Lecture 3

- Applications of de Moivre's theorem
- Curves in the complex plane
- Complex functions as mappings

Uses of de Moivre and complex exponentials

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

- We saw application to trigonometric identities, functional relations for trig. and hyperb. fctns.

- We next see examples of two more kinds of applications:

  ◊ solution of differential equations

  ◊ summation of series

Uses of de Moivre and complex exponentials

Ex. 2 Solving differential equations

The solution to $\frac{d^2 z}{d\theta^2} + z = 0$ is simply given by

$z = Ce^{i\theta}$ where $C = A + \imath B$ is a complex constant.

$$z = x + i\,\mathrm{y}$$

$$\mathrm{y} = \mathrm{Im}(z) = \mathrm{Im}((A + \imath B)(\cos\theta + \imath \sin\theta))$$
$$= A\sin\theta + B\cos\theta$$

c.f. $\frac{d^2 y}{d\theta^2} + y \equiv \mathrm{Im}(\frac{d^2 z}{d\theta^2} + z) = 0,\ \mathrm{v} = A\cos\theta + B\sin\theta$

Uses of de Moivre and complex exponentials

Ex. 3 Summing series

e.g. Show $$\sum_{n=0}^{\infty} r^n \sin(2n+1)\theta = \frac{(1+r)\sin\theta}{1-2r\cos 2\theta + r^2}, \qquad 0 < r < 1$$

$$\sum_{n=0}^{\infty} r^n \sin(2n+1)\theta = \mathrm{Im}\sum_n r^n (\mathrm{e}^{\imath(2n+1)\theta}) = \mathrm{Im}(\mathrm{e}^{\imath\theta}\sum_n (r\mathrm{e}^{2\imath\theta})^n)$$

$$= \mathrm{Im}(\mathrm{e}^{\imath\theta}\frac{1}{1-r\mathrm{e}^{2\imath\theta}})$$

$$= \mathrm{Im}(\frac{\mathrm{e}^{\imath\theta}(1-r\mathrm{e}^{-2\imath\theta})}{(1-r\mathrm{e}^{2\imath\theta})(1-r\mathrm{e}^{-2\imath\theta})})$$

$$= \frac{\sin\theta + r\sin\theta}{1-2r\cos 2\theta + r^2}$$

Homework

(Question from Maths Collection Paper 2006)

◊ Find Re and Im of $1/(1-z)$, where $z = re^{i\theta}$, and hence sum the series $\sum_{n=0}^{\infty} r^n \cos(n\theta)$ for $r < 1$.

$$Answ.: \ \frac{1}{1-z} = \frac{1-z^*}{|1-z|^2} = \underbrace{\frac{1-r\cos\theta}{1+r^2-2r\cos\theta}}_{\text{Re}} + i \ \underbrace{\frac{r\sin\theta}{1+r^2-2r\cos\theta}}_{\text{Im}}$$

$$\sum_{n=0}^{\infty} r^n \cos(n\theta) = \sum_{n=0}^{\infty} r^n \text{Re } e^{in\theta} = \text{Re} \sum_{n=0}^{\infty} (re^{i\theta})^n$$

$$= \text{Re } \frac{1}{1-re^{i\theta}} = \frac{1-r\cos\theta}{1+r^2-2r\cos\theta}$$

◊ Show that $1 + \frac{\cos\theta}{2} + \frac{\cos 2\theta}{4} + \frac{\cos 3\theta}{8} + \ldots = \frac{4-2\cos\theta}{5-4\cos\theta}$.

$$Answ.: \ \sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} \text{ Re } e^{in\theta} = \text{Re} \sum_{n=0}^{\infty} \left(\frac{e^{i\theta}}{2}\right)^n$$

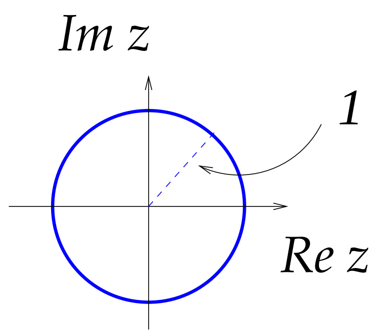
$$= \text{Re } \frac{1}{1-e^{i\theta}/2} = \frac{1-\cos\theta/2}{1+1/4-\cos\theta} = \frac{4-2\cos\theta}{5-4\cos\theta}$$

CURVES IN THE COMPLEX PLANE

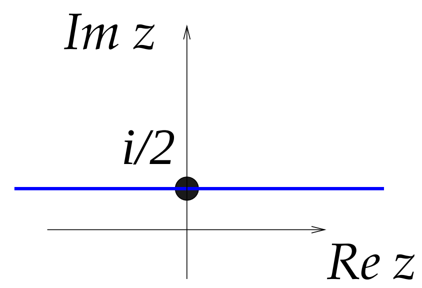
- locus of points satisfying constraint in complex variable z

<u>Examples</u>

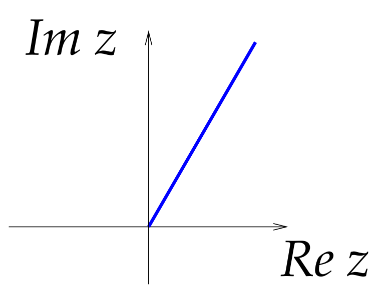
$|z| = 1$



$z - z^{*} = i$



$\arg z = \pi / 3$



Curves in the complex plane

Ex 1 $|z| = 1$

$$z = re^{i\theta} \quad \Rightarrow \quad r = 1, \quad \text{any } \theta$$

or $(a)^2 + (b)^2 = 1, \qquad z = a + ib$

Circle centre (0,0), radius 1

$$|z - z_0| = 1$$

Circle centre z_0, radius 1

$$(a - a_0)^2 + (b - b_0)^2 = 1$$

Curves in the complex plane

Ex 2 $$\left|\frac{z-i}{z+i}\right| = 1$$

$$|z-i| = |z+i|$$

Distance from (0,1) = distance from (0,-1) ⇨ Real axis

or $(\mathrm{a})^2 + (b-1)^2 = (\mathrm{a})^2 + (b+1)^2$ $\qquad z = a + ib$

⇨ $b = 0, \quad a$ arbitrary

Another way to define a curve:

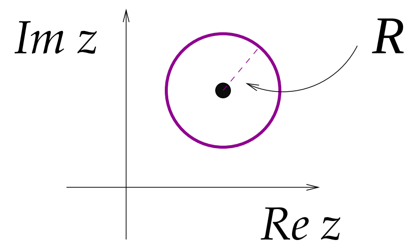
$$\gamma : [a, b] \to \mathbb{C}$$

$$\gamma : t \mapsto z = \gamma(t) = x(t) + iy(t)$$

$x(t) = x_0 + R \cos t$

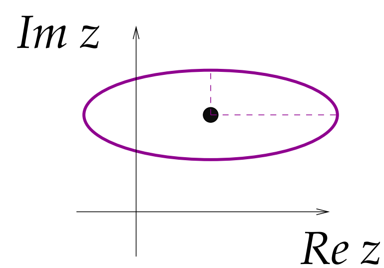
$y(t) = y_0 + R \sin t$

$0 < t < 2\pi$



$x(t) = x_0 + R_1 \cos t$

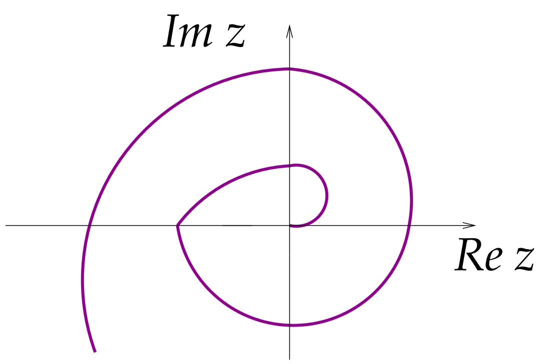
$y(t) = y_0 + R_2 \sin t$



$x(t) = t \cos t$

$y(t) = t \sin t$

i.e., $z = t\, e^{i t}$



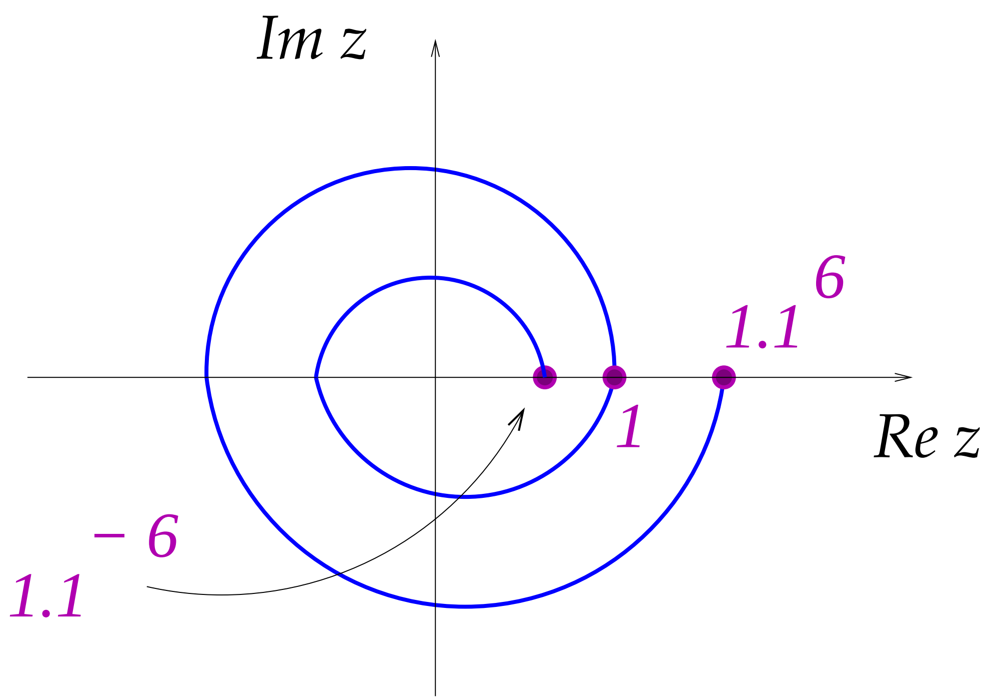
Homework

(Maths Paper 2007)

Sketch the curve in the complex plane

$$z = a^t e^{ibt} \quad , \qquad -6 \leq t \leq 6$$

where a and b are real constants given by $a = 1.1$, $b = \pi/3$.



Curves in the complex plane

Ex 3 $\arg\left(\frac{z}{z+1}\right) = \frac{\pi}{4}$ *i.e.* $\arg(z) - \arg(z+1) = \frac{\pi}{4}$

Take tan of both sides :

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\Rightarrow \quad \frac{\frac{b}{a} - \frac{b}{a+1}}{1 + \frac{b}{a} \cdot \frac{b}{a+1}} = 1 = \frac{b(a+1) - ba}{a(a+1) + b^2} \qquad z = a + ib$$

$$b = a(a+1) + b^2$$

$$(a + \tfrac{1}{2})^2 + (b - \tfrac{1}{2})^2 = \tfrac{1}{2}$$

…but BEWARE…not all of circle satisfies equation…

$$\arg(\tfrac{z}{z+1}) = \tfrac{\pi}{4}$$

$$\frac{z}{z+1} = \frac{z}{z+1} \cdot \frac{z^*+1}{z^*+1} = \frac{z(z^*+1)}{|z+1|^2} = \frac{(a+ib)(a+1-ib)}{|z+1|^2} = \frac{a(a+1)+b^2+ib}{|z+1|^2}$$

$$\Rightarrow \quad b > 0 \text{ since } \arg\left(\frac{z}{z+1}\right) \text{ positive}$$

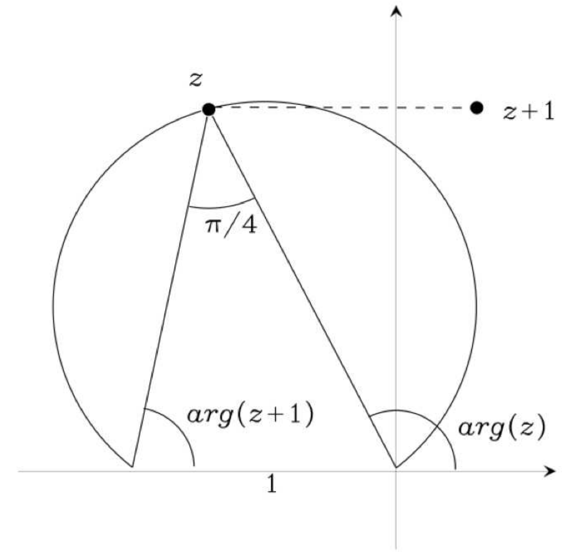
$$(a+\tfrac{1}{2})^2 + (b-\tfrac{1}{2})^2 = \tfrac{1}{2}, \quad b > 0$$

($b < 0$ solution introduced by tangent ambiguity)

Alternative solution

$$\arg\left(\frac{z}{z+1}\right) = \frac{\pi}{4}$$

i.e. $\arg(z) - \arg(z+1) = \frac{\pi}{4}$

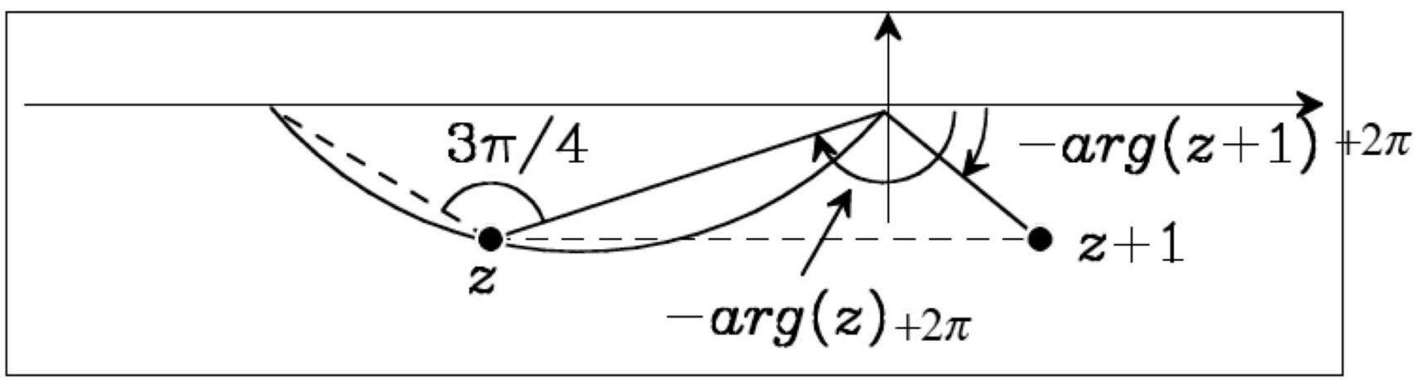


Solution : portion of circle through (0,0) and (-1,0)

Circle centre (-1/2,1/2) and radius $1/\sqrt{2}$

The lower portion of the circle is given by :

$$\arg\left(\frac{z}{z+1}\right) = -\frac{3\pi}{4}$$



Homework

(Question from Maths Collection Paper 2009)

Sketch the locus of points in the complex plane for

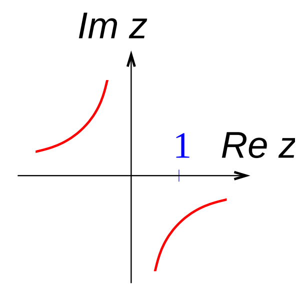
$$(a)\ \mathrm{Im}(z^2+i)=-1 \qquad , \qquad (b)\ \arg(z^2+i)=\pi/2$$

Answ. :

$z = x + iy\ ;\ z^2 = x^2 - y^2 + 2ixy$

(a) $1 + 2xy = -1 \Rightarrow \quad xy = -1$

$y = -1/x$

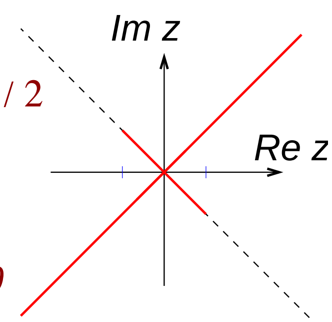


$\theta = arctg\,[(1 + 2xy)/(x^2 - y^2)] = \pi/2$

(b)

$\Rightarrow \quad x^2 = y^2 \quad \Rightarrow \quad x = y$

$x = -y\ ,\ 1 - 2x^2 > 0$



Summary on Curves in the Complex Plane

• locus of points satisfying constraint in complex variable z

♠ Curves can be defined

▷ either directly by constraint in complex variable z

$$\text{ex.}: \quad |z| = R$$

▷ or by parametric form $\gamma(t) = x(t) + iy(t)$

$$\text{ex.}: \quad x(t) = R\cos t$$

$$y(t) = R\sin t \,, \quad 0 \leq t < 2\pi$$

♢ intersection of algebraic and geometric approaches

COMPLEX FUNCTIONS AS MAPPINGS

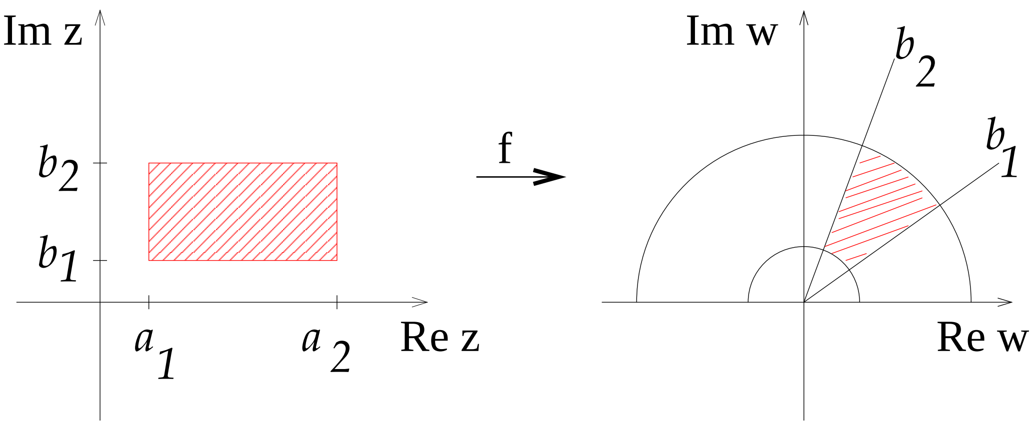
$$f : z \mapsto w = f(z)$$

- Complex functions can be viewed as mapping sets of points in $\mathbb{C}$ (e.g. curves, regions) into other sets of points in $\mathbb{C}$.

Example:

$w = f(z) = e^z$. Set $z = x + iy$; $w = \rho e^{i\phi}$. Then $\rho = e^x$; $\phi = y$.

$$\text{lines } x = a \xrightarrow{f} \text{circles } \rho = e^a$$
$$\text{lines } y = b \xrightarrow{f} \text{rays } \phi = b$$

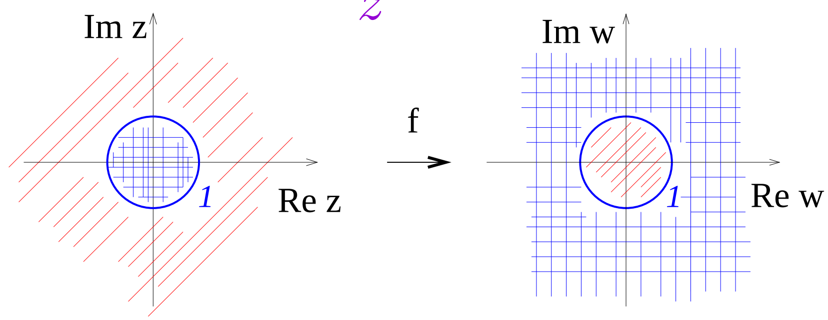


- For any subset A of $\mathbb{C}$, the "image of A through f" is the set of points w such that $w = f(z)$ for some z belonging to A, and is denoted $f(A)$.
- f maps A on to $f(A)$

Examples

- Let $f(z) = 1/z$. What does f do to the interior and exterior of the unit circle $|z| = 1$? What does f do to points on the unit circle?

Answ. : $\quad w = f(z) = \dfrac{1}{z} \;\; ; \quad z = re^{i\theta} \;\Rightarrow\; w = \dfrac{1}{r}\, e^{-i\theta}$



▷ f maps the interior of the unit circle on to the exterior, and viceversa. Unit circle is mapped on to itself.

- Let $f(z) = (z-1)/(z+1)$. What is the image through f of the imaginary axis? [Answ.: unit circle]

  And of the right and left half planes? [Answ.: interior and exterior of unit circle]

- Let $f(z) = \ln z$ (principal branch). What is the image through f of the upper half plane? [Answ.: horizontal strip between 0 and $i\pi$]