Complex numbers and ordinary differential equations

Michaelmas Term 2011

Lecturer: F Hautmann

- Part A: Complex numbers ($\sim$ 4 lectures)
- Part B: Ordinary differential equations ($\sim$ 6 lectures)

- printed lecture notes
- slides will be posted on lecture webpage: http://www-thphys.physics.ox.ac.uk/people/FrancescoHautmann/Cp4/
- suggested problem sheets also on webpage

References

- Course material is covered well in many textbooks on mathematical methods for science students, for example:

[1] Riley, Hobson and Bence: Mathematical methods for Physics and Engineering, CUP

[2] M. Boas: Mathematical Methods in the Physical Sciences, Wiley

Next term (HT2012) "Normal modes and waves"

A. Complex numbers

1 Introduction to complex numbers

2 Fundamental operations with complex numbers

3 Elementary functions of complex variable

4 De Moivre's theorem and applications

5 Curves in the complex plane

6 Roots of complex numbers and polynomials

B. Ordinary differential equations

1 Introduction to differential equations and differential operators

2 First order ordinary differential equations

3 Second order linear ODEs

4 Systems of linear differential equations

Introduction

Why complex nos?

- Natural numbers (positive integers) 1, 2, 3, . . .
- Negative integers $\quad$ e.g. $20 + y = 12 \Rightarrow y = -8$
- Rationals $\quad$ e.g. $4x = 6 \Rightarrow x = \frac{3}{2}$
- Irrationals $\quad$ e.g. $x^2 = 2 \Rightarrow x = \sqrt{2}$
- Complex nos $\quad$ e.g. $x^2 = -1 \Rightarrow x = i \equiv \sqrt{-1}$

Complex numbers

$$\boxed{z = a + ib}$$

$(i^2 = -1)$

where a and b are real

$$\boxed{a = \mathrm{Re}(z),\ b = \mathrm{Im}(z)}$$

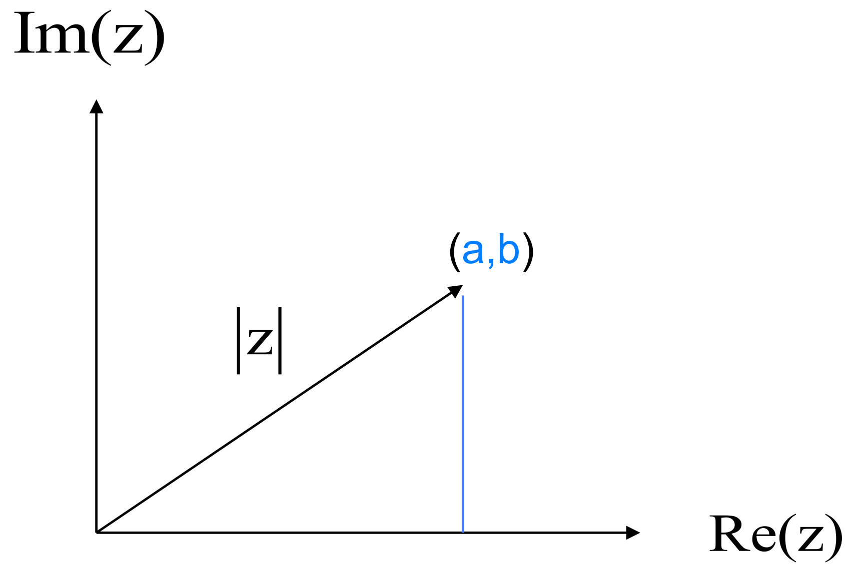
“Real part” “Imaginary part”

(Multiples of i (a=0) are called "pure imaginary" numbers.)

Argand diagram

Each z=a+ib $\rightarrow$ point (a, b) in plane:

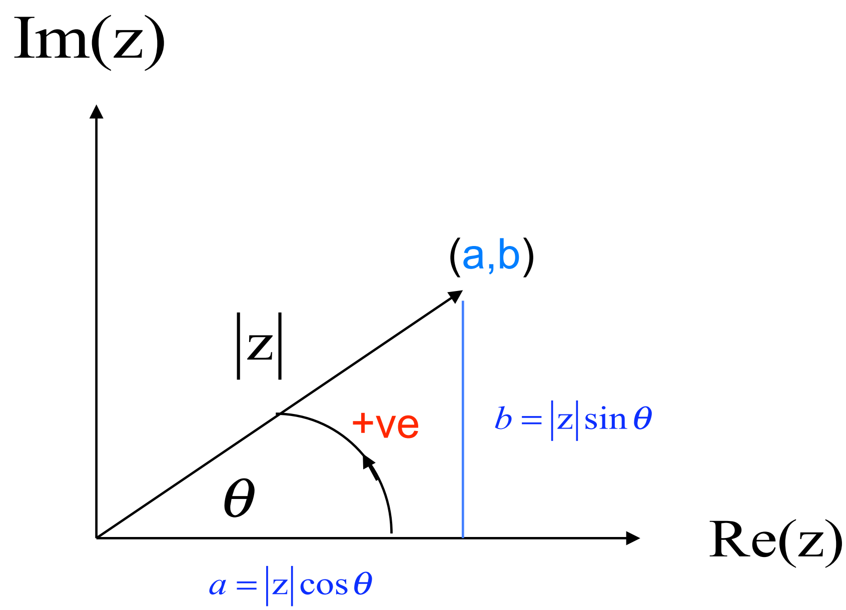
$$|z| \equiv \sqrt{a^2 + b^2}$$ length or "modulus"



Argand diagram

In polar co-ordinates: (r, θ)

$$|z| \equiv r = \sqrt{a^2 + b^2} \quad \text{"modulus"}$$



$$z = |z|(\cos\theta + i \sin\theta)$$
$$\equiv r(\cos\theta + i \sin\theta)$$

Which quadrant?

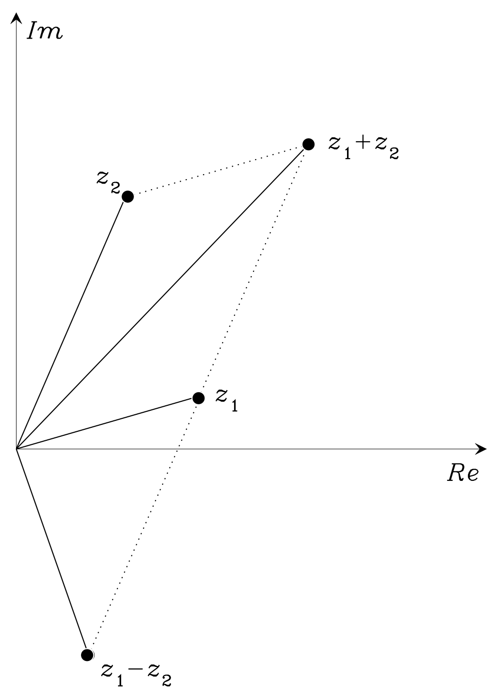
$$\theta \equiv \arg(z) = \arctan(b/a) \equiv \tan^{-1}(b/a)$$

"argument"

Addition :

$$z = a + ib$$

$$\boxed{z_1 \pm z_2 = (a_1 \pm a_2) + i(b_1 \pm b_2)}$$



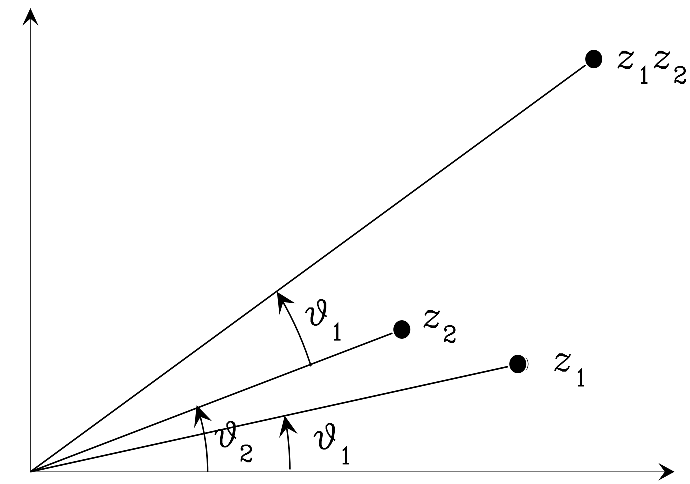
Multiplication

$$z = a + ib$$

$$\boxed{\begin{aligned} z_1 z_2 &= (a_1 + ib_1)(a_2 + ib_2) \\ &= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2) \end{aligned}}$$

Multiplication

Polar co-ordinates



$$z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$$

$$\boxed{z_1\, z_2 = r_1\, r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)}$$

$$|\, z_1 z_2 \,| = |\, z_2 \,|\,|\, z_2 \,|$$

$$Arg[z_1 z_2] = Arg[z_1] + Arg[z_2]$$

Historical note

$\diamondsuit$ Imaginary unit i first introduced by algebrists of 16th century:

- Cardano 1540's: quadratic equation $x^2 = 10x - 40$
- Bombelli 1570's: cubic equation — really the first calculation manipulating imaginary numbers; derived rules of addition and multiplication

$\diamondsuit$ But not until Euler and Gauss (18th century) was power of complex numbers really understood — dormant for nearly two centuries
[Gauss, 1799]: any polynomial of degree n has n roots in $\mathbb{C}$.

$\diamondsuit$ Geometric interpretation: Argand, 19th century
Complex $\leftrightarrow$ ordered pair of real numbers: Hamilton, 19th century

$\diamondsuit$ Theory of complex functions developed by Cauchy, Riemann and others — mid 19th century [see S1 course]

Division $\frac{z_1}{z_2}$?

$z = a + ib$

Define "complex conjugate"

$$\boxed{z^* = a - ib}$$

$$\text{Modulus}^2 \; : \; |z|^2 \equiv zz^* = (a^2 + b^2) \text{ is real (and} > 0)$$

$$\frac{1}{z_2} = \frac{1}{z_2}\frac{z_2^*}{z_2^*} = \frac{1}{|z_2|^2} z_2^*$$

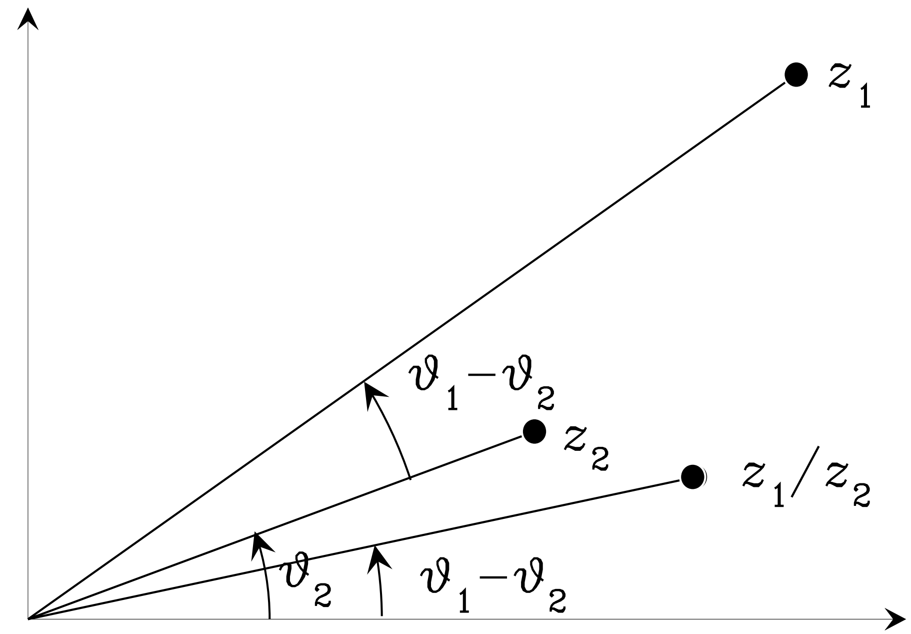
$$\boxed{\frac{z_1}{z_2} = \frac{z_1 z_2^*}{|z_2|^2}}$$

Division

Polar co-ordinates

$$z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$$



$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)}$$

e.g. Find the modulus $\left|\frac{z_1}{z_2}\right| \equiv \frac{r_1}{r_2}$

when $\begin{cases} z_1 = 1+2i \\ z_2 = 1-3i \end{cases}$

Elegant method:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{1+4}}{\sqrt{1+9}} = \frac{1}{\sqrt{2}}$$

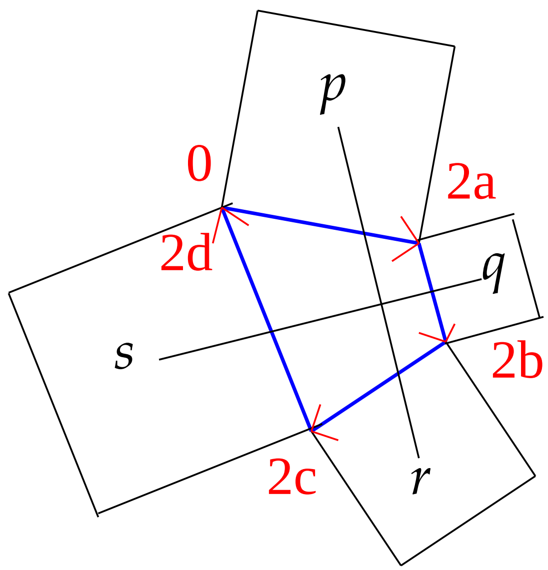
Clumsy method:

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{1+2i}{1-3i}\right| = \frac{|z_1 z_2^*|}{|z_2|^2}$$

$$= \frac{|(1+2i)(1+3i)|}{1+9} = \frac{|(1-6)+i(2+3)|}{10}$$

$$= \frac{\sqrt{25+25}}{10} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

An application of complex algebra to plane geometry



Arbitrary quadrilateral.

Construct squares on each side.

Then

segments joining the centres of opposite squares are perpendicular and of equal length.

2a+2b+2c+2d=0

Center p: $a+a\,e^{i\pi/2} = a(1+i)$; *likewise* $q=2a+b(1+i)$

$r=2a+2b+c(1+i)$; $s=2a+2b+2c+d(1+i)$

Thus $A \equiv s-q = b(1-i)+2c+d(1+i)$; $B \equiv r-p = a(1-i)+2b+c(1+i)$.

A and B perpendicular and of equal length means $B=A\,e^{i\pi/2}$,

i.e. B=iA, i.e. A+ i B=0. Verify that indeed A+iB=0.

Homework

- If two integers can be expressed as the sum of two squares, so can their product.

Prove this statement by using complex algebra.

Hint: Let $n = n_1^2 + n_2^2$, $m = m_1^2 + m_2^2$ and show that $nm = p^2 + q^2$ for integer p, q .

To this end consider complex numbers $n_1 + in_2$, $m_1 + im_2$ and evaluate $|(n_1 + in_2)(m_1 + im_2)|^2$.

- $\cos(\alpha + \beta) = \cos\alpha \ \cos\beta - \sin\alpha \ \sin\beta$
- $\sin(\alpha + \beta) = \sin\alpha \ \cos\beta + \cos\alpha \ \sin\beta$

Prove these trigonometric identities by complex methods.

Hint: start with $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ and use $e^{i\theta} = \cos\theta + i\sin\theta$.