0}{a_n} = (-1)^n \prod_j z_j$$

e.g. quadratic equations : $a_2 x^2 + a_1 x + a_0$

Roots: $$x_{1,2} = \frac{\left(-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}\right)}{2a_2}$$



If complex, roots come in complex conjugate pairs

Sum of roots $\frac{a_1}{a_2} = -(x_1 + x_2)$ Product of roots $\frac{a_0}{a_2} = x_1 . x_2$

for roots

- General solutions not available for higher order polynomials (quartics and above)

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Can find solutions in special cases….

Example 1:

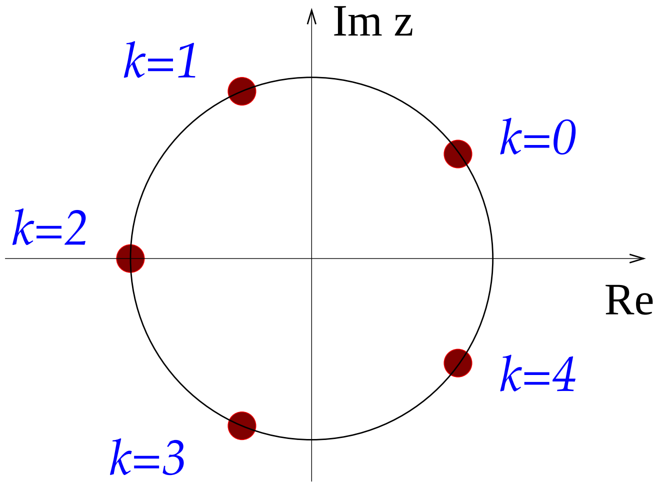
$$z^5 + 32 = 0$$

- The solutions of the given equation are the fifth roots of -32:

$$(-32)^{1/5} = 32^{1/5}\left[\cos\left(\frac{\pi}{5} + \frac{2\pi k}{5}\right) + i\sin\left(\frac{\pi}{5} + \frac{2\pi k}{5}\right)\right] \quad , \qquad k = 0, 1, 2, 3, 4$$

that is,

$$k = 0: \quad 2\left(\cos\tfrac{\pi}{5} + i\sin\tfrac{\pi}{5}\right)$$
$$k = 1: \quad 2\left(\cos\tfrac{3\pi}{5} + i\sin\tfrac{3\pi}{5}\right)$$
$$k = 2: \quad -2$$
$$k = 3: \quad 2\left(\cos\tfrac{7\pi}{5} + i\sin\tfrac{7\pi}{5}\right)$$
$$k = 4: \quad 2\left(\cos\tfrac{9\pi}{5} + i\sin\tfrac{9\pi}{5}\right)$$



Example 2 : Roots of polynomials

$$(z+\imath)^7 + (z-\imath)^7 = 0$$

$$\left(\frac{z+\imath}{z-\imath}\right)^7 = -1 = \mathrm{e}^{(2m+1)\pi\imath}$$

$$\Rightarrow \quad \frac{z+\imath}{z-\imath} = \mathrm{e}^{(2m+1)\pi\imath/7}$$

$$\Rightarrow \quad z(1-\mathrm{e}^{(2m+1)\pi\imath/7}) = -\imath(1+\mathrm{e}^{(2m+1)\pi\imath/7})$$

$$\Rightarrow \quad z = \imath\frac{\mathrm{e}^{(2m+1)\pi\imath/7}+1}{\mathrm{e}^{(2m+1)\pi\imath/7}-1}$$

$$= \imath\frac{\mathrm{e}^{(2m+1)\pi\imath/14}+\mathrm{e}^{-(2m+1)\pi\imath/14}}{\mathrm{e}^{(2m+1)\pi\imath/14}-\mathrm{e}^{-(2m+1)\pi\imath/14}} = \imath\frac{2\cos(\frac{2m+1}{14}\pi)}{2\imath\sin(\frac{2m+1}{14}\pi)} = \cot(\tfrac{2m+1}{14}\pi)$$

m=0,1,2,3,4,5,6

Example 2 : an alternative form

$$(z+\imath)^7 + (z-\imath)^7 = 0$$

We will often need the coefficient of $x^r y^{n-r}$ in $(x+y)^n$

These are conveniently obtained from Pascal's triangle :

$(x+y)^0$						1					
$(x+y)^1$					1		1				
$(x+y)^2$				1		2		1			
$(x+y)^3$			1		3		3		1		
$(x+y)^4$		1		4		6		4		1	
$(x+y)^5$	1		5		10		10		5		1

Solution:

$z = \cot(\frac{2m+1}{14}\pi)$

7th row of Pascal's triangle is 1 7 21 35 35 21 7 1 so

$$(z+\imath)^7 + (z-\imath)^7 = 0 \quad \Rightarrow \quad z^7 - 21z^5 + 35z^3 - 7z = 0$$

Example 2 : yet another form

The original equation $(z+\imath)^7+(z-\imath)^7=0 \quad \Rightarrow \quad z^7-21z^5+35z^3-7z=0$

can be written in another form :

$$z^7-21z^5+35z^3-7z=0$$

$$\Rightarrow \quad z^6-21z^4+35z^2-7=0 \quad or \quad z=0$$

$z_m, \quad m=3$

$$\Rightarrow \quad w^3-21w^2+35w-7=0 \quad (w\equiv z^2)$$

Hence the roots of $w^3-21w^2+35w-7=0$ are

$$w=\cot^2\left(\tfrac{2m+1}{14}\pi\right) \qquad (m=0,1,2)$$

Sum of roots $\Rightarrow \sum_{m=0}^{2}\cot^2\left(\tfrac{2m+1}{14}\pi\right)=21$

Pascal's triangle = table of binomial coefficients $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

i.e., coefficients of $x^r y^{n-r}$ in $(x+y)^n$

[Pascal, 1654]

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}
$$

- r-th element of n-th row given by sum of two elements above it in $(n-1)$-th row:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

[use $(x+y)^n = (x+y)^{n-1}x + (x+y)^{n-1}y$]

Historical note

$\diamondsuit$ binomial coefficients already known in the middle ages:

- "Pascal's triangle" first discovered by Chinese mathematicians of 13th century to find coefficients of $(x+y)^n$
- $\left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{r!(n-r)!}$ in Hebrew writings of 14th century is number of combinations of n objects taken r at a time

$\diamondsuit$ Pascal (1654) rediscovers triangle and most importantly unites algebraic and combinatorial viewpoints

$\longrightarrow$ theory of probability; proof by induction

Ex 3 Another example where the underlying equation is not obvious :

$$z^3 + 7z^2 + 7z + 1 = 0.$$

$(x+y)^0$						1					
$(x+y)^1$					1		1				
$(x+y)^2$				1		2		1			
$(x+y)^3$			1		3		3		1		
$(x+y)^4$		1		4		6		4		1	
$(x+y)^5$	1		5		10		10		5		1

9th row of Pascal's triangle is $1 \;\; 8 \;\; 28 \;\; 56 \;\; 70 \;\; 56 \;\; 28 \;\; 8 \;\; 1$ so

$$\tfrac{1}{2}[(z+1)^8 - (z-1)^8] = 8z^7 + 56z^5 + 56z^3 + 8z$$

$$= 8z[w^3 + 7w^2 + 7w + 1] \quad (w \equiv z^2).$$

Now $(z+1)^8 - (z-1)^8 = 0$ when $\frac{z+1}{z-1} = \mathrm{e}^{2m\pi i/8}$

i.e. when $z = \dfrac{\mathrm{e}^{m\pi\imath/4} + 1}{\mathrm{e}^{m\pi\imath/4} - 1} = -\imath\cot(m\pi/8) \quad (m = 1, 2, \ldots, 7),$

so the roots of the given equation are

$$z = -\cot^2(m\pi/8) \qquad m = 1, 2, 3$$

Example

Question from 2008 Paper

Find all the solutions of the equation

$$\left(\frac{z+i}{z-i}\right)^n = -1\,,$$

and solve

$$z^4 - 10z^2 + 5 = 0\,.$$

$$\left(\frac{z+i}{z-i}\right)^n = -1 = e^{i(\pi+2N\pi)}\,,\quad N \text{ integer} \quad \Rightarrow \quad \frac{z+i}{z-i} = e^{i(\pi/n+2N\pi/n)}\,,\quad N = 0, 1, \ldots, n-1$$

$$\text{Then}: \quad z = i\,\frac{e^{i(\pi/n+2N\pi/n)}+1}{e^{i(\pi/n+2N\pi/n)}-1} = i\,\frac{\cos[\pi(1+2N)/(2n)]}{i\,\sin[\pi(1+2N)/(2n)]} = \operatorname{cotg}\frac{\pi(1+2N)}{2n}\,.$$

For $n=5$: $(z+i)^5 = -(z-i)^5 \Rightarrow z(z^4-10z^2+5)=0$. Then the 4 roots of $z^4-10z^2+5=0$ ar

$$\operatorname{cotg}\frac{\pi}{10}\,,\operatorname{cotg}\frac{3\pi}{10}\,,\operatorname{cotg}\frac{7\pi}{10}\,,\operatorname{cotg}\frac{9\pi}{10}\,.$$

Ex 4 Show that

$$\frac{z^{2m}-a^{2m}}{z^2-a^2}=(z^2-2az\cos\frac{\pi}{m}+a^2)(z^2-2az\cos\frac{2\pi}{m}+a^2)\cdots(z^2-2az\cos\frac{(m-1)\pi}{m}+a^2).$$

i.e. Show that $P(z)=Q(z)$ where

$$P(z)\equiv z^{2m}-a^{2m}$$

(Roots : $z_r = a\mathrm{e}^{r\pi i/m}$)

$$Q(z)\equiv(z^2-a^2)(z^2-2az\cos\frac{\pi}{m}+a^2)(z^2-2az\cos\frac{2\pi}{m}+a^2)\cdots(z^2-2az\cos\frac{(m-1)\pi}{m}+a^2).$$

Roots

$$z_r = a\cos\frac{r\pi}{m}\pm\sqrt{a^2\cos^2\frac{r\pi}{m}-a^2}$$

$$=a(\cos\frac{r\pi}{m}\pm\iota\sqrt{1-\cos^2\frac{r\pi}{m}})=a\mathrm{e}^{\pm\iota r\pi/m}\quad(r=0,1,..,m).$$

Leading coefficient $a_{2m}=1$ → P(z) and Q(z) identical

- This concludes part A of the course.

A. Complex numbers

1 Introduction to complex numbers

2 Fundamental operations with complex numbers

3 Elementary functions of complex variable

4 De Moivre's theorem and applications

5 Curves in the complex plane

6 Roots of complex numbers and polynomials