Complex Numbers and ODE
Lecture 2

- Basic functions of complex variable

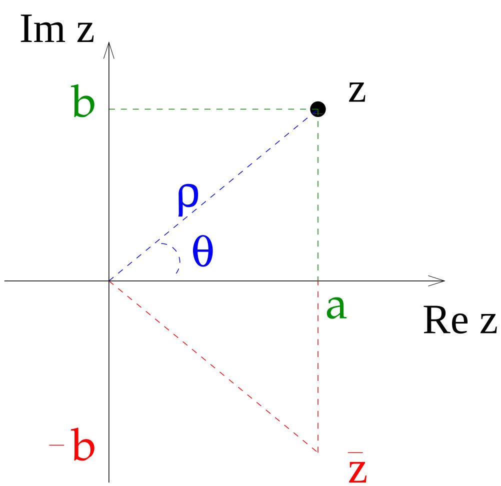
- De Moivre's theorem

The Complex Plane — Recap of Lecture 1

$$z = a + ib \quad , \qquad i^2 = -1 \quad , \qquad a, b \ \text{ real}$$

$$a = \text{Re}z \quad , \qquad b = \text{Im}z$$

♠ Argand diagram representation



$\rho = \text{mod}(z) = |z|$

$\theta = \arg(z)$

$$z = a + ib = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta)$$

♠ $\mathbb{C}$ = set of complex nos with $+$, $\times$ operations

Elementary functions of complex variable

- Polynomials. Rational functions.

- Exponential

$\Longrightarrow$ Trigonometric fctns. Hyperbolic fctns.

- Logarithm

$\Longrightarrow$ Complex powers

FUNCTIONS OF COMPLEX $z = x + iy$

$$f : \mathbb{C} \to \mathbb{C}$$

$$f : z \mapsto w = f(z)$$

Polynomials

• defined via algebraic rule for complex multiplication

e.g: $f(z) = z^2 = (x + iy)(x + iy) = x^2 - y^2 + 2ixy$

Rational functions = quotients of polynomials

• also defined via algebraic operations on $\mathbb{C}$:

$$\text{e.g.} \quad f(z) = \frac{1}{1+z} = \frac{1+z^*}{|1+z|^2} = \frac{1+x}{(1+x)^2+y^2} - i\,\frac{y}{(1+x)^2+y^2}$$

Exponential

- defined by power series expansion
- we will define other functions (trig., etc.) in terms of exponential

Functions of complex numbers

- The complex exponential

Functions defined by power series :

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

Define the complex exponential

$$e^\alpha = 1 + \alpha + \frac{\alpha^2}{2!} + \ldots + \frac{\alpha^n}{n!} + \ldots$$

$\alpha = a + ib$

Special case $\alpha = i\theta$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

$$= \cos\theta + i\sin\theta$$

$$e^{i\theta} = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots) + i\,(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots)$$

$$= \cos\theta + i\sin\theta$$

Used in :

$$z = |z|(\cos\theta + \imath\sin\theta) = |z|\,\mathrm{e}^{\imath\theta} \equiv re^{i\theta}$$

$$z^* = |z|(\cos\theta - \imath\sin\theta) = |z|\,\mathrm{e}^{-\imath\theta} \equiv r\mathrm{e}^{-\imath\theta}$$

$$\frac{1}{z} = \frac{z^*}{zz^*} = \frac{\mathrm{e}^{-\imath\theta}}{|z|} \equiv \frac{\mathrm{e}^{-\imath\theta}}{r}$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

$$= \cos\theta + i \sin\theta$$

Used in :

$$z = |z|(\cos\theta + \imath \sin\theta) = |z|\, e^{\imath\theta} \equiv re^{i\theta}$$

Multiplication

$$z_1\, z_2 = r_1 e^{i\theta_1}\; r_2 e^{i\theta_2} = r_1\, r_2\, e^{i\theta_1} e^{i\theta_2} = r_1\, r_2\, e^{i(\theta_1 + \theta_2)} = r_1\, r_2 \left(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\right)$$

Division

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2}\,\frac{e^{i\theta_1}}{e^{i\theta_2}} = \frac{r_1}{r_2}\, e^{i\theta_1} e^{-i\theta_2} = \frac{r_1}{r_2}\, e^{i(\theta_1 - \theta_2)} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\right)$$

$$e^{i\theta} = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots) + i\,(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots)$$

$$= \cos\theta + i\sin\theta$$

Can invert the relation :

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

The complex exponential function

$$e^{\alpha} = 1+\alpha + \frac{\alpha^2}{2!} + \ldots + \frac{\alpha^n}{n!} + \ldots$$

General case $\alpha = z = a + ib, \quad a, b$ real

$$e^{\imath z} = (1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots) + \imath(z - \frac{z^3}{3!} + \cdots)$$
$$= \cos z + \imath \sin z$$

Similarly one has

$$\cos z = \frac{1}{2}(e^{\imath z} + e^{-\imath z})$$
$$\sin z = \frac{1}{2i}(e^{\imath z} - e^{-\imath z})$$

Hence

$$\cos(\imath b) = \frac{1}{2}(e^{-b} + e^{b}) = \cosh b$$
$$\sin(\imath b) = \frac{1}{2i}(e^{-b} - e^{b}) = \imath \sinh b$$

- For complex z

$$\cosh z = \frac{1}{2}\ \left(e^{z} + e^{-z}\right)$$

$$\sinh z = \frac{1}{2}\ \left(e^{z} - e^{-z}\right)$$

hyperbolic functions

▷ Verify that

$$\cos^2 z + \sin^2 z = 1$$

$$\cosh^2 z - \sinh^2 z = 1$$

An application of complex exponential:

Complex exponential and trig identities

$$\cos(a+b)+\imath\sin(a+b)=e^{\imath(a+b)}=e^{\imath a}e^{\imath b}$$
$$=(\cos a+\imath\sin a)(\cos b+\imath\sin b)$$
$$=(\cos a\cos b-\sin a\sin b)+\imath(\cos a\sin b+\sin a\cos b)$$

Equating real and imaginary parts

$$\cos(a+b)=\cos a\cos b-\sin a\sin b$$
$$\sin(a+b)=\sin a\cos b+\cos a\sin b$$

de Moivre's theorem and trigonometric identities

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$$

For r=1

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

De Moivre

e.g. n=2 :

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\cos\theta\sin\theta$$

Uses of de Moivre and complex exponentials

Ex. 1 Find $(1+\imath)^8$

. Taking powers is much simpler in polar form so we write

$$(1+\imath) = \sqrt{(2)}\, e^{\imath\pi/4}$$

. Hence

.

$$(1+\imath)^8 = (\sqrt{(2)}\, e^{\imath\pi/4})^8 = 16e^{2\pi\imath} = 16$$

- Ex. 2: Prove the trigonometric identity

$$\sin^3\theta = [3\sin\theta - \sin 3\theta]/4$$

$$\begin{aligned}\sin^3\theta &= \left[\left(e^{i\theta} - e^{-i\theta}\right)/(2i)\right]^3 \\ &= -(1/(8i))\left[e^{3i\theta} - 3e^{2i\theta}e^{-i\theta} + 3e^{i\theta}e^{-2i\theta} - e^{-3i\theta}\right] \\ &= -\left[\left(e^{3i\theta} - e^{-3i\theta}\right) - 3\left(e^{i\theta} - e^{-i\theta}\right)\right]/(8i) \\ &= -[\sin 3\theta - 3\sin\theta]/4\end{aligned}$$

- Similarly to higher powers, e.g.:

$$\cos^4\theta = [\cos 4\theta + 4\cos 2\theta + 3]/8$$

- Complex trigonometric functions can be expressed in terms of real trigonometric and hyperbolic functions

For the case $z = a + ib$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\begin{aligned}\cos z &= \cos(a + ib) \\ &= \frac{1}{2}(e^{(ia-b)} + e^{(-ia+b)}) \\ &= \frac{1}{2}(e^{-b}(\cos a + i\sin a) + e^{b}(\cos a - i\sin a))\end{aligned}$$

i.e.

$$\cos z = \cos a \cosh b - i \sin a \sinh b.$$

and analogously

$$\sin z = \sin a \cosh b + i \cos a \sinh b.$$

- <u>Homework</u>: Determine Re and Im of $\cos i$, $\sin i$.

Solution of equations by complex function methods

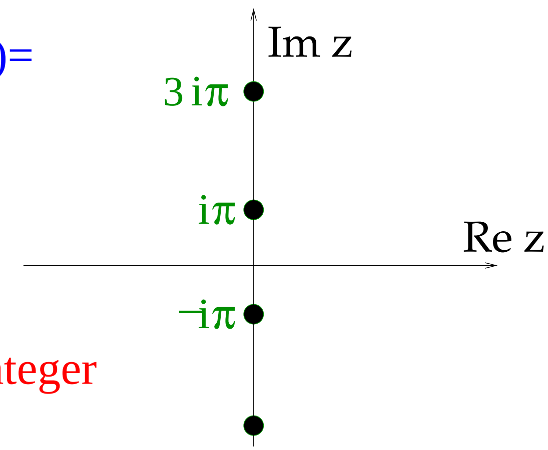
- Solve the equation

$$e^z = -1$$

$$e^z = e^{x+iy} = e^x (\cos y + i \sin y) = -1 = e^{i\pi(1+2n)}$$

$$\Longrightarrow \quad x = 0, \quad y = (2n+1)\pi$$

Thus $z = i\pi(2n+1)$, n integer



Homework

- Solve the equation $e^z = -2$.
- Solve the equations

$$\sin z = 2 \quad , \qquad \cosh z = 0 \quad .$$

The complex logarithm

$\ln z$

$$e^{\ln z} = z \quad = |z| e^{\imath\theta} = e^{\ln|z|} e^{\imath\theta} = e^{\ln|z| + \imath\theta}$$

$$\Rightarrow \quad \boxed{\ln z = \ln|z| + \imath \arg(z)}$$

Need to know θ including 2πn phase ambiguity in z

$$\ln z = \ln|z| + i(\theta + 2\pi n) \ , \qquad n \ \text{integer}$$

- different $n \Rightarrow$ different "branches" of the logarithm
  - $n = 0$: "principal" branch
- $\ln z$ is our first example of "multi-valued" functions

Note

♢ 1712: exchange of letters between Bernoulli and Leibniz on the meaning of $\ln(-1)$. Neither one got it right. [Bernoulli: It is 0; Leibniz: No, it must be < 0.]

♢ Sorted out by Euler, 1749: $\ln(-1) = i\pi$. $(\text{for } n = 0)$

$$\ln(-1) = \ln e^{i\pi} = \ln 1 + i(\pi + 2\pi n) = i\pi + 2\pi i n \ , \quad n \ \text{ integer}$$

Homework

Find Re and Im of $\ln z$ ($n = 0$ branch) for

$a)\ z = i \qquad b)\ z = 1 + i \qquad c)\ z = 1 - i$

Remark

◇ $e^{\ln z}$ always equals z, while $\ln e^z$ does not always equal z.

Verify:

$z = a + ib = re^{i\theta}$

Then $\ln z = \ln r + i(\theta + 2\pi n)$, n integer.

So $e^{\ln z} = e^{\ln r + i(\theta+2\pi n)} = re^{i\theta} \underbrace{e^{2\pi n i}}_{1} = re^{i\theta} = z$.

On the other hand $e^z = e^{a+ib} = e^a e^{ib}$

So $\ln e^z = \ln e^a + i(b + 2\pi n) = \underbrace{a + ib}_{z} + 2\pi i n = z + 2\pi i n$ which may be $\neq z$.

Complex powers

$f(z) = z^\alpha$, when both z and α are complex,
can now be defined using the complex logarithm:

$$z^\alpha = e^{\alpha \ln z}$$

Because $\ln z$ is multi-valued, so is z^α.

Example

Show that i^i is real and the principal-branch value is

$$i^i = 1/\sqrt{e^\pi}$$

[Euler, 1746]

$$i^i = e^{i \ln i} = e^{i[\ln 1 + i(\pi/2 + 2\pi n)]} = e^{-\pi/2 - 2\pi n}$$

- Homework: by similar method, find all the values of $(1+i)^i$.

Overview of elementary functions on $\mathbb{C}$

- Complex polynomials and rational functions defined by algebraic operations in $\mathbb{C}$

- Complex exponential: $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$

$\longrightarrow$ complex trigon. and hyperb. fctns in terms of exp.

e.g. $\sin z = (e^{iz} - e^{-iz})/(2i)$

$\sinh z = (e^z - e^{-z})/2$

- Complex logarithm $\ln z$: $e^{\ln z} = z$

$\Rightarrow \ln z = \ln |z| + i(\theta + 2n\pi)$, $n = 0, \pm 1, \ldots$ ($\leftarrow$ multi-valued)

$\longrightarrow$ complex powers: $z^\alpha = e^{\alpha \ln z}$ (α complex)