Lecture 3

• Applications of de Moivre’s theorem

• Curves in the complex plane

• Complex functions as mappings
Uses of de Moivre and complex exponentials

\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\]

- We saw application to trigonometric identities, functional relations for trig. and hyperb. fctns.

- We next see examples of two more kinds of applications:
  - solution of differential equations
  - summation of series
Uses of de Moivre and complex exponentials

Ex. 2  Solving differential equations

The solution to $\frac{d^2 z}{d\theta^2} + z = 0$ is simply given by

$$z = Ce^{i\theta} \quad \text{where } C = A + i B \text{ is a complex constant.}$$

$$z = x + i y$$

$$y = \text{Im}(z) = \text{Im}((A + i B)(\cos \theta + i \sin \theta))$$

$$= A \sin \theta + B \cos \theta$$

\[c.f.\]  $\frac{d^2 y}{d\theta^2} + y \equiv \text{Im}(\frac{d^2 z}{d\theta^2} + z) = 0, \quad v = A \cos \theta + B \sin \theta$
Uses of de Moivre and complex exponentials

Ex. 3  Summing series

e.g. Show \[ \sum_{n=0}^{\infty} r^n \sin(2n+1)\theta = \frac{(1 + r) \sin \theta}{1 - 2r \cos 2\theta + r^2}, \quad 0 < r < 1 \]

\[
\sum_{n=0}^{\infty} r^n \sin(2n+1)\theta = \text{Im} \sum_{n} r^n (e^{i(2n+1)\theta}) = \text{Im}(e^{i\theta} \sum_{n} (re^{2i\theta})^n)
\]

\[
= \text{Im}(e^{i\theta} \frac{1}{1 - re^{2i\theta}})
\]

\[
= \text{Im}\left(\frac{e^{i\theta} (1 - re^{-2i\theta})}{(1 - re^{2i\theta})(1 - re^{-2i\theta})}\right)
\]

\[
= \frac{\sin \theta + r \sin \theta}{1 - 2r \cos 2\theta + r^2}
\]
Homework
(Question from Maths Collection Paper 2006)

◊ Find Re and Im of $1/(1 - z)$, where $z = re^{i\theta}$, and hence sum the series $\sum_{n=0}^{\infty} r^n \cos(n\theta)$ for $r < 1$.

Answ. : \[
\frac{1}{1 - z} = \frac{1 - z^*}{|1 - z|^2} = \underbrace{\frac{1 - r \cos \theta}{1 + r^2 - 2r \cos \theta}}_{\text{Re}} + i \underbrace{\frac{r \sin \theta}{1 + r^2 - 2r \cos \theta}}_{\text{Im}}
\]

\[
\sum_{n=0}^{\infty} r^n \cos(n\theta) = \sum_{n=0}^{\infty} r^n \text{Re} \ e^{in\theta} = \text{Re} \sum_{n=0}^{\infty} (re^{i\theta})^n
\]

= \text{Re} \ \frac{1}{1 - re^{i\theta}} = \frac{1 - r \cos \theta}{1 + r^2 - 2r \cos \theta}

◊ Show that $1 + \frac{\cos \theta}{2} + \frac{\cos 2\theta}{4} + \frac{\cos 3\theta}{8} + \ldots = \frac{4 - 2 \cos \theta}{5 - 4 \cos \theta}$.

Answ. : \[
\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} \text{Re} \ e^{in\theta} = \text{Re} \sum_{n=0}^{\infty} \left(\frac{e^{i\theta}}{2}\right)^n
\]

= \text{Re} \ \frac{1}{1 - e^{i\theta}/2} = \frac{1 - \cos \theta/2}{1 + 1/4 - \cos \theta} = \frac{4 - 2 \cos \theta}{5 - 4 \cos \theta}$
CURVES IN THE COMPLEX PLANE

• locus of points satisfying constraint in complex variable $z$

Examples

$|z| = 1$

$z - z^* = i$

$\arg z = \pi / 3$
Curves in the complex plane

Ex 1 \[ |z| = 1 \]

\[ z = re^{i\theta} \implies r = 1, \text{ any } \theta \]

or \[ (a)^2 + (b)^2 = 1, \quad z = a + ib \]

Circle centre (0,0), radius 1

\[ |z - z_0| = 1 \]

Circle centre \( z_0 \), radius 1

\[ (a - a_0)^2 + (b - b_0)^2 = 1 \]
Curves in the complex plane

Ex 2

\[ \left| \frac{z-i}{z+i} \right| = 1 \]

\[ |z - i| = |z + i| \]

Distance from (0,1) = distance from (0,-1) \[\Longleftrightarrow\] Real axis

or \( (a)^2 + (b-1)^2 = (a)^2 + (b+1)^2 \) \( z = a + ib \)

\[\Longleftrightarrow\] \( b = 0, \quad a \) arbitrary
Another way to define a curve:

\[\gamma : [a, b] \rightarrow \mathbb{C}\]

\[\gamma : t \mapsto z = \gamma(t) = x(t) + iy(t)\]

- \[x(t) = x_0 + R \cos t\]
- \[y(t) = y_0 + R \sin t\]
  
  \[0 < t < 2\pi\]

- \[x(t) = x_0 + R_1 \cos t\]
- \[y(t) = y_0 + R_2 \sin t\]

- \[x(t) = t \cos t\]
- \[y(t) = t \sin t\]

i.e., \[z = t e^{i t}\]

\[Im\ z\]

\[Re\ z\]
Homework
(Maths Paper 2007)

Sketch the curve in the complex plane

\[ z = a^t e^{ibt}, \quad -6 \leq t \leq 6 \]

where \( a \) and \( b \) are real constants given by \( a = 1.1, \ b = \pi/3 \).
Curves in the complex plane

Ex 3 \[ \arg\left(\frac{z}{z+1}\right) = \frac{\pi}{4} \]

i.e. \[ \arg(z) - \arg(z+1) = \frac{\pi}{4} \]

Take tan of both sides:

\[ \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \]

\[ \frac{\frac{b}{a} - \frac{b}{a+1}}{1 + \frac{b}{a} \cdot \frac{b}{a+1}} = 1 = \frac{b(a+1) - ba}{a(a+1) + b^2} \]

\[ b = a(a+1) + b^2 \]

\[ (a + \frac{1}{2})^2 + (b - \frac{1}{2})^2 = \frac{1}{2} \]

…but BEWARE…not all of circle satisfies equation…
\[
\arg\left(\frac{z}{z+1}\right) = \frac{\pi}{4}
\]

\[
\frac{z}{z+1} = \frac{z}{z+1} \cdot \frac{z^* + 1}{z^* + 1} = \frac{z(z^* + 1)}{|z+1|^2} = \frac{(a+ib)(a+1-ib)}{|z+1|^2} = \frac{a(a+1) + b^2 + ib}{|z+1|^2}
\]

\[
\Rightarrow \quad b > 0 \text{ since } \arg\left(\frac{z}{z+1}\right) \text{ positive}
\]

\[
(a + \frac{1}{2})^2 + (b - \frac{1}{2})^2 = \frac{1}{2}, \quad b > 0
\]

\[
(b < 0 \text{ solution introduced by tangent ambiguity})
\]
Alternative solution

\[ \text{arg}\left(\frac{z}{z+1}\right) = \frac{\pi}{4} \]

\[ \text{i.e. } \text{arg}(z) - \text{arg}(z+1) = \frac{\pi}{4} \]

Solution: portion of circle through (0,0) and (-1,0)

Circle centre (-1/2,1/2) and radius 1/\sqrt{2}
The lower portion of the circle is given by:

$$\arg\left(\frac{z}{z+1}\right) = -\frac{3\pi}{4}$$
Homework
(Question from Maths Collection Paper 2009)

Sketch the locus of points in the complex plane for

(a) \( \text{Im}(z^2 + i) = -1 \),
(b) \( \text{arg}(z^2 + i) = \pi/2 \)

Answ. :

\[
z = x + iy ; z = x - y + 2ixy
\]

(a) \( 1 + 2xy = -1 \Rightarrow xy = -1 \)
\( y = -1/x \)

(b) \[
\theta = \arctg \left[ \frac{1 + 2xy}{x^2 - y^2} \right] = \frac{\pi}{2}
\]
\( x^2 = y^2 \Rightarrow x = y \)
\( x = -y, 1 - 2x^2 > 0 \)
Summary on Curves in the Complex Plane

- locus of points satisfying constraint in complex variable $z$

♠ Curves can be defined

▷ either directly by constraint in complex variable $z$

ex. : $|z| = R$

▷ or by parametric form $\gamma(t) = x(t) + iy(t)$

ex. : $x(t) = R \cos t$

$y(t) = R \sin t , \quad 0 \leq t < 2\pi$

♦ intersection of algebraic and geometric approaches
COMPLEX FUNCTIONS AS MAPPINGS

\[ f : z \mapsto w = f(z) \]

• Complex functions can be viewed as mapping sets of points in \( \mathbb{C} \) (e.g. curves, regions) into other sets of points in \( \mathbb{C} \).

Example:

\[ w = f(z) = e^z \]. Set \( z = x + iy \); \( w = \rho e^{i\phi} \). Then \( \rho = e^x \); \( \phi = y \).

lines \( x = a \) \( \mapsto \) circles \( \rho = e^a \)
lines \( y = b \) \( \mapsto \) rays \( \phi = b \)

• For any subset \( A \) of \( \mathbb{C} \), the “image of \( A \) through \( f \)” is the set of points \( w \) such that \( w = f(z) \) for some \( z \) belonging to \( A \), and is denoted \( f(A) \).

• \( f \) maps \( A \) on to \( f(A) \)
Examples

- Let $f(z) = 1/z$. What does $f$ do to the interior and exterior of the unit circle $|z| = 1$? What does $f$ do to points on the unit circle?

  **Answ.**

  $w = f(z) = \frac{1}{z}$ ; $z = re^{i\theta} \Rightarrow w = \frac{1}{r} e^{-i\theta}$

  ▶ $f$ maps the interior of the unit circle on to the exterior, and vice versa. Unit circle is mapped on to itself.

- Let $f(z) = (z - 1)/(z + 1)$. What is the image through $f$ of the imaginary axis? And of the right and left half planes?

  [Answ.: unit circle] [Answ.: interior and exterior of unit circle]

- Let $f(z) = \ln z$ (principal branch). What is the image through $f$ of the upper half plane?

  [Answ.: horizontal strip between 0 and $i\pi$]