Complex Numbers
and
Ordinary Differential Equations

F Hautmann

Oxford University
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Books: The material of this course is covered well in many texts on mathematical methods for science students, for example *Mathematical Methods for Physics and Engineering,* Riley, Hobson, Bence (Cambridge University Press) or *Mathematical Methods in the Physical Sciences,* Boas (Wiley), both of which provide many examples. A more elementary book is Stephenson, *Mathematical Methods for Science Students* (Longmans). The book by Louis Lyons, *All you ever wanted to know about Mathematics but were afraid to ask* (Vol I, CUP, 1995) also provides an excellent treatment of the subject. I am grateful to James Binney and Graham Ross for providing past courses' material on which these lecture notes are based.

1 Complex Numbers I : Friendly Complex Numbers

Complex numbers are widely used in physics. The solution of physical equations is often made simpler through the use of complex numbers and we will study examples of this when solving differential equations later in this course. Another particularly important application of complex numbers is in quantum mechanics where they play a central role representing the state, or wave function, of a quantum system. In this course I will give a straightforward introduction to complex numbers and to simple functions of a complex variable. The first Section "Friendly Complex Numbers" is intended to provide a presentation of basic definitions and properties of complex numbers suitable for those who have not studied the subject.

1.1 Why complex numbers?

The obvious first question is "Why introduce complex numbers?". The logical progression follows simply from the need to solve equations of increasing complexity. Thus we start with natural numbers $\mathcal{N}$ (positive integers) 1, 2, 3, ...

But $20 + y = 12 \quad \Rightarrow \quad y = -8 \quad \rightarrow$ integers $\mathcal{Z} \quad \ldots, -3, -2, -1, 0, 1, 2, \ldots$

But $4x = 6 \quad \Rightarrow \quad x = \frac{3}{2} \quad \rightarrow$ rationals $\mathcal{Q}$

But $x^2 = 2 \quad \Rightarrow \quad x = \sqrt{2} \quad \rightarrow$ irrationals $\quad \rightarrow \quad$ reals $\mathcal{R}$ (rationals and irrationals)

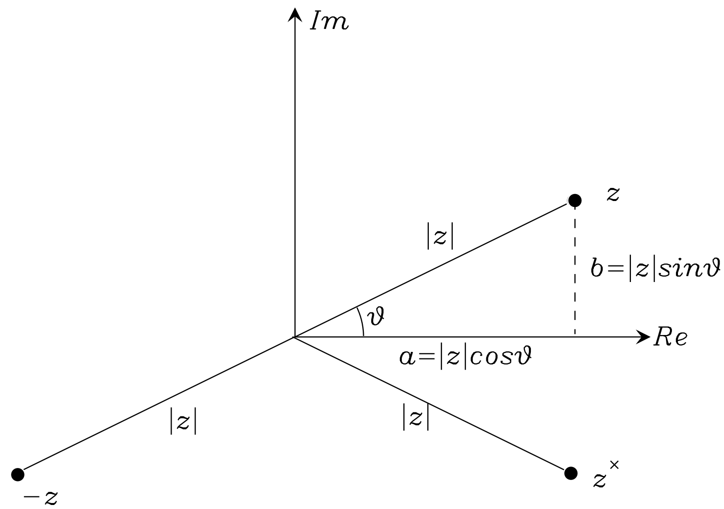
But $x^2 = -1 \quad \Rightarrow \quad x = \mathrm{i} \quad \rightarrow$ complex numbers $\mathcal{C}$

Multiples of i are called **pure imaginary** numbers. A general complex number is the sum of a multiple of 1 and a multiple of i such as $z = 2 + 3\mathrm{i}$. We often use the notation $z = a + \mathrm{i}b$, where a and b are real. (Sometimes the symbol j instead if i is used - for example in circuit theory where i is reserved for a current.)

We define operators for extracting a, b from z: $a \equiv \Re\mathrm{e}(z)$, $b \equiv \Im\mathrm{m}(z)$. We call a the **real part** and b the **imaginary part** of z.

1.2 Argand diagram (complex plane)

Complex numbers can be represented in the (x,y) plane. Thus the complex number $z = a + \mathrm{i}b \ \rightarrow$ point (a, b) in the "complex" plane (or "Argand diagram"):



Using polar co-ordinates the point (a, b) can equivalently be represented by its (r, θ) values. Thus with $\arg(z) \equiv \theta = \arctan(b/a)$ we have

$$z = |z|(\cos\theta + \mathrm{i}\sin\theta) \equiv r(\cos\theta + \mathrm{i}\sin\theta) \qquad (1.1).$$

Note that the **length** or **modulus** of the vector from the origin to the point (a, b) is given by

$$|z| \equiv r = \sqrt{a^2 + b^2} \qquad (1.2).$$

As we will show in the next section, $\cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta}$, the exponential of a *complex* argument. So an equivalent way of writing the polar form is

$$z = r\mathrm{e}^{\mathrm{i}\theta}. \qquad (1.3)$$

It is important to get used to this form as it proves to be very useful in many applications. Note that there are an infinite number of values of θ which give the same values of $\cos\theta$ and $\sin\theta$ because adding an integer multiple of 2π to θ does not change them. Often one gives only one value of θ when specifying the complex number in polar form but, as we shall see, it is important to include this ambiguity when for instance taking roots or logarithms of a complex number.

It also proves useful to define the **complex conjugate** z^* of z by reversing the sign of i, i.e.

$$z^* \equiv a - \mathrm{i}b \qquad (1.4).$$

The complex numbers z^* and $-z$ are also shown in the figure. We see that taking the complex conjugate z^* of z can be represented by reflection with respect to the real axis.

Example 1.1

Express $z \equiv a + ib = -1 - i$ in polar form. Here $r = \sqrt{2}$ and $\arctan(b/a) = \arctan 1 = \pi/4$. However it is necessary to identify the correct quadrant for θ.

Since a and b are both negative so too are $\cos\theta$ and $\sin\theta$. Thus θ lies in the third quadrant $\theta = \frac{5\pi}{4} + 2n\pi$ where n is any positive or negative integer. Thus finally we have $z = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{5\pi}{4}+2n\pi}, n = 0, \pm1, \pm2, \cdots$, where we have made the ambiguity in the phase explicit.

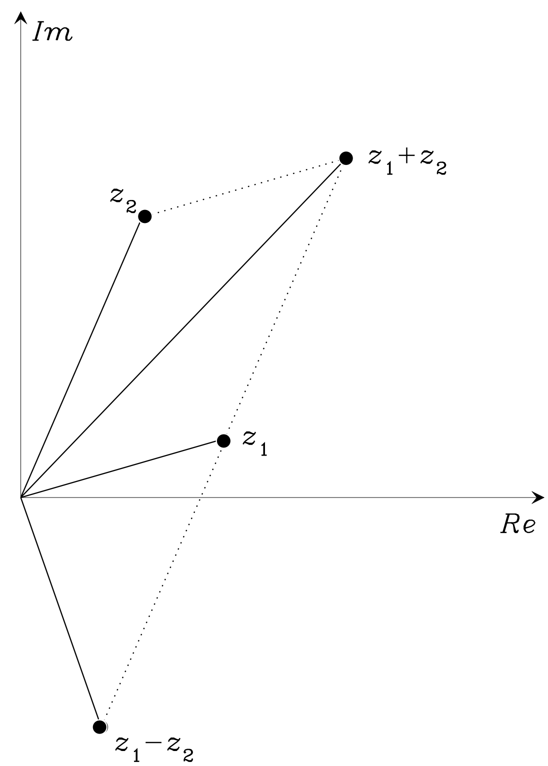
In the next two subsections we describe the basic operations of addition and multiplication with complex numbers.

1.3 Addition and subtraction

Addition and subtraction of complex numbers follow the same rules as for ordinary numbers except that the real and imaginary parts are treated separately:

$$z_1 \pm z_2 \equiv (a_1 \pm a_2) + \mathrm{i}(b_1 \pm b_2) \tag{1.5}$$

Since the complex numbers can be represented in the Argand diagram by vectors, addition and subtraction of complex numbers is the same as addition and subtraction of vectors as is shown in the figure. Adding z_2 to any z amounts to translating z by z_2.



1.4 Multiplication and division

Remembering that $\mathrm{i}^2 = -1$ it is easy to define multiplication for complex numbers :

$$\begin{aligned} z_1 z_2 &= (a_1 + \mathrm{i}b_1)(a_2 + \mathrm{i}b_2) \\ &\equiv (a_1 a_2 - b_1 b_2) + \mathrm{i}(a_1 b_2 + b_1 a_2) \end{aligned} \tag{1.6}$$

Note that the product of a complex number and its complex conjugate, $|z|^2 \equiv zz^* = (a^2+b^2)$, is real (and ≥ 0) and, c.f. eq (1.2), is given by the square of the length of the vector representing the complex number $zz^* \equiv |z|^2 = (a^2+b^2)$.

It is necessary to define division also. This is done by multiplying the numerator and denominator of the fraction by the complex conjugate of the denominator :

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} \tag{1.7}$$

One may see that division by a complex number has been changed into multiplication by a complex number. The denominator in the right hand side of eq (1.7) has become a real number and all we now need to define complex division is the rule for multiplication of complex numbers.

Multiplication and division are particularly simple when the polar form of the complex number is used. If $z_1 = |z_1|\mathrm{e}^{\mathrm{i}\theta_1}$ and $z_2 = |z_2|\mathrm{e}^{\mathrm{i}\theta_2}$, then their product is given by

$$z_1 * z_2 = |z_1| * |z_2|\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}. \tag{1.8}$$

To multiply any z by $z_2 = |z_2|\mathrm{e}^{\mathrm{i}\theta_2}$ means to rotate z by angle θ_2 and to dilate its length by $|z_2|$.

To determine $\frac{z_1}{z_2}$ note that

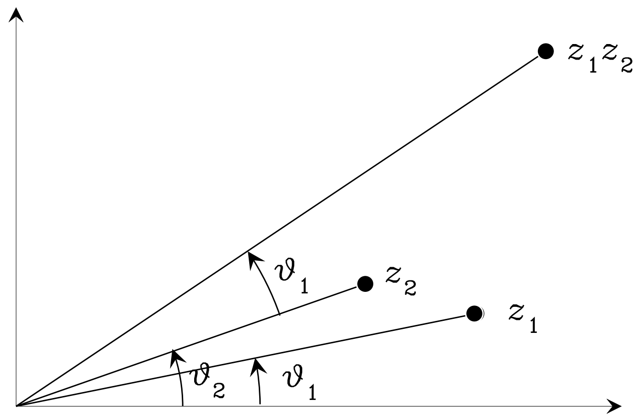
$$\begin{aligned} z &= |z|(\cos\theta + \mathrm{i}\sin\theta) = |z|\mathrm{e}^{\mathrm{i}\theta} \\ z^* &= |z|(\cos\theta - \mathrm{i}\sin\theta) = |z|\mathrm{e}^{-\mathrm{i}\theta} \\ \frac{1}{z} &= \frac{z^*}{zz^*} = \frac{\mathrm{e}^{-\mathrm{i}\theta}}{|z|}. \end{aligned} \tag{1.9}$$

Thus

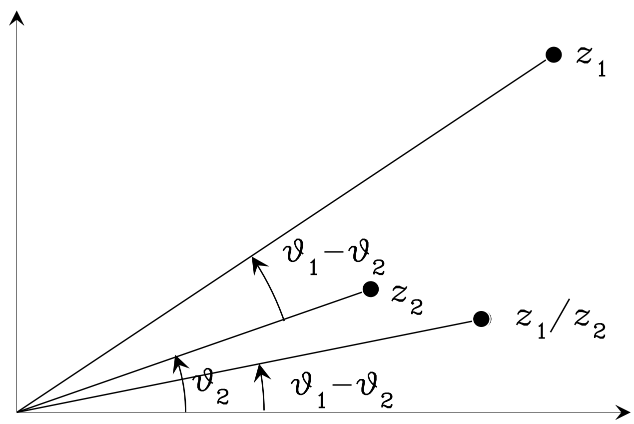
$$\begin{aligned} \frac{z_1}{z_2} &= \frac{|z_1|\mathrm{e}^{\mathrm{i}\theta_1} * \mathrm{e}^{-\mathrm{i}\theta_2}}{|z_2|} \\ &= \frac{|z_1|}{|z_2|}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)} \end{aligned} \tag{1.10}$$

1.4.1 Graphical representation of multiplication & division

$$z_1 z_2 = |z_1||z_2|\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}$$



$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)}$$



Example 1.2

Find the modulus $|z_1/z_2|$ when $\begin{cases} z_1 = 1+2\mathrm{i} \\ z_2 = 1-3\mathrm{i} \end{cases}$

Clumsy method:

$$\begin{aligned} \left|\frac{z_1}{z_2}\right| = \left|\frac{1+2\mathrm{i}}{1-3\mathrm{i}}\right| &= \frac{|z_1 z_2^*|}{|z_2|^2} \\ &= \frac{|(1+2\mathrm{i})(1+3\mathrm{i})|}{1+9} = \frac{|(1-6)+\mathrm{i}(2+3)|}{10} \\ &= \frac{\sqrt{25+25}}{10} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \end{aligned}$$

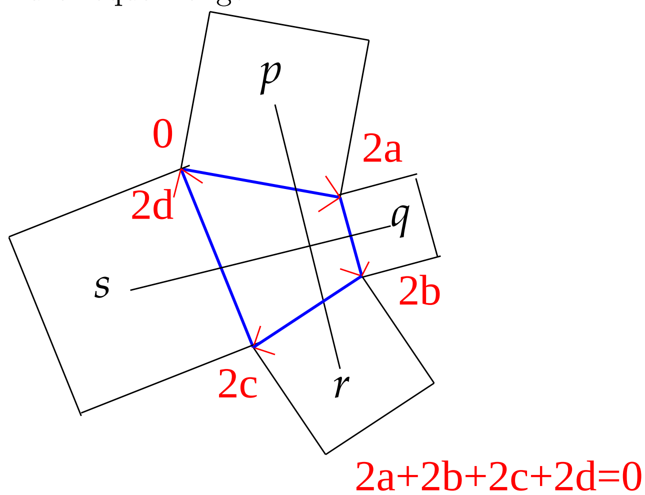
Elegant method:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{1+4}}{\sqrt{1+9}} = \frac{1}{\sqrt{2}}$$

Methods based on complex addition and multiplication can be useful to analyze plane geometry problems as in the following example.

Example 1.3

Consider an arbitrary quadrilateral and construct squares on each side as in the figure below. Show that segments joining the centres of opposite squares are perpendicular and of equal length.



Consider the complex plane and let the vertices of the quadrilateral be at points $2a$, $2a+2b$, $2a+2b+2c$, and $2a+2b+2c+2d=0$. The centre of the square on the first side is at

$$p = a + ae^{i\pi/2} = a(1+i) \ .$$

Likewise, the centres of the other squares are at

$$q = 2a + b(1+i) \ \ , \quad r = 2a + 2b + c(1+i) \ \ , \quad s = 2a + 2b + 2c + d(1+i) \ \ .$$

Thus

$$A \equiv s - q = b(1-i) + 2c + d(1+i) \ \ , \quad B \equiv r - p = a(1-i) + 2b + c(1+i) \ \ .$$

A and B perpendicular and of equal length means $B = Ae^{i\pi/2}$, i.e., $B = iA$, i.e., $A + iB = 0$. We verify that this is indeed the case:

$$A + iB = b(1-i) + 2c + d(1+i) + ia(1-i) + 2ib + ic(1+i)$$

$$= b(1+i) + c(1+i) + d(1+i) + a(1+i) = (1+i)(a+b+c+d) = 0 \ \ .$$

2 Complex Numbers II

This section is devoted to basic functions of complex variable and simple applications. We give de Moivre's theorem and show examples of its uses. We introduce the notion of curves in the complex plane. We end by discussing applications of complex variable to finding roots of polynomials.

2.1 Elementary functions of complex variable

We may define polynomials and rational functions of complex variable z based on the algebraic operations of multiplication and addition of complex numbers introduced in the previous section. For example, separating real and imaginary parts, $z = x + iy$, we have

$$f(z) = z^2 = (x + iy)(x + iy) = x^2 - y^2 + 2ixy .$$

Similarly,

$$f(z) = \frac{1}{z} = \frac{z^*}{|z|^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} .$$

To define the complex exponential and related functions such as trigonometric and hyperbolic functions, we use power series expansion.

2.1.1 The complex exponential function

The definition of the exponential, cosine and sine functions of a *real* variable can be done by writing their series expansions :

$$\begin{aligned} \mathrm{e}^x &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \end{aligned} \tag{2.1}$$

For small x a few of terms may be sufficient to provide a good approximation. Thus for very small x, $\sin x \approx x$.

In a similar manner we may define functions of the complex variable z. The complex exponential is defined by

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2!} + \cdots \tag{2.2}$$

A special case is if z is purely imaginary $z = \mathrm{i}\theta$. Using the fact that $\mathrm{i}^{2n} = 1$ or -1 for n even or odd and $\mathrm{i}^{2n+1} = i$ or $-i$ for n even or odd we may write

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\theta} &= \Big(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\Big) + \mathrm{i}\Big(\theta - \frac{\theta^3}{3!} + \cdots\Big) \\ &= \cos\theta + \mathrm{i}\sin\theta \end{aligned} \tag{2.3}$$

This is the relation that we used in writing a complex number in polar form, c.f. eq (1.3). Thus

$$\begin{aligned} z &= |z|(\cos\theta + \mathrm{i}\sin\theta) = |z|\mathrm{e}^{\mathrm{i}\theta} \\ z^* &= |z|(\cos\theta - \mathrm{i}\sin\theta) = |z|\mathrm{e}^{-\mathrm{i}\theta} \\ \frac{1}{z} &= \frac{z^*}{zz^*} = \frac{\mathrm{e}^{-\mathrm{i}\theta}}{|z|}. \end{aligned} \tag{2.4}$$

We may find a useful relation between sines and cosines and complex exponentials. Adding and then subtracting the first two of equations (2.4) we find that

$$\begin{aligned} \cos\theta &= \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) \\ \sin\theta &= \tfrac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}) \end{aligned} \tag{2.5}$$

2.1.2 The complex sine and cosine functions

In a similar manner we can define $\cos z$ and $\sin z$ by replacing the argument x in (2.1) by the complex variable z. The analogue of de Moivre's theorem is

$$\begin{aligned} \mathrm{e}^{\mathrm{i}z} &= \Big(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\Big) + \mathrm{i}\Big(z - \frac{z^3}{3!} + \cdots\Big) \\ &= \cos z + \mathrm{i}\sin z \end{aligned} \tag{2.6}$$

Similarly one has

$$\begin{aligned} \cos z &= \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) \\ \sin z &= \tfrac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}) \end{aligned} \tag{2.7}$$

From this we learn that the cosine and the sine of an imaginary angle are

$$\begin{aligned} \cos(\mathrm{i}b) &= \tfrac{1}{2}(\mathrm{e}^{-b} + \mathrm{e}^{b}) = \cosh b \\ \sin(\mathrm{i}b) &= \tfrac{1}{2\mathrm{i}}(\mathrm{e}^{-b} - \mathrm{e}^{b}) = \mathrm{i}\sinh b, \end{aligned} \tag{2.8}$$

where we have used the definitions of the hyperbolic functions

$$\begin{aligned} \cosh b &\equiv \tfrac{1}{2}(\mathrm{e}^{b} + \mathrm{e}^{-b}) \\ \sinh b &\equiv \tfrac{1}{2}(\mathrm{e}^{b} - \mathrm{e}^{-b}). \end{aligned} \tag{2.9}$$

Note:

Hyperbolic functions get their name from the identity $\cosh^2\theta - \sinh^2\theta = 1$, which is readily proved from (2.9) and is reminiscent of the equation of a hyperbola, $x^2 - y^2 = 1$.

2.1.3 Complex hyperbolic sine and cosine functions

We define complex hyperbolic functions in a similar manner as done above for complex trigonometric functions, by replacing the real argument in the power series expansion by complex variable z. Then we have

$$\begin{aligned} \cosh z &= \tfrac{1}{2}(\mathrm{e}^{z} + \mathrm{e}^{-z}) \\ \sinh z &= \tfrac{1}{2}(\mathrm{e}^{z} - \mathrm{e}^{-z}). \end{aligned} \tag{2.10}$$

2.2 de Moivre's theorem and trigonometric identities

Using the rules for multiplication of complex numbers gives the general form of **de Moivre's** theorem :

$$z^n = (r\mathrm{e}^{\mathrm{i}\theta})^n = r^n \mathrm{e}^{in\theta} = r^n(\cos n\theta + i\sin n\theta) \tag{2.11}$$

for any integer n. That is,

$$(\mathrm{e}^{\mathrm{i}\theta})^n = (\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta. \tag{2.12}$$

2.2.1 Trigonometric identities

Eq (2.12) generates simple identities for $\cos n\theta$ and $\sin n\theta$. For example, for $n = 2$ we have, equating the real and imaginary parts of the equation

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\cos\theta\sin\theta \end{aligned} \tag{2.13}$$

The complex exponential is very useful in establishing trigonometric identities. We have

$$\begin{aligned} \cos(a+b) + \mathrm{i}\sin(a+b) &= \mathrm{e}^{\mathrm{i}(a+b)} = \mathrm{e}^{\mathrm{i}a}\mathrm{e}^{\mathrm{i}b} \\ &= (\cos a + \mathrm{i}\sin a)(\cos b + \mathrm{i}\sin b) \\ &= (\cos a\cos b - \sin a\sin b) + \mathrm{i}(\cos a\sin b + \sin a\cos b) \end{aligned} \tag{2.14}$$

where we have used the property of exponentials that $\mathrm{e}^{\mathrm{i}(a+b)} = \mathrm{e}^{\mathrm{i}a}\mathrm{e}^{\mathrm{i}b}$. This is an example of a complex equation relating a complex number on the left hand size (LHS) to a complex number on the right hand side (RHS). To solve it we must equate the real parts of the LHS and the RHS and separately the imaginary parts of the LHS and RHS. Thus a complex equation is equivalent to two real equations. Comparing real and imaginary parts on the two sides of (2.14), we deduce that

$$\begin{aligned} \cos(a+b) &= \cos a\cos b - \sin a\sin b \\ \sin(a+b) &= \sin a\cos b + \cos a\sin b \end{aligned}$$

2.2.2 Identities for complex sines and cosines

We may use the result of (2.7) to evaluate the cosine of a complex number:

$$\begin{aligned} \cos z &= \cos(a + \mathrm{i}b) \\ &= \tfrac{1}{2}(\mathrm{e}^{(\mathrm{i}a-b)} + \mathrm{e}^{(-\mathrm{i}a+b)}) \\ &= \tfrac{1}{2}(\mathrm{e}^{-b}(\cos a + \mathrm{i}sina) + \mathrm{e}^{b}(\cos a - \mathrm{i}\sin a)) \\ &= \cos a\cosh b - \mathrm{i}\sin a\sinh b. \end{aligned} \tag{2.15}$$

Analogously

$$\sin z = \sin a\cosh b + \mathrm{i}\cos a\sinh b. \tag{2.16}$$

2.3 Uses of de Moivre's theorem

It is often much easier and more compact to work with the complex exponential rather than with sines and cosines. Here I give just three examples; you will encounter more in the discussion of differential equations and in the problem sets.

Example 2.1

Find $(1+\mathrm{i})^8$. Taking powers is much simpler in polar form so we write $(1+\mathrm{i}) = \sqrt(2)\mathrm{e}^{\mathrm{i}\pi/4}$. Hence $(1+\mathrm{i})^8 = (\sqrt(2)\mathrm{e}^{\mathrm{i}\pi/4})^8 = 16\mathrm{e}^{2\pi\mathrm{i}} = 16$.

Example 2.2

Solving differential equations is often much simpler using complex exponentials as we shall discuss in detail in later lectures. As an introductory example I consider here the solution of simple harmonic motion, $\frac{\mathrm{d}^2 y}{\mathrm{d}\theta^2} + y = 0$. The general solution is well known $y = A\cos\theta + B\sin\theta$ where A and B are real constants. To solve it using the complex exponential we first write $y = \Re\mathrm{e} z$ so that the equation becomes $\frac{\mathrm{d}^2\Re\mathrm{e} z}{\mathrm{d}\theta^2} + \Re\mathrm{e} z = \Re\mathrm{e}(\frac{\mathrm{d}^2 z}{\mathrm{d}\theta^2} + z) = 0$ The solution to the equation $\frac{d^2 z}{\mathrm{d}\theta^2} + z = 0$ is simply $z = C\mathrm{e}^{i\theta}$ where C is a (complex) constant. (You may check that this is the case simply by substituting the answer in the original equation). Writing $C = A - \mathrm{i}B$ one finds, using de Moivre,

$$\begin{aligned} y &= \Re\mathrm{e} z = \Re\mathrm{e}((A - \mathrm{i}B)(\cos\theta + \mathrm{i}\sin\theta)) \\ &= A\cos\theta + B\sin\theta \end{aligned} \tag{2.17}$$

Thus we have derived the general solution in one step - there is no need to look for the sine and cosine solutions separately. Although the saving in effort through using complex exponentials is modest in this simple example, it becomes significant in the solution of more general differential equations.

Example 2.3

Series involving sines and cosines may often be summed using de Moivre. As an example we will prove that for $0 < r < 1$

$$\sum_{n=0}^{\infty} r^n \sin(2n+1)\theta = \frac{(1+r)\sin\theta}{1 - 2r\cos 2\theta + r^2}$$

Proof:

$$\begin{aligned} \sum_{n=0}^{\infty} r^n \sin(2n+1)\theta &= \sum_n r^n \Im\mathrm{m}(\mathrm{e}^{\mathrm{i}(2n+1)\theta}) = \Im\mathrm{m}\Big(\mathrm{e}^{\mathrm{i}\theta} \sum_n (r\mathrm{e}^{2\mathrm{i}\theta})^n\Big) \\ &= \Im\mathrm{m}\left(\mathrm{e}^{\mathrm{i}\theta} \frac{1}{1 - r\mathrm{e}^{2\mathrm{i}\theta}}\right) \\ &= \Im\mathrm{m}\left(\frac{\mathrm{e}^{\mathrm{i}\theta}(1 - r\mathrm{e}^{-2\mathrm{i}\theta})}{(1 - r\mathrm{e}^{2\mathrm{i}\theta})(1 - r\mathrm{e}^{-2\mathrm{i}\theta})}\right) \\ &= \frac{\sin\theta + r\sin\theta}{1 - 2r\cos 2\theta + r^2} \end{aligned}$$

2.4 Complex logarithm

The logarithmic function $f(z) = \ln z$ is the inverse of the exponential function meaning that if one acts on z by the logarithmic function and then by the exponential function one gets just z, $\mathrm{e}^{\ln z} = z$. We may use this property to define the logarithm of a complex variable :

$$\begin{aligned} \mathrm{e}^{\ln z} = z = |z|e^{\mathrm{i}\theta} = \mathrm{e}^{\ln|z|}\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\ln|z|+\mathrm{i}\theta} \\ \Rightarrow \quad \ln z = \ln|z| + \mathrm{i}\arg(z) \\ (a) \qquad\quad (b) \end{aligned} \tag{2.18}$$

Part (a) is just the normal logarithm of a real variable and gives the real part of the logarithmic fiunction while part (b) gives its imaginary part. Note that the infinite ambiguity in the phase of z is no longer present in $\ln z$ because the addition of an integer multiple of 2π to the argument of z changes the imaginary part of the logarithm by the same amount. Thus it is essential, when defining the logarithm, to know precisely the argument of z. We can rewrite Eq (2.18) explicitly as

$$\ln z = \ln|z| + \mathrm{i}(\theta + 2\pi n) \quad , \quad n \text{ integer} \quad . \tag{2.19}$$

For different n we get different values of the complex logarithm. So we need to assign n to fully specify the logarithm.

The different values corresponding to different n are called "branches" of the logarithm. $n = 0$ is called the principal branch.

A function of z which may take not one but multiple values for a given value of z is called multi-valued. The logarithm is our first example of a multi-valued function.

Example 2.4

Find all values of $\ln(-1)$.

$$\ln(-1) = \ln e^{i\pi} = \ln 1 + i(\pi + 2\pi n) = i\pi + 2\pi i n \ , \ \ n \text{ integer} \ .$$

For the principal branch $n = 0$

$$\ln(-1) = i\pi \qquad (n = 0) \quad .$$

Note:

$e^{\ln z}$ always equals z, while $\ln e^z$ does not always equal z.

Let $z = a + ib = re^{i\theta}$. Then $\ln z = \ln r + i(\theta + 2\pi n)$, n integer. So

$$e^{\ln z} = e^{\ln r + i(\theta+2\pi n)} = re^{i\theta}\underbrace{e^{2\pi n i}}_{1} = re^{i\theta} = z \quad .$$

On the other hand $e^z = e^{a+ib} = e^a e^{ib}$. Therefore

$$\ln e^z = \ln e^a + i(b + 2\pi n) = \underbrace{a + ib}_{z} + 2\pi i n = z + 2\pi i n \ \text{ which may be } \neq z \quad .$$

2.4.1 Complex powers

Once we have the complex logarithm, we can define complex powers $f(z) = z^\alpha$, where both z and α are complex:

$$f(z) = z^\alpha = e^{\alpha \ln z} \quad . \tag{2.20}$$

Since the logarithm is multi-valued, complex powers also are multi-valued functions.

Example 2.5

Show that i^i is real and the principal-branch value is $i^i = 1/\sqrt{e^\pi}$.

$$i^i = e^{i \ln i} = e^{i[\ln 1 + i(\pi/2 + 2\pi n)]} = e^{-\pi/2 - 2\pi n}.$$

These values are all real. For $n = 0$ we have $i^i = e^{-\pi/2} = 1/\sqrt{e^\pi}$.

2.5 Curves in the complex plane

The locus of points satisfying some constraint on a complex parameter traces out a curve in the complex plane. For example the constraint $|z| = 1$ requires that the length of the vector from the origin to the point z is constant and equal to 1. This clearly corresponds to the set of points lying on a circle of unit radius.

Instead of determining the geometric structure of the constraint one may instead solve the constraint equation algebraically and look for the equation of the curve. This has the advantage that the method is in principle straightforward although the details may be algebraically involved whereas the geometrical construction may not be obvious. In Cartesian coordinates the algebraic constraint corresponding to $|z| = 1$ is $|z|^2 = a^2 + b^2 = 1$ which is the equation of a circle as expected. In polar coordinates the calculation is even simpler $|z| = r = 1$.

As a second example consider the constraint $|z - z_0| = 1$. This is the equation of a unit circle centre z_0 as may be immediately seen by changing the coordinate system to $z' = (z - z_0)$.

Alternatively one may solve the constraint algebraically to find $|z - z_0|^2 = (a - a_0)^2 + (b - b_0)^2 = 1$ which is the equation of the unit circle centred at the point (a_0, b_0). The solution in polar coordinates is not so straightforward in this case, showing that it is important to try the alternate forms when looking for the algebraic solution. To illustrate the techniques for finding curves in the complex plane in more complicated cases I present some further examples:

Example 2.6

What is the locus in the Argand diagram that is defined by $\left|\dfrac{z - \mathrm{i}}{z + \mathrm{i}}\right| = 1$?

Equivalently we have $|z - \mathrm{i}| = |z + \mathrm{i}|$, so the distance to z from $(0, 1)$ is the same as the distance from $(0, -1)$. Hence the solution is the “real axis”.

Alternatively we may solve the equation

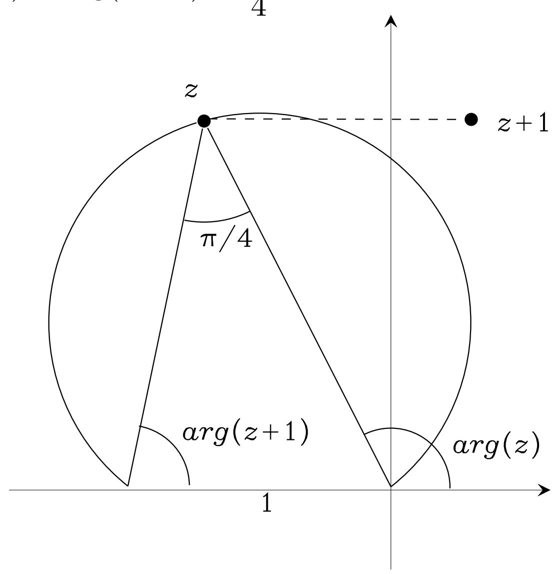
$$a^2 + (b-1)^2 = a^2 + (b+1)^2$$

which gives $b = 0$, a arbitrary, corresponding to the real axis.

Example 2.7

What is the locus in the Argand diagram that is defined by $\arg\left(\dfrac{z}{z+1}\right) = \dfrac{\pi}{4}$?

Equivalently $\arg(z) - \arg(z+1) = \dfrac{\pi}{4}$

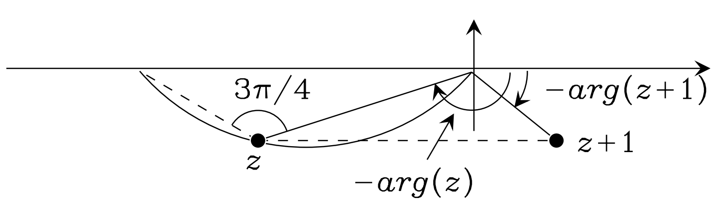


Solution: "portion of circle through $(0,0)$ and $(-1,0)$"

The x-coordinate of the centre is $-\frac{1}{2}$ by symmetry. The angle subtended by a chord at the centre is twice that subtended at the circumference, so here it is $\pi/2$. With this fact it follows that the y-coordinate of the centre is $\frac{1}{2}$.

Try solving this example algebraically.

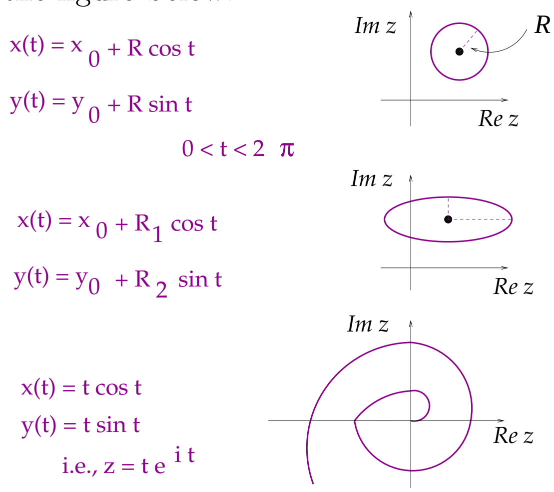
The lower portion of circle is $\arg\left(\dfrac{z}{z+1}\right) = -\dfrac{3\pi}{4}$



Another way to specify a curve in the complex plane is to give its parametric form, namely, a function $z = \gamma(t)$ that maps points on a real axis interval $[a, b]$ on to points in the complex plane $\mathcal{C}$:

$$\gamma : t \mapsto z = \gamma(t) = x(t) + iy(t) \quad , \quad a \leq t \leq b \quad .$$

Examples are given in the figure below.



The top example is a representation of the circle centred in $z_0 = x_0 + iy_0$ of radius R, since it gives

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \quad .$$

The next example is the ellipse

$$\frac{(x - x_0)^2}{R_1^2} + \frac{(y - y_0)^2}{R_2^2} = 1 \quad .$$

The third example is a spiral obtained by letting the radius vary with t.

2.5.1 Complex functions as mappings

As discussed earlier on, the topic of this section highlights the intersection of algebraic and geometric viewpoints on complex variable. In this perspective it is worth observing that functions of complex variable

$$f : z \mapsto w = f(z)$$

can be usefully viewed as mapping sets of points in the complex plane $\mathcal{C}$ (e.g. curves, or regions delimited by curves) into other sets of points in $\mathcal{C}$.

For example, take the exponential function

$$w = f(z) = e^z$$

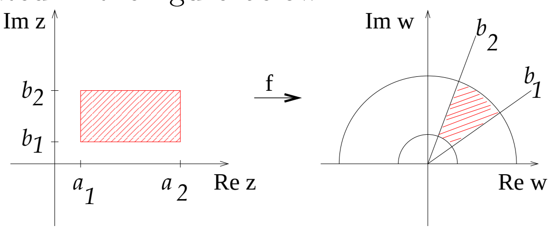
and ask on to which curves in the complex w plane this function maps straight lines parallel to the real and to the imaginary axes in the complex z plane. To see this, set

$$z = x + iy \quad ; \quad w = \rho e^{i\phi} \quad .$$

Then

$$\rho = e^x \quad ; \phi = y \quad .$$

Thus the exponential maps horizontal lines $x = a$ in the z plane on to circles $\rho = e^a$ in the w plane, and vertical lines $y = b$ in the z plane on to rays $\phi = b$ in the w plane. The mapping is illustrated in the figure below.

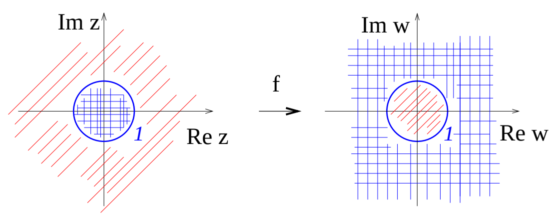


Given a function of complex variable f, for any subset A of $\mathcal{C}$ we call the "image of A through f" the set of points w such that $w = f(z)$ for some z belonging to A, and we denote this by $f(A)$. We say that f maps A on to $f(A)$.

Example 2.8

Let $f(z) = 1/z$. Where does f map the interior and exterior of the unit circle $|z| = 1$? What does f do to points on the unit circle?

$$\text{Let } \ w = f(z) = \frac{1}{z} \quad ; \qquad z = re^{i\theta} \ \Rightarrow \ w = \frac{1}{r}\, e^{-i\theta}$$



f maps the interior of the unit circle on to the exterior, and viceversa. The unit circle is mapped on to itself.

2.6 Roots of polynomials

Complex numbers enable us to find roots for any polynomial

$$P(z) \equiv a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0. \tag{2.21}$$

That is, there is at least one, and possibly as many as n complex numbers z_i such that $P(z_i) = 0$. Many physical problems involve such roots.

In the case $n = 2$ you already know a general formula for the roots. There is a similar formula for the case $n = 3$ and historically this is important because it lead to the invention of complex numbers. However, it can be shown that such general formulae do not exist for equations of higher order. We can, however, find the roots of specially simple polynomials.

In this section we start by defining and evaluating roots of complex numbers. Then we discuss how to characterize polynomials by their roots. Finally we give examples of how to use complex variable techniques to solve polynomial equations in special cases.

2.6.1 Roots of complex numbers

A number u is defined to be an n-th root of complex number z if $u^n = z$. Then we write $u = z^{1/n}$. The following result obtains.

Every complex number has exactly n distinct n-th roots.

Proof. Let $z = r(\cos\theta + i\sin\theta)$; $u = \rho(\cos\alpha + i\sin\alpha)$. Then, using de Moivre's theorem,

$$r(\cos\theta + i\sin\theta) = \rho^n(\cos\alpha + i\sin\alpha)^n = \rho^n(\cos n\alpha + i\sin n\alpha)$$

$$\Rightarrow \rho^n = r \quad , \quad n\alpha = \theta + 2\pi k \qquad (k \text{ integer})$$

$$\text{Therefore } \rho = r^{1/n} \quad , \quad \alpha = \theta/n + 2\pi k/n \ .$$

We thus see that we get n distinct values for k, from 0 to $n-1$, corresponding to n distinct values of u for $z \neq 0$. So

$$u = z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right], \ k = 0, 1, \ldots, n-1 \ . \qquad (2.22)$$

We note that the function $f(z) = z^{1/n}$ is a multi-valued function.

Example 2.9

Find all cubic roots of $z = -1 + i$.

Applying Eq (2.22) we have

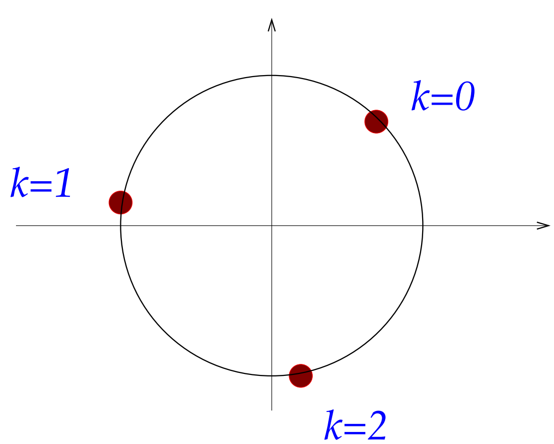
$$u = (-1+i)^{1/3}$$
$$= (\sqrt{2})^{1/3}\left[\cos\left(\frac{3\pi}{4}\frac{1}{3} + \frac{2\pi k}{3}\right) + i\sin\left(\frac{3\pi}{4}\frac{1}{3} + \frac{2\pi k}{3}\right)\right] \quad , \quad k = 0, 1, 2 \ .$$

That is, the three cubic roots of $-1+i$ are

$$2^{1/6}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad (k = 0) \quad ,$$

$$2^{1/6}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) \quad (k = 1) \quad ,$$

$$2^{1/6}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) \quad (k = 2) \quad .$$



Equivalently, using Eq (2.20),

$$u = (-1+i)^{1/3} = e^{(1/3)\ln(-1+i)} = e^{(1/3)[\ln\sqrt{2}+i(3\pi/4+2k\pi)]}$$
$$= (\sqrt{2})^{1/3} e^{i(\pi/4+2k\pi/3)} \quad .$$

2.6.2 Characterizing a polynomial by its roots

Consider a polynomial of degree n in the complex variable z

$$P(z) \equiv a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where a_k are complex coefficients. $z = z_i$ is root of $P(z)$ if $P(z_i) = 0$.

For $n = 2$ we have the quadratic formula for roots

$$z_{1,2} = -\frac{a_1}{2a_2} \pm \frac{1}{2a_2}\sqrt{a_1^2 - 4a_0a_2} \, .$$

The sum and product of the roots are given respectively by

$$\text{sum of roots} = -\frac{a_1}{a_2} \quad , \quad \text{product of roots} = \frac{a_0}{a_2} \quad . \tag{2.23}$$

For $n = 3$ there exists a formula for roots as well. For $n > 3$ no explicit formula exists, but there exists a general theorem, the fundamental theorem of algebra (discussed in the course on "Functions of a complex variable"), which states that every nonzero single-variable polynomial $P(z)$ with complex coefficients has exactly as many complex roots as its degree (counting repeated roots with their multiplicity). Here we use this result to notice that we can characterize $P(z)$ equivalently by its coefficients or by its roots.

More precisely, knowledge of a polynomial's roots enables us to express the polynomial as a product of linear terms

$$\begin{aligned} a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 &= a_n (z - r_1)(z - r_2)\cdots(z - r_n) \\ &= a_n \Big(z^n - z^{n-1} \sum_{j=1}^{n} r_j + \cdots + (-1)^n \prod_{j=1}^{n} r_j \Big). \end{aligned} \tag{2.24}$$

Comparing the coefficients of z^{n-1} and z^0 on the two sides, we deduce that

$$\frac{a_{n-1}}{a_n} = -\sum_j r_j \quad ; \quad \frac{a_0}{a_n} = (-1)^n \prod_j r_j \tag{2.25}$$

i.e. the two ratios on the left hand sides in Eq (2.25) are related to the sum and the product of the roots respectively, of which Eq (2.23) is a special case for $n = 2$.

In what follows we use complex variable techniques to find roots of polynomials in special cases.

2.6.3 Special polynomials

Let us start with a simple case. Consider

$x^n = 1 \quad \Rightarrow \quad x = 1^{1/n}$

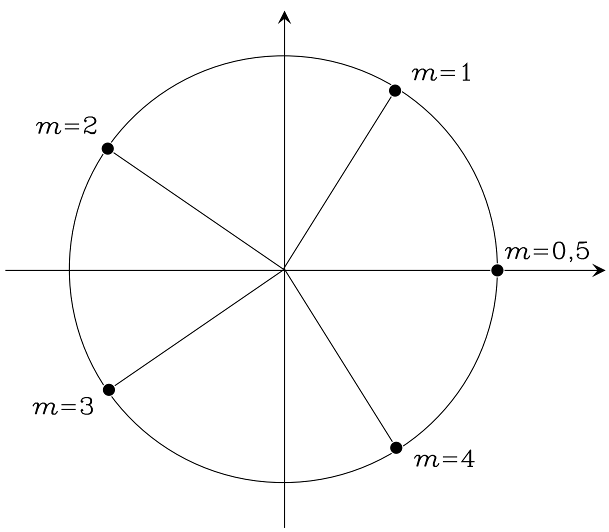
The solution is given by the nth roots of unity. In taking roots it is crucial to allow for the ambiguity in the phase of a (complex) number

$$1 = \mathrm{e}^{2m\pi \mathrm{i}} \quad \Rightarrow \quad 1^{1/n} = \mathrm{e}^{2m\pi \mathrm{i}/n} = \cos\left(\frac{2m\pi}{n}\right) + \mathrm{i}\sin\left(\frac{2m\pi}{n}\right) \tag{2.26}$$

E.g. for $n = 5$

$$1^{1/5} = \cos\left(\frac{2m\pi}{5}\right) + \mathrm{i}\sin\left(\frac{2m\pi}{5}\right) \quad (m = 0, 1, 2, 3, 4).$$

The roots may be drawn in the Argand plane and correspond to five equally spaced points in the plane :



Example 2.10

Consider the equation

$$z^5 + 32 = 0\ .$$

This is similar to the previous case. The solutions are the fifth roots of -32:

$$(-32)^{1/5} = 32^{1/5}\left[\cos\left(\frac{\pi}{5} + \frac{2\pi k}{5}\right) + i\sin\left(\frac{\pi}{5} + \frac{2\pi k}{5}\right)\right] \quad , \qquad k = 0, 1, 2, 3, 4$$

that is,

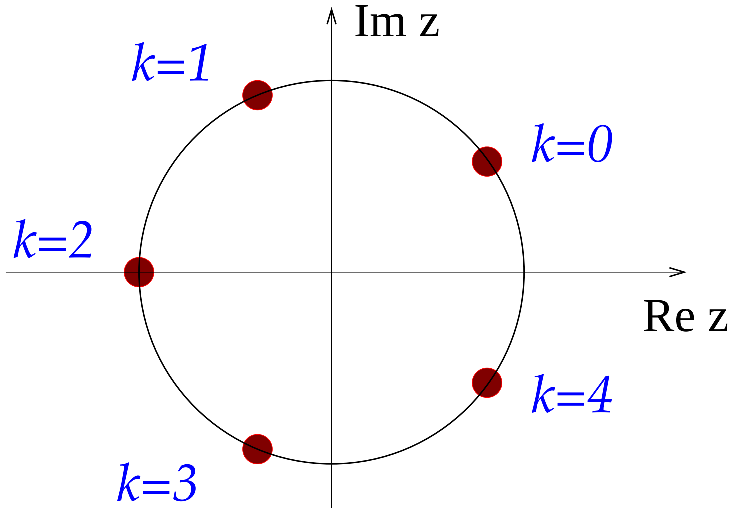
$k = 0: \quad 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

$k = 1: \quad 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$

$k = 2: \quad -2$

$k = 3: \quad 2\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right)$

$k = 4: \quad 2\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right)$



In what follows I will illustrate the techniques of taking roots in more complicated cases by a series of examples. In this we shall often need the coefficients of $x^r y^{n-r}$ in $(x+y)^n$. These are given by the binomial coefficients

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad .$$

Useful mnemonics for these coefficients is represented by the **Pascal's triangle**

$$\begin{array}{lccccccccccc}
(x+y)^0 & & & & & & 1 & & & & & \\
(x+y)^1 & & & & & 1 & & 1 & & & & \\
(x+y)^2 & & & & 1 & & 2 & & 1 & & & \\
(x+y)^3 & & & 1 & & 3 & & 3 & & 1 & & \\
(x+y)^4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(x+y)^5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

in which each row is obtained from the one above by adding the numbers to right and left of the position to be filled in. That is, the r-th element of n-th row is given by the sum of the two elements above it in the $(n-1)$-th row:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \quad . \tag{2.27}$$

Example 2.11

Consider the equation

$$(z+\mathrm{i})^7 + (z-\mathrm{i})^7 = 0.$$

This may be solved by the techniques just discussed as follows:

$$\left(\frac{z+\mathrm{i}}{z-\mathrm{i}}\right)^7 = -1 = \mathrm{e}^{(2m+1)\pi\mathrm{i}}$$

$$\Rightarrow \quad \frac{z+\mathrm{i}}{z-\mathrm{i}} = \mathrm{e}^{(2m+1)\pi\mathrm{i}/7} \quad \Rightarrow \quad z\Big(1-\mathrm{e}^{(2m+1)\pi\mathrm{i}/7}\Big) = -\mathrm{i}\Big(1+\mathrm{e}^{(2m+1)\pi\mathrm{i}/7}\Big)$$

$$\Rightarrow \quad z = \mathrm{i}\frac{\mathrm{e}^{(2m+1)\pi\mathrm{i}/7}+1}{\mathrm{e}^{(2m+1)\pi\mathrm{i}/7}-1} = \mathrm{i}\frac{\mathrm{e}^{(2m+1)\pi\mathrm{i}/14}+\mathrm{e}^{-(2m+1)\pi\mathrm{i}/14}}{\mathrm{e}^{(2m+1)\pi\mathrm{i}/14}-\mathrm{e}^{-(2m+1)\pi\mathrm{i}/14}} = \mathrm{i}\frac{2\cos\left(\frac{2m+1}{14}\pi\right)}{2\mathrm{i}\sin\left(\frac{2m+1}{14}\pi\right)}.$$

Thus

$$z = \cot\left(\frac{2m+1}{14}\pi\right) \quad , \qquad m = 0,1,2,3,4,5,6 \quad .$$

The original equation can be written in another form

$$\begin{aligned} &\Rightarrow \quad z^7 - 21z^5 + 35z^3 - 7z = 0 \\ &\Rightarrow \quad z^6 - 21z^4 + 35z^2 - 7 = 0 \quad \mathit{or} \quad z = 0 \\ &\Rightarrow \quad w^3 - 21w^2 + 35w - 7 = 0 \quad (w \equiv z^2) \end{aligned}$$

Thus, using our solution for the roots of the original equation, we see the roots of $w^3 - 21w^2 + 35w - 7 = 0$ are $w = \cot^2\left(\frac{2m+1}{14}\pi\right) \quad (m = 0,1,2)$.

Example 2.12

Sometimes the underlying equation which can be solved by these techniques is not obvious. For example - find the roots of

$$z^3 + 7z^2 + 7z + 1 = 0.$$

If one writes three more rows in the Pascal's triangle given in the previous page using the rule in Eq. (2.27), one finds that the ninth row is

$$1\ 8\ 28\ 56\ 70\ 56\ 28\ 8\ 1,$$

so

$$\begin{aligned} \tfrac{1}{2}[(z+1)^8 - (z-1)^8] &= 8z^7 + 56z^5 + 56z^3 + 8z \\ &= 8z[w^3 + 7w^2 + 7w + 1] \quad (w \equiv z^2). \end{aligned}$$

Now $(z+1)^8 - (z-1)^8 = 0$ when $\dfrac{z+1}{z-1} = \mathrm{e}^{2m\pi i/8}$, i.e. when

$$z = \frac{\mathrm{e}^{m\pi\mathrm{i}/4}+1}{\mathrm{e}^{m\pi\mathrm{i}/4}-1} = -\mathrm{i}\cot(m\pi/8) \quad (m = 1,2,\ldots,7),$$

so the roots of the given equation are $z = -\cot^2(m\pi/8)$, $m = 1,2,3$.

Example 2.13

Show that $\sum_{m=0}^{2} \cot^2\left(\frac{2m+1}{14}\pi\right) = 21$

Solution: From Example 2.11 we have that the left hand side is the sum of the roots of $w^3 - 21w^2 + 35w - 7 = 0$. Then the result follows from the first equation in (2.25).

Example 2.14

Note that based on eq (2.24) a polynomial may be characterized by (i) its roots and (ii) any a_n. To illustrate the use of this representation show that

$$\frac{z^{2m} - a^{2m}}{z^2 - a^2} = \left(z^2 - 2az\cos\frac{\pi}{m} + a^2\right)\left(z^2 - 2az\cos\frac{2\pi}{m} + a^2\right)\cdots\left(z^2 - 2az\cos\frac{(m-1)\pi}{m} + a^2\right).$$

Solution: Consider $P(z) \equiv z^{2m} - a^{2m}$, a polynomial of order $2m$ with leading term $a_{2m} = 1$ and roots $z_r = a\mathrm{e}^{r\pi i/m}$. Define

$$Q(z) \equiv (z^2 - a^2)\left(z^2 - 2az\cos\frac{\pi}{m} + a^2\right)\left(z^2 - 2az\cos\frac{2\pi}{m} + a^2\right)\cdots\left(z^2 - 2az\cos\frac{(m-1)\pi}{m} + a^2\right).$$

This polynomial is of order $2m$ with leading coeff. $a_{2m} = 1$ and with roots that are the numbers

$$\begin{aligned} z_r &= a\cos\frac{r\pi}{m} \pm \sqrt{a^2\cos^2\frac{r\pi}{m} - a^2} \\ &= a\left(\cos\frac{r\pi}{m} \pm \mathrm{i}\sqrt{1 - \cos^2\frac{r\pi}{m}}\right) = a\mathrm{e}^{\pm \mathrm{i}r\pi/m} \quad (r = 0, 1, \ldots, m). \end{aligned}$$

Thus P and Q are identical.

3 Differential Equations I

A **differential equation** is an equation in which an expression involving derivatives of an unknown function is set equal to a known function. For example

$$\frac{\mathrm{d}f}{\mathrm{d}x} + xf = \sin x \tag{3.1}$$

is a differential equation for $f(x)$. To determine a unique solution of a differential equation we require some initial data; in the case of (3.1), the value of f at some point x. These data are often called **initial conditions**. Below we'll discuss how many initial conditions one needs.

Differential equations enable us to encapsulate physical laws: the equation governs events everywhere and at all times; the rich variety of experience arises because at different places and times different initial conditions select different solutions. Since differential equations are of such transcending importance for physics, let's talk about them in some generality. In this course we will talk of *ordinary* differential equations (ODE): namely, equations involving derivatives with respect to one single variable (as opposed to partial differential equations, involving derivatives with respect to more than one variable).

3.1 Differential operators

Every differential equation involves a **differential operator**.

functions: numbers → numbers (e.g. $x \to \mathrm{e}^x$)

operators: functions → functions (e.g. $f \to \alpha f;\ f \to 1/f;\ f \to f + \alpha; \ldots$)

A differential operator does this mapping by differentiating the function one or more times (and perhaps adding in a function, or multiplying by one, etc).

$$\left(\text{e.g. } f \to \frac{\mathrm{d}f}{\mathrm{d}x};\ f \to \frac{\mathrm{d}^2 f}{\mathrm{d}x^2};\ f \to 2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f\frac{\mathrm{d}f}{\mathrm{d}x}; \ldots\right)$$

It is useful to name the operators. For example we could denote by $L(f)$ the operation $f \to \dfrac{\mathrm{d}f}{\mathrm{d}x}$.

3.1.1 Order of a differential operator

The **order** of a differential operator is the order of the highest derivative contained in it. So

$$L_1(f) \equiv \frac{\mathrm{d}f}{\mathrm{d}x} + 3f \quad \text{is first order,}$$

$$L_2(f) \equiv \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 3f \quad \text{is second order,}$$

$$L_3(f) \equiv \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 4\frac{\mathrm{d}f}{\mathrm{d}x} \quad \text{is second order.}$$

3.1.2 Linear operators

L is a **linear operator** if

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g), \tag{3.2}$$

where α and β are (possibly complex) numbers.

$$\left(\text{e.g. } f \to \frac{\mathrm{d}f}{\mathrm{d}x} \text{ and } f \to \alpha f \text{ are linear, but } f \to \frac{1}{f} \text{ and } f \to f + \alpha \text{ are not.}\right)$$

3.2 Linearity

An expression of the type $\alpha f + \beta g$ that is a sum of multiples of two or more functions is called a **linear combination** of the functions.

To a good approximation, many physical systems are described by linear differential equations $L(f) = 0$. Classical electrodynamics provides a notable example: the equations (Maxwell's) governing electric and magnetic fields in a vacuum are linear. The related equation governing the generation of a Newtonian gravitational field is also linear. Similarly in quantum mechanics the differential equations governing the evolution of the system, such as the Schrödinger equation, are linear.

Suppose f and g are two solutions of the linear equation $L(y) = 0$ for different initial conditions. For example, if L symbolizes Maxwell's equations, f and g might describe the electric fields generated by different distributions of charges. Then since L is linear, $L(f + g) = 0$, so $(f + g)$ describes the electric field generated by both charge distributions taken together. This idea, that if the governing equations are linear, then the response to two stimuli taken together is just the sum of the responses to the stimuli taken separately, is known as the **principle of superposition**. This principle is widely used to find the required solution to linear differential equations: we start by finding some very simple solutions that individually don't satisfy our initial conditions and then we look for linear combinations of them that do.

Linearity is almost always an approximation that breaks down if the stimuli are very large. For example, in consequence of the linearity of Maxwell's equations, the beam from one torch will pass right through the beam of another torch without being affected by it. But the beam from an extremely strong source of light would scatter a torch beam because the vacuum contains 'virtual' electron-positron pairs which respond non-negligibly to the field of a powerful beam, and the excited electron-positron pairs can then scatter the torch beam. In a similar way, light propagating through a crystal (which is full of positive and negative charges) can rather easily modify the electrical properties of a crystal in a way that affects a second light beam – this is the idea behind non-linear optics, now an enormously important area technologically. Gravity too is non-linear for very strong fields.

While non-linearity is the generic case, the regime of weak stimuli in which physics is to a good approximation linear is often a large and practically important one. Moreover, when we do understand non-linear processes quantitatively, this is often done using concepts that arise in the linear regime. For example, any elementary particle, such as an electron or a quark, is a weak-field, linear-response construct of quantum field theory.

3.3 Inhomogeneous terms

We've so far imagined the stimuli to be encoded in the initial conditions. It is sometimes convenient to formulate a physical problem so that at least some of the stimuli are encoded by a function that we set our differential operator equal to. Thus we write

$$\underset{\substack{\text{given}\\\text{homogeneous}}}{L}\ \underset{\text{sought}}{(f)} = \underset{\substack{\text{given}\\\text{inhomogeneous}}}{h(x)} \tag{3.3}$$

If $h(x) = 0$, Eq. (3.3) is homogeneous. If $h(x) \neq 0$, it is nonhomogeneous.

Suppose f_1 is the general solution of $Lf = 0$ and f_0 is any solution of $Lf = h$. We call f_1 the **complementary function** and f_0 the **particular integral** and then the general solution of $Lf = h$ is

$$\underset{\text{Complementary fn}}{f_1} + \underset{\text{Particular integral}}{f_0}. \tag{3.4}$$

How many initial conditions do we need to specify to pick out a unique solution of $L(f) = 0$? It is easy to determine this in a hand-waving way because the solution to a differential equation requires integration. With a single derivative one needs to perform one integration which introduces one integration constant which in turn must be fixed by one initial condition. Thus the number of integration constants needed, or equivalently the number of initial conditions, is just the order of the differential equation. A more rigorous justification of this may be found in Sec. 3.5.

3.4 First-order linear equations

Any first-order linear equation can be written in the form

$$\frac{\mathrm{d}f}{\mathrm{d}x} + q(x)f = h(x). \tag{3.5}$$

Since the solution to this equation implies an integration to remove the derivative the general solution will have one arbitrary constant. The solution can be found by seeking a function $I(x)$ such that

$$I\frac{\mathrm{d}f}{\mathrm{d}x} + Iqf = \frac{\mathrm{d}If}{\mathrm{d}x} = Ih \quad \Rightarrow \quad f(x) = \frac{1}{I(x)}\left[\int I(x)h(x)\mathrm{d}x + C\right]. \tag{3.6}$$

C is the required arbitrary constant in the solution, and I is called the **integrating factor**. We need $Iq = \mathrm{d}I/\mathrm{d}x$, so

$$\ln I = \int q\,\mathrm{d}x \quad \Rightarrow \quad I = \mathrm{e}^{\int q\,\mathrm{d}x}. \tag{3.7}$$

Example 3.1

Solve

$$2x\frac{\mathrm{d}f}{\mathrm{d}x} - f = x^2.$$

In standard form the equation reads

$$\frac{\mathrm{d}f}{\mathrm{d}x} - \frac{f}{2x} = \tfrac{1}{2}x$$

so $q = -\dfrac{1}{2x}$ and by (3.7) $I = \mathrm{e}^{-\frac{1}{2}\ln x} = \dfrac{1}{\sqrt{x}}$.

Plugging this into (3.6) we have $f = \sqrt{x}[\int \frac{1}{2}\sqrt{x}\,\mathrm{d}x + C] = \frac{1}{3}x^2 + Cx^{1/2}$.

Example 3.2

Solve the equation

$$\frac{df}{dx} + 2f = x^2$$

with the initial condition

$$f(3) = 0 \ .$$

Solution. The integrating factor is given by

$$I(x) = e^{\int 2 \ dx} = e^{2x} \ .$$

Thus the general solution is

$$f(x) = e^{-2x}[\int dx \ x^2 e^{2x} + \text{const.}]$$

$$= e^{-2x} \ [e^{2x} \frac{x^2 - x + 1/2}{2} + \text{const.}] \ .$$

The initial condition $f(3) = 0$ requires

$$e^{-6} \ [e^6 \ \frac{13}{4} + \text{const.}] = 0 \ ,$$

that is, const. $= -e^6 \ 13/4$. Therefore

$$f(x) = \frac{x^2 - x + 1/2}{2} - \frac{13}{4} \ e^{6-2x} \ .$$

3.5 Arbitrary constants & general solutions

We have seen that the general solution of a first-order linear ODE depends on one arbitrary constant, and one initial condition is needed to determine the solution uniquely. How many initial conditions do we need to specify to pick out a unique solution of an nth-order linear ODE $L(f) = 0$? Arrange $Lf \equiv a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_0 f = 0$ as

$$f^{(n)}(x) = -\left(\frac{a_{n-1}}{a_n} f^{(n-1)}(x) + \cdots + \frac{a_0}{a_n} f\right). \tag{3.8}$$

If we differentiate both sides of this equation with respect to x, we obtain an expression for $f^{(n+1)}(x)$ in terms of $f^{(n)}(x)$ and lower derivatives. With the help of (3.8) we can eliminate $f^{(n)}(x)$ from this new equation, and thus obtain an expression for $f^{(n+1)}(x)$ in terms of $f(x)$ and derivatives up to $f^{(n-1)}(x)$. By differentiating both sides of our new equation and again using (3.8) to eliminate $f^{(n)}$ from the resulting equation, we can obtain an expression for $f^{(n+2)}(x)$ in terms of $f(x)$ and derivatives up to $f^{(n-1)}(x)$. Repeating this procedure a sufficient number of times we can obtain an expression for *any* derivative of f in terms of $f(x)$ and derivatives up to $f^{(n-1)}$. Consequently, if the values of these n functions are given at any point x_0 we can evaluate the Taylor series

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \tfrac{1}{2}(x - x_0)^2 f''(x_0) + \cdots \tag{3.9}$$

for any value of x that lies near enough to x_0 for the series to converge. Consequently, the functional form of $f(x)$ is determined by the original n^{th} order differential equation and the n initial conditions $f(x_0), \ldots, f^{(n-1)}(x_0)$. Said another way, to pick out a unique solution to an nth order equation, we need n initial conditions.

The **general solution** of a differential equation is one that contains a sufficient supply of arbitrary constants to allow it to become *any* solution of the equation if these constants are assigned appropriate values. We have seen that once the n numbers $f^{(r)}(x_0)$ for $r = 0, \ldots, n-1$ have been specified, the solution to the linear nth-order equation $Lf = 0$ is uniquely determined. This fact suggests that the general solution of $Lf = 0$ should include n arbitrary constants, one for each derivative. This is true, although the constants don't have to be the values of individual derivatives; all that is required is that appropriate choices of the constants will cause the rth derivative of the general solution to adopt any specified value.

Given the general solution we can construct n particular solutions $f_1, \ldots, f_n$ as follows: let f_1 be the solution in which the first arbitrary constant, k_1, is unity and the others zero, f_2 be the solution in which the second constant, k_2, is unity and the other zero, etc. It is easy to see that the general solution is

$$f(x) = \sum_{r=1}^{n} k_r f_r(x). \tag{3.10}$$

That is, the general solution is a linear combination of n **particular solutions**, that is, solutions with no arbitrary constant.

3.6 First-order non-linear equations

The general form of a first-order differential equation for a function $y = f(x)$ can be written as a relationship between x, y, and the first derivative $y' = dy/dx$,

$$G(x, y, y') = 0 \ .$$

If this can solved for y', we may write the first-order differential equation in the *normal* form

$$y' = F(x, y) \ .$$

Writing F as the quotient of two functions $F(x,y) = -M(x,y)/N(x,y)$, this can be put in the *differential* form

$$M(x,y)dx + N(x,y)dy = 0 \ .$$

In this section we discuss first-order differential equations of non-linear kind. Non-linear equations are generally not solvable analytically – in large measure because their solutions display richer structure than analytic functions can describe. There are some interesting special cases, however, in which analytic solutions of nonlinear equations can be derived.†

3.6.1 Separable equations

The general form of a separable differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x)}{g(y)}$$

which is readily solved by

$$\int g(y)\mathrm{d}y = \int f(x)\mathrm{d}x.$$

Example 3.3

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \mathrm{e}^x$$

Separating variables gives

$$\int \mathrm{d}y/y^2 = \int \mathrm{e}^x \mathrm{d}x$$

with solution

$$\frac{-1}{y} = \mathrm{e}^x + c$$

or

$$y = \frac{-1}{(\mathrm{e}^x + c)}$$

where c is a constant.

† These techniques may also provide simple ways of solving particular linear equations.

3.6.2 Almost separable equations

The general form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(ax + by),$$

where f is an arbitrary function and a and b are constants, can be solved by the change of variables $z = ax + by$. Using $\frac{\mathrm{d}z}{\mathrm{d}x} = a + b\frac{\mathrm{d}y}{\mathrm{d}x}$, one finds

$$\frac{\mathrm{d}z}{\mathrm{d}x} = a + bf(z),$$

which is trivially separable and can be solved to give

$$x = \int \frac{1}{(a + bf(z))}\mathrm{d}z.$$

Example 3.4

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (-4x + y)^2$$

In this case the right hand side is a function of $-4x+y$ only, so we change variable to $z = y - 4x$, giving

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -4 + \frac{\mathrm{d}y}{\mathrm{d}x} = z^2 - 4$$

with solution

$$x = \int \frac{1}{((z-2)(z+2))}\mathrm{d}z.$$

So $x = \frac{1}{4}\ln(\frac{(z-2)}{(z+2)}) + C$, where C is a constant. Solving for y we find $y = 4x + 2\frac{(1+k\mathrm{e}^{4x})}{(1-k\mathrm{e}^{4x})}$, where k is a constant.

Example 3.5

Another example is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - y}{x - y + 1}.$$

We define

$$u \equiv x - y + 1 \quad \text{and have} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1 - \frac{u-1}{u} \quad \Rightarrow \quad u\frac{\mathrm{d}u}{\mathrm{d}x} = 1,$$

which is trivially solvable.

3.6.3 Homogeneous equations

Consider equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y/x). \tag{3.11}$$

Such equations are called **homogeneous** because they are invariant under a rescaling of both variables: that is, if $X = sx$, $Y = sy$ are rescaled variables, the equation for $Y(X)$ is identical to that for $y(x)$.† These equations are readily solved by the substitution

$$y = vx \quad \Rightarrow \quad y' = v'x + v. \tag{3.12}$$

We find

$$\int \frac{\mathrm{d}v}{f(v) - v} = \int \frac{\mathrm{d}x}{x} = \ln x + \text{constant}. \tag{3.13}$$

Example 3.6

Solve

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = (x+y)^2 \mathrm{e}^{-y/x}.$$

Solution: Dividing through by xy and setting $y = vx$ we have

$$(v'x + v) - v = \frac{(1+v)^2}{v}\mathrm{e}^{-v} \quad \Rightarrow \quad \ln x = \int \frac{\mathrm{e}^v v \mathrm{d}v}{(1+v)^2}.$$

The substitution $u \equiv 1 + v$ transforms the integral to

$$\mathrm{e}^{-1}\int \Big(\frac{1}{u} - \frac{1}{u^2}\Big)\mathrm{e}^u \mathrm{d}u = \mathrm{e}^{-1}\Big[\frac{\mathrm{e}^u}{u}\Big].$$

3.6.4 Homogeneous but for constants

These are equations which are not homogeneous but can be brought to homogeneous form by shifting variables by constants. Consider

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + 2y + 1}{x + y + 2}.$$

Changing variables to $x' = x - a$, $y' = y - b$ yields

$$\frac{\mathrm{d}y'}{\mathrm{d}x'} = \frac{x' + 2y' + 1 + a + 2b}{x' + y' + 2 + a + b}.$$

Setting $a = -3$, $b = 1$ gives

$$\frac{\mathrm{d}y'}{\mathrm{d}x'} = \frac{x' + 2y'}{x' + y'}$$

which is homogeneous and can be solved by the technique of the previous subsection.

† Note that the word "homogeneous" is used with two different, unrelated meanings in the context of differential equations: one usage is that defined here; the other is that defined after Eq. (3.3).

3.6.5 The Bernoulli equation

Bernoulli's equation has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n. \tag{3.14}$$

This is nonlinear but can readily be reduced to a linear equation by the change of variable

$$z = y^{1-n}. \tag{3.15}$$

Then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence

$$\frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)P(x)z = (1-n)Q(x).$$

Having converted the equation to a linear equation, this can be solved by the methods of Sec. 3.4. After this, the answer for y is obtained by $y = z^{1/(1-n)}$.

Example 3.7

Solve the differential equation

$$y' + y = y^{2/3}.$$

Changing variable to $z = y^{1/3}$ leads to the equation $z' + z/3 = 1/3$. The integrating factor is $\mathrm{e}^{x/3}$ and so the solution is $z\mathrm{e}^{x/3} = \int \mathrm{e}^{x/3}dx/3$. This implies $z = y^{1/3} = 1/3 + c\mathrm{e}^{-x/3}$ where c is a constant.

3.6.6 Exact equations

Suppose x, y are related by $\phi(x,y) =$ constant. Then $0 = \mathrm{d}\phi = \phi_x \mathrm{d}x + \phi_y \mathrm{d}y$, where $\phi_x \equiv \partial\phi/\partial x$, $\phi_y \equiv \partial\phi/\partial y$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\phi_x}{\phi_y}\ .$$

Conversely, given a first-order differential equation $y' = F(x,y)$, we can ask if there exists a function $\phi(x,y)$ such that $F = -\phi_x/\phi_y$. If such a function exists, the differential equation is called **exact**, and the solution to the differential equation is determined by $\phi(x,y) =$ constant. That is, if we write the equation in the differential form

$$M(x,y)dx + N(x,y)dy = 0\ ,$$

as at the beginning of this section, the equation is exact if there exists ϕ such that

$$\frac{\partial\phi}{\partial x} = M\ , \quad \frac{\partial\phi}{\partial y} = N\ . \tag{3.16}$$

A test for exactness that is useful in practice arises from taking derivatives of the first equation in Eq. (3.16) with respect to y and of the second equation in Eq. (3.16) with respect to x. Since the order of partial derivatives can be interchanged for smooth ϕ, we get

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \; .$$

This gives a necessary condition for exactness. (It is also a sufficient condition if the domain on which the functions are defined has no holes.)

Note that a differential equation may not be exact as it stands, but it can become so once we multiply it through by a certain factor, namely, once we multiply both M and N by the same function. It may not be easy to discover what the required factor is though.

Example 3.8

Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(3x^2 + 2xy + y^2)\tan x - (6x + 2y)}{(2x + 2y)}.$$

Solution: Notice that

$$\text{top} \times \cos x = -\frac{\partial}{\partial x}\left[(3x^2 + 2xy + y^2)\cos x\right]$$

and

$$\text{bottom} \times \cos x = \frac{\partial}{\partial y}\left[(3x^2 + 2xy + y^2)\cos x\right]$$

so the solution is $(3x^2 + 2xy + y^2)\cos x = \text{constant}$.

3.6.7 Equations solved by interchange of variables

Consider

$$y^2\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0.$$

As it stands the equation is non-linear and may appear insoluble. But when we interchange the rôles of the dependent and independent variables, it becomes linear: on multiplication by $(\mathrm{d}x/\mathrm{d}y)$ we get

$$y^2 + x - 2y\frac{\mathrm{d}x}{\mathrm{d}y} = 0,$$

which can be solved by the techniques we have developed for linear equations.

3.6.8 Equations solved by linear transformation

Consider

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x - y)^2.$$

In terms of $u \equiv y - x$ the equation reads $\mathrm{d}u/\mathrm{d}x = u^2 - 1$, which is trivially soluble.

Similarly, given the equation of Example 3.5

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-y}{x-y+1},$$

we define

$$u \equiv x - y + 1 \quad \text{and have} \quad 1 - \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u-1}{u} \quad \Rightarrow \quad u\frac{\mathrm{d}u}{\mathrm{d}x} = 1,$$

which is trivially soluble.

3.7 Geometric meaning of first-order ODEs

In this section we illustrate the geometrical interpretation of the solutions of differential equations in terms of families of curves.

3.7.1 Differential equations and families of curves

We have noted that the general solution of a first-order differential equation

$$y' = f(x, y) \tag{3.17}$$

contains one arbitrary constant:

$$y = y(x, c)\ . \tag{3.18}$$

For each value of the constant c Eq. (3.18) specifies a curve in the (x, y) plane. The general solution (3.18) can be thought of as a family of curves, obtained by varying the parameter c. We can write this one-parameter family of curves in the general form

$$F(x, y, c) = 0\ . \tag{3.19}$$

We next illustrate this by examples.

Example 3.9

Consider for example

$$y' = -\frac{x}{y}\ .$$

This is a separable equation, whose general solution is given by

$$\int y\ dy = -\int x\ dx\ ,$$

that is,

$$\frac{y^2}{2} = -\frac{x^2}{2} + c\ ,$$

or

$$x^2 + y^2 = \text{constant} \ , \tag{3.20}$$

which is the equation of the family of all circles centered at the origin (Fig. 1).

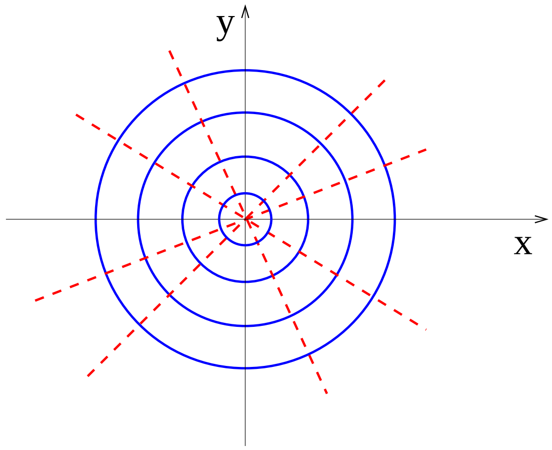


Fig.1

Example 3.10

The equation

$$y' = \frac{y^2 - x^2}{2xy}$$

is a homogeneous equation,

$$y' = f(y/x) \quad \text{with} \ \ f(y/x) = (y/x - x/y)/2 \ .$$

This can be solved as in Subsec. 3.6.3 by the change of variable $y \to v = y/x$ ($y' = xv' + v$), which yields

$$xv' + v = (v - 1/v)/2 \ .$$

Then by separation of variables

$$\int dv \frac{2v}{1+v^2} = -\int dx \ \frac{1}{x} = -\ln x + \text{constant} \ .$$

That is,

$$\ln(1 + v^2) = \ln \frac{c}{x} \ ,$$

i.e.,

$$1 + \frac{y^2}{x^2} = \frac{c}{x} \ ,$$

or

$$x^2 + y^2 = cx \ . \tag{3.21}$$

This is the equation of the family of all circles tangent to the y-axis at the origin (solid-line circles in Fig. 2).

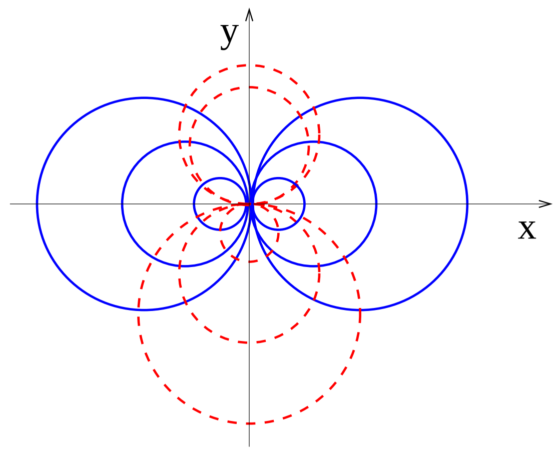


Fig.2

3.7.2 Orthogonal trajectories

Given a family of curves, represented by solutions of the differential equation

$$y' = f(x, y) \ ,$$

the orthogonal trajectories are given by a second family of curves such that each curve in either family is perpendicular to every curve in the other family. Since the slope y' of any curve in the first family is given by $f(x, y)$ at every point in the (x, y) plane, orthogonality requires that orthogonal trajectories are solutions of the differential equation

$$y' = -1/f(x, y) \ . \tag{3.22}$$

Example 3.11

Find the orthogonal trajectories of the family of circles centered at the origin in Example 3.9.

Solution. We need to solve the equation

$$y' = \frac{y}{x} \ .$$

By separation of variables

$$\int \frac{dy}{y} = \int \frac{dx}{x} \ ,$$

$$\text{so} \quad \ln y = \ln x + \text{constant},$$

i.e.,

$$y = cx \ , \tag{3.23}$$

which is the family of all straight lines through the origin (dashed lines in Fig. 1). These are the orthogonal trajectories of the family of circles in Eq. (3.20).

Example 3.12

Find the orthogonal trajectories of the family of circles tangent to the y-axis at the origin in Example 3.10.

Solution. We need to solve the equation

$$y' = -\frac{2xy}{y^2 - x^2} \ .$$

Proceeding as in Example 3.10 with the change of variable $y \to v = y/x$ we get

$$xv' + v = 2/(1/v - v) \ .$$

By separation of variables

$$\int dv \frac{1 - v^2}{v(1 + v^2)} = \int dx \ \frac{1}{x} = \ln x + \text{constant} \ .$$

That is,

$$\ln \frac{v}{1 + v^2} = \ln x + \text{constant} \ ,$$

i.e.,

$$\frac{y/x}{1 + y^2/x^2} = \text{const. } x,$$

or

$$x^2 + y^2 = ky. \tag{3.24}$$

This is the equation of the family of all circles tangent to the x-axis at the origin (dashed circles in Fig. 2), which are the orthogonal trajectories of the family of circles in Eq. (3.21).

3.8 Applications of solution methods for first-order ODEs to higher order equations

In some special cases, the methods of solution that we have seen for first-order differential equations can be usefully exploited to treat equations of higher order. If in a second-order equation $F(x, y, y', y'') = 0$ the variable y is not explicitly present, setting $y' = q$, and so $y'' = q'$, turns the original equation into a first-order equation for q, $G(x, q, q') = 0$. If this can be solved, then the solution for y may be obtained by inserting the solution for q into $y' = q$ and solving this.

Another case is when in a second-order equation $F(x, y, y', y'') = 0$ the variable x is not explicitly present. In this case, setting $y' = q$ and expressing y'' in terms of the derivative of q with respect to y as

$$y'' = \frac{dq}{dx} = \frac{dq}{dy}\frac{dy}{dx} = \frac{dq}{dy} \ q \tag{3.25}$$

turns the original equation into a first-order equation for q of the form

$$G(y, q, dq/dy) = 0 \ , \tag{3.26}$$

in which y plays the role of the independent variable. If we can find the solution to Eq. (3.26), then we can insert this solution for q into $y' = q$ and what is left to do is to solve the resulting first-order equation.

In both the cases above, one is able to reduce the problem of solving one second-order equation to that of solving two first-order equations in succession.

We will learn later that this type of correspondence can actually be considered in a more systematic manner. Differential equations of order n can be related to *systems* of n first-order differential equations. See e.g. Sec. 5 on systems of linear equations. This goes to emphasize how central first-order solution methods are in the theory of ordinary differential equations.

The following is a mechanical example of a second-order equation $F(x, y, y', y'') = 0$ in which the variable y is not explicitly present, and we can thus solve by first-order methods as described at the beginning of this section.

Example

We consider a mechanical system given by a homogeneous, flexible chain hanging between two points under its own weight and ask: what is the shape assumed by the chain?

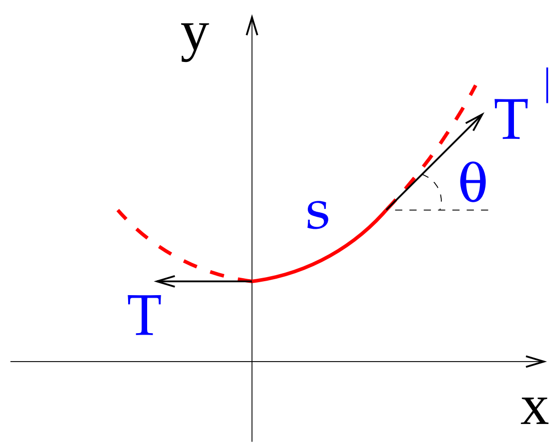


Fig.3

With reference to Fig. 3, we consider the portion of the chain between the lowest point (at which we set the zero of the x-axis) and point (x, y). We obtain the answer to the question from the fact that the system is in equilibrium under the action of the tension T at one end, the tension T' at the other end, and the gravitational force. The chain being flexible means that the tension is tangent to the chain itself at every point. Therefore, in the notation of the figure, the x and y components of Newton's law give

$$T = T' \cos\theta \ \ , \quad T' \sin\theta = \int_0^s \rho(s) \ g \ ds \ ,$$

where g is the gravitational acceleration, s is the arc length and $\rho(s)$ is the linear mass density of the chain. We thus have

$$\int_0^s \rho(s)\ g\ ds = \frac{T}{\cos\theta}\ \sin\theta = T\tan\theta = T\ \frac{dy}{dx} = Ty'\ .$$

Differentiating both sides with respect to x gives

$$Ty'' = \frac{d}{dx}\int_0^s \rho(s)\ g\ ds = (\frac{d}{ds}\int_0^s \rho(s)\ g\ ds)\frac{ds}{dx} = \rho\ g\ \sqrt{1+(y')^2}\ ,$$

where in the last step we have used that the chain is homogeneous, i.e., $\rho(s) = \rho$ is constant, and $ds = \sqrt{dx^2+dy^2} = dx\sqrt{1+(dy/dx)^2}$. The differential equation describing the shape $y(x)$ of the chain is thus

$$y'' = a\sqrt{1+(y')^2}\ , \tag{3.27}$$

where

$$a \equiv \rho g/T\ .$$

Eq. (3.27) is a second-order equation in which the variable y does not appear explicitly. We proceed as described at the beginning of this section. On setting $y' = q$, and then $y'' = dq/dx = q'$, Eq. (3.27) reduces to the first-order nonlinear equation

$$q' = a\sqrt{1+q^2}\ ,$$

which can be solved by separation of variables,

$$\int dq\ \frac{1}{\sqrt{1+q^2}} = \int a\ dx\ .$$

Using that $q = dy/dx = 0$ at $x = 0$, we get

$$\ln(q+\sqrt{1+q^2}) = ax\ ,$$

and, solving for q,

$$q = \frac{dy}{dx} = \frac{e^{ax}-e^{-ax}}{2}\ .$$

Thus

$$y(x) = \frac{1}{a}\ \frac{e^{ax}+e^{-ax}}{2} + \text{constant} = \frac{1}{a}\ \cosh ax + \text{constant}\ . \tag{3.28}$$

The solution (3.28) gives the shape assumed by the homogeneous flexible chain. This curve is called a catenary. Catenaries arise in several problems in mechanics and geometry.

4 Differential Equations II: Second-order linear ODEs

The general second-order linear equation can be written in the form

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}f}{\mathrm{d}x} + q(x)f = h(x). \tag{4.1}$$

We can start by asking: Is there an integrating factor? Suppose $\exists\ I(x)$ s.t. $\frac{\mathrm{d}^2 If}{\mathrm{d}x^2} = Ih$. Then

$$2\frac{\mathrm{d}I}{\mathrm{d}x} = Ip \quad \text{and} \quad \frac{\mathrm{d}^2 I}{\mathrm{d}x^2} = Iq. \tag{4.2}$$

These equations are unfortunately incompatible in most cases. Thus, unlike the case of first-order linear ODEs, we cannot count on there being an integrating factor.

Sec. 4.1 is devoted to discussing the general structure of solutions in the second order case. We give the general solution in Subsec. 4.1.1 in terms of a sum of two contributions, the complementary function and the particular integral, and focus on the notion of linearly independent solutions in Subsec. 4.1.2. In Subsec. 4.1.3 we observe that, although we cannot count on an integrating factor to find the general solution for second-order equations, we can nevertheless use the integrating factor technique if one particular solution of the homogeneous equation is known.

In the special case in which the coefficients $p(x)$ and $q(x)$ in Eq. (4.1) are constants, there exist general methods to solve the equation. Second-order linear ODEs with constant coefficients arise in many physical situations and so their solution is of great practical importance. We focus on such class of equations in Sec. 4.2 and Sec. 4.3, where we address respectively the question of finding the complementary function and the question of finding the particular integral. In Sec. 4.4 we give a physical application of these results by considering a forced, damped oscillator.

4.1 Generalities on second-order linear equations

4.1.1 Structure of the general solution

According to the discussion of Sec. 3.3, the general solution f of Eq. (4.1) is the sum of a particular solution f_0 (the "particular integral", PI) and the general solution f_1 of the associated homogeneous equation (the "complementary function", CF):

$$f = f_0 + f_1\ , \tag{4.3}$$

$$\text{i.e.,} \quad \text{general solution} \ = \ \text{PI} \ + \ \text{CF}\ . \tag{4.4}$$

The associated homogeneous equation is defined by setting $h(x)$ to zero in Eq. (4.1),

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}f}{\mathrm{d}x} + q(x)f = 0. \tag{4.5}$$

We have also seen in Sec. 3.2 that linearity implies the principle of superposition, namely, if u_1 and u_2 are two solutions of the linear homogeneous equation (4.5) then any linear combination of them $\alpha u_1 + \beta u_2$ is also solution. A converse of this also applies (see Subsec. 4.1.2 for proof of this statement), in the sense that *any* solution of Eq. (4.5) can be expressed as a linear combination of two fundamental solutions, where "fundamental" means two *linearly independent* solutions according to the following definition.

Two functions $u_1(x)$ and $u_2(x)$ are *linearly independent* if the relation $\alpha u_1(x) + \beta u_2(x) = 0$ implies $\alpha = \beta = 0$.

This means that, if u_1 and u_2 are two linearly independent solutions of Eq. (4.5), the complementary function CF in Eq. (4.4) can be written in general as

$$\mathrm{CF} = c_1 u_1(x) + c_2 u_2(x) \ , \tag{4.6}$$

where c_1 and c_2 are arbitrary constants. That is, as c_1 and c_2 vary, all solutions of Eq. (4.5) precisely span the whole set of linear combinations of two independent functions u_1 and u_2. We see the proof of Eq. (4.6) in the next Subsec. 4.1.2.

Example 4.1

Verify the following statement: The functions $u_1(x) = \sin x$ and $u_2(x) = \cos x$ are linearly independent on the interval $[0, 2\pi]$.

Answer. According to the definition given above, let us write

$$\alpha \sin x + \beta \cos x = 0 \ . \tag{4.7}$$

One can verify the statement above by noting that there are no two constants α and β, other than $\alpha = \beta = 0$, such that the expression $\alpha \sin x + \beta \cos x$ vanishes identically at every point in $[0, 2\pi]$.

Alternatively, one can show this as follows. Suppose α and β exist such that Eq. (4.7) is verified. Differentiating Eq. (4.7) gives

$$\alpha \cos x - \beta \sin x = 0 \ .$$

Therefore

$$\alpha = \beta \, \frac{\sin x}{\cos x} \ \Rightarrow \ \beta \left(\frac{\sin^2 x}{\cos x} + \cos x \right) = 0 \ \Rightarrow \ \beta \, \frac{1}{\cos x} = 0 \ \Rightarrow \ \beta = 0 \ .$$

Thus $\alpha = \beta = 0$.

4.1.2 Linearly independent solutions

In this section we show how the result in Eq. (4.6) arises. To this end, it is convenient to characterize in general linearly independent functions as follows. Consider two functions $u_1(x)$ and $u_2(x)$, and let

$$\alpha u_1(x) + \beta u_2(x) = 0 \ . \tag{4.8}$$

By differentiation Eq. (4.8) implies

$$\alpha u_1'(x) + \beta u_2'(x) = 0 \ . \tag{4.9}$$

If the determinant of the system of equations (4.8), (4.9) in α and β is nonzero, i.e.,

$$W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_2 u_1' \neq 0 \ , \tag{4.10}$$

then $\alpha = \beta = 0$, and the functions u_1 and u_2 are linearly independent.

If on the other hand the determinant $W(u_1, u_2)$ in Eq. (4.10) vanishes, then

$$u_1 u_2' - u_2 u_1' = 0 \ ,$$

which can be integrated to give

$$u_2 = \text{constant} \ \times \ u_1 \ .$$

This implies that Eq. (4.8) can be satisfied with α and β not all zero, and therefore u_1 and u_2 are not linearly independent.

The determinant $W(u_1, u_2)$ is called wronskian determinant of functions u_1 and u_2.

Example 4.2

The wronskian determinant of the functions $u_1(x) = \sin x$ and $u_2(x) = \cos x$ is given by

$$W(u_1, u_2) = u_1 u_2' - u_2 u_1' = -\sin^2 x - \cos^2 x = -1 \ ,$$

which says that $\sin x$ and $\cos x$ are linearly independent (as we know from direct calculation in Example 4.1).

We are now in a position to see that the general solution of Eq. (4.5) can be written in the form of Eq. (4.6). Let us suppose that we have a solution u of Eq. (4.5), and let us verify that it can be expressed as a linear combination of linearly independent solutions u_1 and u_2. Since u, u_1 and u_2 all solve Eq. (4.5), with nonzero coefficients of the second-derivative, first-derivative and no-derivative terms, we must have vanishing determinant

$$\begin{vmatrix} u & u_1 & u_2 \\ u' & u_1' & u_2' \\ u'' & u_1'' & u_2'' \end{vmatrix} = 0 \ . \tag{4.11}$$

Analogously to Eq. (4.10) for the case of two functions, Eq. (4.11) gives the wronskian determinant for three functions, and its vanishing for the three solutions u, u_1 and u_2 implies that

$$\alpha u + \beta u_1 + \gamma u_2 = 0 \tag{4.12}$$

is satisfied with α, β and γ not all zero. If α was zero, β and γ nonzero would contradict the linear independence of u_1 and u_2, so $\alpha \neq 0$. Then if u is nontrivial β and γ cannot both be zero. Thus, solving Eq. (4.12) for u expresses the solution u as a linear combination of u_1 and u_2.

Example 4.3

The general solution of the 2nd-order linear ODE

$$y'' + y = 0 \tag{4.13}$$

is

$$A \sin x + B \cos x \ . \tag{4.14}$$

To show this, it is sufficient to show that i) $\sin x$ and $\cos x$ solve the equation (for instance, by direct computation) and ii) $\sin x$ and $\cos x$ are linearly independent (as done by two methods in Example 4.1 and in Example 4.2). Then the result in Eq. (4.6) guarantees that the general solution has the form (4.14).

4.1.3 An application of the integrating factor technique: finding the general solution once one solution of the homogeneous equation is known

This subsection illustrates that, if one particular solution of the homogeneous equation is known, then one can find the general solution by applying the integrating factor technique.

Suppose we have a solution u of the homogeneous equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}u}{\mathrm{d}x} + q(x)u = 0, \tag{4.15}$$

Then write $f = uv$ and $u' \equiv \dfrac{\mathrm{d}u}{\mathrm{d}x}$ etc. so that

$$f' = u'v + uv' \quad ; \quad f'' = u''v + 2u'v' + uv''. \tag{4.16}$$

Substituting these results into (4.1) we obtain

$$\begin{aligned} h &= f'' + pf' + qf \\ &= u''v + 2u'v' + uv'' + pu'v + puv' + quv \\ &= v(u'' + pu' + qu) + uv'' + 2u'v' + puv' \\ &= \qquad 0 \qquad\quad + uv'' + 2u'v' + puv'. \end{aligned} \tag{4.17}$$

Now define $w \equiv v'$ and find

$$uw' + (2u' + pu)w = h \quad \Rightarrow \quad \begin{cases} \mathrm{IF} = \exp\left[\displaystyle\int \left(2\frac{u'}{u} + p\right) \mathrm{d}x\right] \\ \quad = u^2 \mathrm{e}^{\int p \, \mathrm{d}x}. \end{cases} \tag{4.18}$$

Finally can integrate

$$v'(x) = w(x) = u^{-2}(x)\mathrm{e}^{-\int^x p \, \mathrm{d}x} \int_{x_0}^{x} \mathrm{e}^{\int^{x'} p \, \mathrm{d}x} hu \, \mathrm{d}x'. \tag{4.19}$$

Thus if we can find one solution, u, of any second-order linear equation, we can find the general solution $f(x) = \alpha u(x) + u(x)v(x, x_0)$.

4.2 2nd-order linear ODE's with constant coefficients: The Complementary Function

Suppose the coefficients of the unknown function f and its derivatives are mere constants:

$$Lf = a_2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x). \tag{4.20}$$

In this case there exist general methods of solution. The solution proceeds through a combination of the complementary function and the particular integral. In this section we deal with the complementary function. In Sec. 4.3 we deal with the particular integral.

4.2.1 Auxiliary equation

We look for a complementary function $y(x)$ that satisfies $Ly = 0$. We try $y = \mathrm{e}^{\alpha x}$. Substituting this into $a_2y'' + a_1y' + a_0y = 0$ we find that the equation is satisfied $\forall x$ provided

$$a_2\alpha^2 + a_1\alpha + a_0 = 0. \tag{4.21}$$

This condition for the exponent α is called the **auxiliary equation**. It has two roots

$$\alpha_\pm \equiv \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2}, \tag{4.22}$$

so the CF is

$$y = A_+\mathrm{e}^{\alpha_+ x} + A_-\mathrm{e}^{\alpha_- x}. \tag{4.23}$$

Example 4.4

Solve

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 0.$$

The auxiliary equation is $(\alpha+3)(\alpha+1) = 0$, so the CF is $y = A\mathrm{e}^{-3x} + B\mathrm{e}^{-x}$.

Example 4.5

Solve

$$Ly = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0.$$

The auxiliary equation is $\alpha = \frac{1}{2}(2\pm\sqrt{4-20}) = 1\pm 2\mathrm{i}$, so $y = A\mathrm{e}^{(1+2\mathrm{i})x} + B\mathrm{e}^{(1-2\mathrm{i})x}$. But this is complex!

However, L is real operator. So $0 = \Re\mathrm{e}(Ly) = L[\Re\mathrm{e}(y)]$ and $\Re\mathrm{e}(y)$ is also a solution. Ditto $\Im\mathrm{m}(y)$. Consequently the solution can be written

$$y = \mathrm{e}^x\left[A'\cos(2x) + B'\sin(2x)\right].$$

Example 4.6

Find the solutions to the equation of Example 4.5 for which $y(0) = 1$ and $(\mathrm{d}y/\mathrm{d}x)_0 = 0$.

Solution: We obtain simultaneous equations for A' and B' by evaluating the general solution and its derivative at $x=0$:

$$\begin{aligned}1&=A'\\ 0&=A'+2B'\end{aligned}\quad\Rightarrow\quad B'=-\tfrac{1}{2}\quad\Rightarrow\quad y=\mathrm{e}^x\big[\cos(2x)-\tfrac{1}{2}\sin(2x)\big].$$

4.2.2 Factorization of operators & repeated roots

The auxiliary equation (4.21) is just the differential equation $Lf=0$ with $\mathrm{d}/\mathrm{d}x$ replaced by α. So just as the roots of a polynomial enables us to express the polynomial as a product of terms linear in the variable, so the knowledge of the roots of the auxiliary equation allows us to express L as a product of two first-order differential operators:

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha_-\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha_+\Big)f&=\frac{\mathrm{d}^2f}{\mathrm{d}x^2}-(\alpha_-+\alpha_+)\frac{\mathrm{d}f}{\mathrm{d}x}+\alpha_-\alpha_+f\\ &=\frac{\mathrm{d}^2f}{\mathrm{d}x^2}+\frac{a_1}{a_2}\frac{\mathrm{d}f}{\mathrm{d}x}+\frac{a_0}{a_2}\equiv\frac{Lf}{a_2},\end{aligned}\tag{4.24}$$

where we have used our formulae (2.25) for the sum and product of the roots of a polynomial. The CF is made up of exponentials because

$$\Big(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha_-\Big)\mathrm{e}^{\alpha_-x}=0\quad;\quad\Big(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha_+\Big)\mathrm{e}^{\alpha_+x}=0.$$

What happens if $a_1^2-4a_2a_0=0$? Then $\alpha_-=\alpha_+=\alpha$ and

$$Lf=\Big(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\Big)f.\tag{4.25}$$

It follows that

$$\begin{aligned}L\big(x\mathrm{e}^{\alpha x}\big)&=\Big(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\Big)x\mathrm{e}^{\alpha x}\\ &=\Big(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\Big)\mathrm{e}^{\alpha x}=0,\end{aligned}$$

and the CF is $y=A\mathrm{e}^{\alpha x}+Bx\mathrm{e}^{\alpha x}$.

Example 4.7

Solve

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+y=0.$$

The auxiliary equation is $(\alpha-1)^2=0$, so $y=A\mathrm{e}^x+Bx\mathrm{e}^x$.

4.2.3 Extension to higher orders

These results we have just derived generalize easily to linear equations with constant coeffs of any order.

Example 4.8

Solve

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4}-2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+y=0.$$

The auxiliary equation is $(\alpha-1)^2(\alpha-\mathrm{i})(\alpha+\mathrm{i})=0$, so

$$y=\mathrm{e}^x(A+Bx)+C\cos x+D\sin x.$$

4.3 2nd-order linear ODE's with constant coefficients: The Particular Integral

Recall that the general solution of $Lf = h$ is $\mathrm{CF} + f_0$ where the particular integral f_0 is *any* function for which $Lf_0 = h$. There is a general technique for finding PIs. This technique, which centres on **Green's functions**, lies beyond the syllabus although it is outlined in Chapter 6. For simple inhomogeneous part h we can get by with the use of trial functions. The type of function to be tried depends on the nature of h.

4.3.1 Polynomial h Suppose h is a sum of some powers of x,

$$h(x) = b_0 + b_1 x + b_2 x^2 + \cdots \tag{4.26}$$

Then we try

$$\begin{aligned} f(x) &= c_0 + c_1 x + c_2 x^2 + \cdots \\ \Rightarrow \quad f' &= c_1 + 2c_2 x + \cdots \\ f'' &= 2c_2 + \cdots \end{aligned} \tag{4.27}$$

so

$$\begin{aligned} h(x) = a_2 f'' + a_1 f' + a_0 f =&(a_0 c_0 + a_1 c_1 + a_2 2c_2 + \cdots) \\ &+ (a_0 c_1 + a_1 2c_2 + \cdots)x \\ &+ (a_0 c_2 + \cdots)x^2 \\ &+ \cdots \end{aligned} \tag{4.28}$$

Comparing powers of x^0, $x^1, \ldots$ on the two sides of this equation, we obtained coupled linear equations for the c_r in terms of the b_r. We solve these equations from the bottom up; e.g. for quadratic h

$$\begin{aligned} c_2 &= \frac{b_2}{a_0}, \\ c_1 &= \frac{b_1 - 2a_1 c_2}{a_0}, \\ c_0 &= \frac{b_0 - a_1 c_1 - 2a_2 c_2}{a_0}. \end{aligned} \tag{4.29}$$

Notice that the procedure doesn't work if $a_0 = 0$; the orders of the polynomials on left and right then inevitably disagree. This difficulty may be resolved by recognizing that the equation is then a first-order one for $g \equiv f'$ and using a trial solution for g that contains a term in x^2.

Example 4.9

Find the PI for

$$f'' + 2f' + f = 1 + 2x + 3x^2.$$

Try $f = c_0 + c_1 x + c_2 x^2$; have

$$\left.\begin{array}{lr} x^2: & c_2 = 3 \\ x^1: & 4c_2 + c_1 = 2 \\ x^0: & 2c_2 + 2c_1 + c_0 = 1 \end{array}\right\} \quad \Rightarrow \quad \begin{array}{l} c_1 = 2(1 - 2c_2) = -10 \\ c_0 = 1 - 2(c_2 + c_1) = 1 - 2(3 - 10) = 15 \end{array}$$

Check

$$\begin{aligned} f &= 15 - 10x + 3x^2, \\ 2f' &= (-10 + 6x) \times 2, \\ f'' &= 6, \\ L(f) &= 1 + 2x + 3x^2. \end{aligned}$$

4.3.2 Exponential f When $h = H\mathrm{e}^{\gamma x}$, we try $f = P\mathrm{e}^{\gamma x}$. Substituting this into the general second-order equation with constant coefficients we obtain

$$P(a_2\gamma^2 + a_1\gamma + a_0)\mathrm{e}^{\gamma x} = H\mathrm{e}^{\gamma x}. \tag{4.30}$$

Cancelling the exponentials, solving for P, and substituting the result into $f = P\mathrm{e}^{\gamma x}$, we have finally

$$\begin{aligned} f &= \frac{H\mathrm{e}^{\gamma x}}{a_2\gamma^2 + a_1\gamma + a_0} \\ &= \frac{H\mathrm{e}^{\gamma x}}{a_2(\gamma - \alpha_-)(\gamma - \alpha_+)} \quad \text{where} \quad \mathrm{CF} = A_\pm \mathrm{e}^{\alpha_\pm x}. \end{aligned} \tag{4.31}$$

Example 4.10

Find the PI for

$$f'' + 3f' + 2f = \mathrm{e}^{2x}.$$

So the PI is $f = \dfrac{\mathrm{e}^{2x}}{4+6+2} = \frac{1}{12}\mathrm{e}^{2x}$.

If h contains two or more exponentials, we find separate PIs for each of them, and then add our results to get the overall PI.

Example 4.11

Find the PI for

$$f'' + 3f' + 2f = \mathrm{e}^{2x} + 2\mathrm{e}^{x}.$$

Reasoning as above we conclude that $f_1 \equiv \frac{1}{12}\mathrm{e}^{2x}$ satisfies $f_1'' + 3f_1' + 2f_1 = \mathrm{e}^{2x}$.

and $f_2 \equiv \dfrac{2\mathrm{e}^x}{1+3+2} = \frac{1}{3}\mathrm{e}^x$ satisfies $f_2'' + 3f_2' + 2f_2 = \mathrm{e}^x$,

so $\frac{1}{12}\mathrm{e}^{2x} + \frac{1}{3}\mathrm{e}^x$ satisfies the given equation.

From equation (4.31) it is clear that we have problem when part of h is in the CF because then one of the denominators of our PI vanishes. The problem we have to address is the solution of

$$Lf = a_2\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_1\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_2\Big)f = H\mathrm{e}^{\alpha_2 x}. \tag{4.32}$$

$P\mathrm{e}^{\alpha_2 x}$ is not a useful trial function for the PI because $L\mathrm{e}^{\alpha_2 x} = 0$. Instead we try $Px\mathrm{e}^{\alpha_2 x}$. We have

$$\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_2\Big)Px\mathrm{e}^{\alpha_2 x} = P\mathrm{e}^{\alpha_2 x}, \tag{4.33}$$

and

$$L\big(Pxe^{\alpha_2 x}\big) = a_2\Big(\frac{d}{dx} - \alpha_1\Big)Pe^{\alpha_2 x} = a_2 P(\alpha_2 - \alpha_1)e^{\alpha_2 x}. \tag{4.34}$$

Hence, we can solve for P so long as $\alpha_2 \neq \alpha_1$: $P = \dfrac{H}{a_2(\alpha_2 - \alpha_1)}$.

Example 4.12

Find the PI for

$$f'' + 3f' + 2f = e^{-x}.$$

The CF is $Ae^{-2x} + Be^{-x}$, so we try $f = Pxe^{-x}$. We require

$$\begin{aligned} e^{-x} &= \Big(\frac{d}{dx} + 2\Big)\Big(\frac{d}{dx} + 1\Big)Pxe^{-x} = \Big(\frac{d}{dx} + 2\Big)Pe^{-x} \\ &= Pe^{-x}. \end{aligned}$$

Thus $P = 1$ and $f = xe^{-x}$.

What if $\alpha_1 = \alpha_2 = \alpha$ and $h = He^{\alpha x}$? Then we try $f = Px^2e^{\alpha x}$:

$$\begin{aligned} He^{\alpha x} &= a_2\Big(\frac{d}{dx} - \alpha\Big)^2 Px^2e^{\alpha x} = a_2\Big(\frac{d}{dx} - \alpha\Big)2Pxe^{\alpha x} \\ &= 2a_2Pe^{\alpha x} \quad \Rightarrow \quad P = \frac{H}{2a_2} \end{aligned}$$

4.3.3 Sinusoidal h

Suppose $h = H\cos x$, so $Lf \equiv a_2f'' + a_1f' + a_0f = H\cos x$.

Clumsy method:

$$f = A\cos x + B\sin x$$

$$\cdots\cdots$$

Elegant method: Find solutions $z(x)$ of the *complex* equation

$$Lz = He^{ix}. \tag{4.35}$$

Since L is real

$$\Re e(Lz) = L[\Re e(z)] = \Re e(He^{ix}) = H\Re e(e^{ix}) = H\cos x, \tag{4.36}$$

so the real part of our solution z will answer the given problem.

Set $z = Pe^{ix}$ (P complex)

$$Lz = (-a_2 + ia_1 + a_0)Pe^{ix} \quad \Rightarrow \quad P = \frac{H}{-a_2 + ia_1 + a_0}. \tag{4.37}$$

Finally,

$$\begin{aligned} f &= H\Re e\Big(\frac{e^{ix}}{(a_0 - a_2) + ia_1}\Big) \\ &= H\frac{(a_0 - a_2)\cos x + a_1\sin x}{(a_0 - a_2)^2 + a_1^2}. \end{aligned} \tag{4.38}$$

Note:

We shall see below that in many physical problems explicit extraction of the real part is unhelpful; more physical insight can be obtained from the first than the second of equations (4.38). But don't forget that $\Re\mathrm{e}$ operator! It's especially important to include it when evaluating the arbitrary constants in the CF by imposing initial conditions.

Example 4.13

Find the PI for

$$f'' + 3f' + 2f = \cos x.$$

We actually solve

$$z'' + 3z' + 2z = \mathrm{e}^{\mathrm{i}x}.$$

Hence

$$z = P\mathrm{e}^{\mathrm{i}x} \quad \text{where} \quad P = \frac{1}{-1 + 3\mathrm{i} + 2}.$$

Extracting the real part we have finally

$$f = \Re\mathrm{e}\Big(\frac{\mathrm{e}^{\mathrm{i}x}}{1 + 3\mathrm{i}}\Big) = \tfrac{1}{10}(\cos x + 3\sin x).$$

What do we do if $h = H\sin x$? We solve $Lz = H\mathrm{e}^{\mathrm{i}x}$ and take imaginary parts of both sides.

Example 4.14

Find the PI for

$$f'' + 3f' + 2f = \sin x.$$

Solving $z'' + 3z' + 2z = \mathrm{e}^{\mathrm{i}x}$ with $z = P\mathrm{e}^{\mathrm{i}x}$ we have

$$P = \frac{1}{1 + 3\mathrm{i}} \qquad \Rightarrow \qquad f = \Im\mathrm{m}\Big(\frac{\mathrm{e}^{\mathrm{i}x}}{1 + 3\mathrm{i}}\Big) = \tfrac{1}{10}(\sin x - 3\cos x).$$

Note:

It is often useful to express $A\cos\theta + B\sin\theta$ as $\widetilde{A}\cos(\theta + \phi)$. We do this by noting that $\cos(\theta + \phi) = \cos\phi\cos\theta - \sin\phi\sin\theta$, so

$$\begin{aligned} A\cos\theta + B\sin\theta &= \sqrt{A^2 + B^2}\Big(\frac{A}{\sqrt{A^2 + B^2}}\cos\theta + \frac{B}{\sqrt{A^2 + B^2}}\sin\theta\Big) \\ &= \sqrt{A^2 + B^2}\cos(\theta + \phi), \end{aligned}$$

where $\cos\phi = A/\sqrt{A^2 + B^2}$ and $\sin\phi = -B/\sqrt{A^2 + B^2}$.

Example 4.15

Find the PI for

$$f'' + 3f' + 2f = 3\cos x + 4\sin x.$$

The right-hand side can be rewritten $5\cos(x+\phi) = 5\Re\mathrm{e}(\mathrm{e}^{\mathrm{i}(x+\phi)})$, where $\phi = \arctan(-4/3)$. So our trial solution of the underlying complex equation is $z = P\mathrm{e}^{\mathrm{i}(x+\phi)}$. Plugging this into the equation, we find

$$P = \frac{5}{-1+3\mathrm{i}+2} = \frac{5}{1+3\mathrm{i}},$$

so the required PI is

$$f_0 = 5\Re\mathrm{e}\Big(\frac{\mathrm{e}^{\mathrm{i}(x+\phi)}}{1+3\mathrm{i}}\Big) = \tfrac{1}{2}\big[\cos(x+\phi) + 3\sin(x+\phi)\big].$$

The last three examples are rather easy because $\mathrm{e}^{\mathrm{i}x}$ does not occur in the CF (which is $A\mathrm{e}^{-x} + B\mathrm{e}^{-2x}$). What if $\mathrm{e}^{\mathrm{i}x}$ is in the CF? Then we try $z = Px\mathrm{e}^{\mathrm{i}x}$.

Example 4.16

Find the PI for

$$f'' + f = \cos x \quad \Rightarrow \quad z'' + z = \mathrm{e}^{\mathrm{i}x}$$

From the auxiliary equation we find that the equation can be written

$$\Big(\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}\Big)z = \mathrm{e}^{\mathrm{i}x}.$$

For the PI $Px\mathrm{e}^{\mathrm{i}x}$ we require

$$\mathrm{e}^{\mathrm{i}x} = \Big(\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}\Big)Px\mathrm{e}^{\mathrm{i}x} = \Big(\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}\Big)P\mathrm{e}^{\mathrm{i}x} = 2\mathrm{i}P\mathrm{e}^{\mathrm{i}x}$$

$$\Rightarrow \quad P = \frac{1}{2\mathrm{i}} \quad \Rightarrow \quad f = \Re\mathrm{e}\Big(\frac{x\mathrm{e}^{\mathrm{i}x}}{2\mathrm{i}}\Big) = \tfrac{1}{2}x\sin x$$

4.3.4 Exponentially decaying sinusoidal h Since we are handling sinusoids by expressing them in terms of exponentials, essentially nothing changes if we are confronted by a combination of an exponential and sinusoids:

Example 4.17

Find the PI for

$$f'' + f = \mathrm{e}^{-x}\big(3\cos x + 4\sin x\big).$$

The right-hand side can be rewritten $5\mathrm{e}^{-x}\cos(x+\phi) = 5\Re\mathrm{e}(\mathrm{e}^{(\mathrm{i}-1)x+\mathrm{i}\phi})$, where $\phi = \arctan(-4/3)$. So our trial solution of the underlying complex equation is $z = P\mathrm{e}^{(\mathrm{i}-1)x+\mathrm{i}\phi}$. Plugging this into the equation, we find

$$P = \frac{5}{(\mathrm{i}-1)^2+1} = \frac{5}{1-2\mathrm{i}}.$$

Finally the required PI is

$$f_0 = 5\Re\mathrm{e}\Big(\frac{\mathrm{e}^{(\mathrm{i}-1)x+\mathrm{i}\phi}}{1-2\mathrm{i}}\Big) = \mathrm{e}^{-x}\big[\cos(x+\phi) - 2\sin(x+\phi)\big].$$

4.4 Application to Oscillators

Second-order differential equations with constant coefficients arise from all sorts of physical systems in which something undergoes small oscillations about a point of equilibrium. It is hard to exaggerate the importance for physics of such systems. Obvious examples include the escapement spring of a watch, the horn of a loudspeaker and an irritating bit of trim that makes a noise at certain speeds in the car. Less familiar examples include the various fields that the vacuum sports, which include the electromagnetic field and the fields whose excitations we call electrons and quarks.

The equation of motion of a mass that oscillates in response to a periodic driving force $mF\cos\omega t$ is

$$m\ddot{x} = -\underset{\text{spring}}{m\omega_0^2 x} - \underset{\text{friction}}{m\gamma\dot{x}} + \underset{\text{forcing}}{mF\cos\omega t}. \tag{4.39}$$

Gathering the homogeneous and inhomogeneous terms onto the left- and right-hand sides, respectively, we see that the associated complex equation is

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = F\mathrm{e}^{\mathrm{i}\omega t}. \tag{4.40}$$

4.4.1 Transients The auxiliary equation of (4.40) is

$$\begin{aligned}\alpha^2 + \gamma\alpha + \omega_0^2 = 0 \quad \Rightarrow \quad \alpha &= -\tfrac{1}{2}\gamma \pm \mathrm{i}\sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2}\\ &= -\tfrac{1}{2}\gamma \pm \mathrm{i}\omega_\gamma \quad \text{where} \quad \omega_\gamma \equiv \omega_0\sqrt{1 - \tfrac{1}{4}\gamma^2/\omega_0^2}.\end{aligned}$$

Here we concentrate on the case that $\omega_0^2 - \frac{1}{4}\gamma^2 > 0$ which corresponds to the case there are oscillating solutions. Using the solutions for α we may determine the CF

$$x = \mathrm{e}^{-\gamma t/2}\big[A\cos(\omega_\gamma t) + B\sin(\omega_\gamma t)\big] = \mathrm{e}^{-\gamma t/2}\widetilde{A}\cos(\omega_\gamma t + \psi), \tag{4.41}$$

where ψ, the **phase angle**, is an arbitrary constant. Since $\gamma > 0$, we have that the CF $\to 0$ as $t \to \infty$. Consequently, the part of motion that is described by the CF is called the **transient** response.

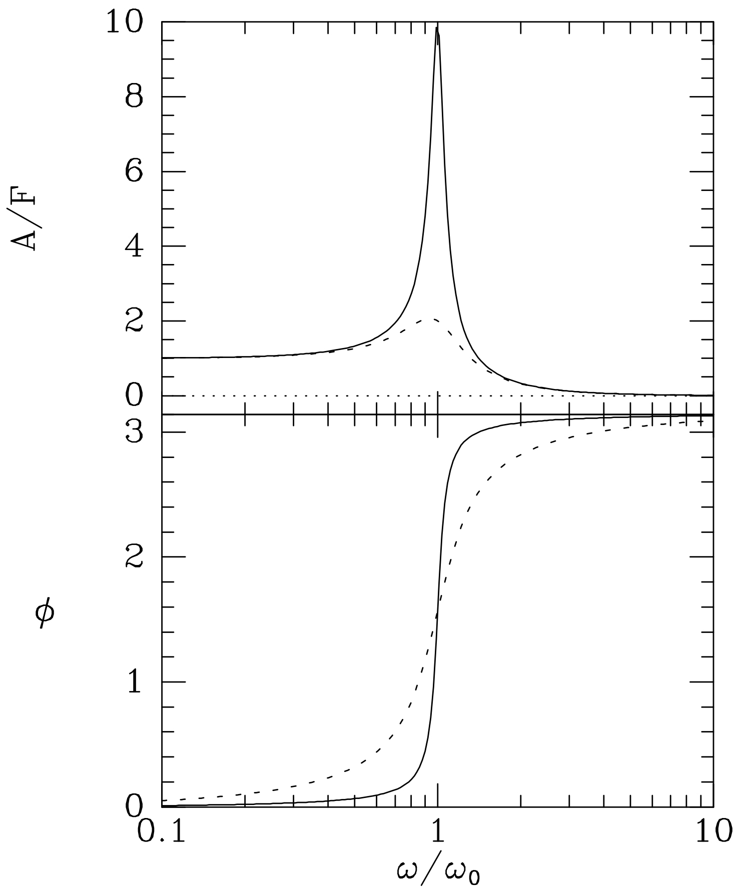
4.4.2 Steady-state solutions The PI of equation (4.40) is

$$x = \Re\mathrm{e}\Big(\frac{F\mathrm{e}^{\mathrm{i}\omega t}}{\omega_0^2 - \omega^2 + \mathrm{i}\omega\gamma}\Big). \tag{4.42}$$

The PI describes the steady-state response that remains after the transient has died away.

In (4.42) the bottom $= \sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\,\mathrm{e}^{\mathrm{i}\phi}$, where $\phi \equiv \arctan\Big(\dfrac{\omega\gamma}{\omega_0^2 - \omega^2}\Big)$, so the PI may also be written

$$x = \frac{F\Re\mathrm{e}\big(\mathrm{e}^{\mathrm{i}(\omega t - \phi)}\big)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} = \frac{F\cos(\omega t - \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}. \tag{4.43}$$



Amplitude and phase of a driven oscillator. Full lines are for $\gamma = 0.1\omega_0$, dashed lines for $\gamma = 0.5\omega_0$.

For $\phi > 0$, x achieves the same phase as F at t greater by ϕ/ω, so ϕ is called the **phase lag** of the response.

The amplitude of the response is

$$A = \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}, \tag{4.44}$$

which peaks when

$$0 = \frac{\mathrm{d}A^{-2}}{\mathrm{d}\omega} \propto -4(\omega_0^2 - \omega^2)\omega + 2\omega\gamma^2 \quad \Rightarrow \quad \omega^2 = \omega_0^2 - \tfrac{1}{2}\gamma^2. \tag{4.45}$$

$\omega_R \equiv \sqrt{\omega_0^2 - \gamma^2/2}$ is called the **resonant** frequency. Note that the frictional coefficient γ causes ω_R to be smaller than the natural frequency ω_0 of the undamped oscillator.

The figure shows that the amplitude of the steady-state response becomes very large at $\omega = \omega_R$ if γ/ω_0 is small. The figure also shows that the phase lag of the response increases from small values at $\omega < \omega_R$ to π at high frequencies. Many important physical processes, including dispersion of light in glass, depend on this often rapid change in phase with frequency.

4.4.3 Power input Power in is $W = \mathcal{F}\dot{x}$, where $\mathcal{F} = mF\cos\omega t$. Since $\Re\mathrm{e}(z_1) \times \Re\mathrm{e}(z_2) \neq \Re\mathrm{e}(z_1 z_2)$, we have to extract real bits before multiplying them together

$$\begin{aligned}
W = \mathcal{F}\dot{x} &= \Re\mathrm{e}(mF\mathrm{e}^{\mathrm{i}\omega t}) \times \frac{\Re\mathrm{e}\big(i\omega F\mathrm{e}^{\mathrm{i}(\omega t-\phi)}\big)}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} \\
&= \frac{\omega mF^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}[-\cos(\omega t)\sin(\omega t-\phi)] \\
&= -\frac{\frac{1}{2}\omega mF^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}[\sin(2\omega t-\phi)+\sin(-\phi)].
\end{aligned} \tag{4.46}$$

Averaging over an integral number of periods, the mean power is

$$\overline{W} = \frac{\frac{1}{2}\omega mF^2\sin\phi}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}. \tag{4.47}$$

4.4.4 Energy dissipated Let's check that the mean power input is equal to the rate of dissipation of energy by friction. The dissipation rate is

$$\overline{D} = m\gamma\overline{\dot{x}\dot{x}} = \frac{m\gamma\omega^2F^2\frac{1}{2}}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}. \tag{4.48}$$

It is equal to work done because $\sin\phi = \gamma\omega/\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$.

4.4.5 Quality factor Now consider the energy content of the transient motion that the CF describes. By (4.41) its energy is

$$\begin{aligned}
E &= \tfrac{1}{2}(m\dot{x}^2 + m\omega_0^2x^2) \\
&= \tfrac{1}{2}mA^2\mathrm{e}^{-\gamma t}\big[\tfrac{1}{4}\gamma^2\cos^2\eta + \omega_\gamma\gamma\cos\eta\sin\eta + \omega_\gamma^2\sin^2\eta + \omega_0^2\cos^2\eta\big] \quad (\eta \equiv \omega_\gamma t + \psi)
\end{aligned} \tag{4.49}$$

For small γ/ω_0 this becomes

$$E \simeq \tfrac{1}{2}m(\omega_0 A)^2\mathrm{e}^{-\gamma t}. \tag{4.50}$$

We define the **quality factor** Q to be

$$\begin{aligned}
Q &\equiv \frac{E(t)}{E(t-\pi/\omega_0)-E(t+\pi/\omega_0)} \simeq \frac{1}{\mathrm{e}^{\pi\gamma/\omega_0}-\mathrm{e}^{-\pi\gamma/\omega_0}} = \tfrac{1}{2}\,\mathrm{csch}(\pi\gamma/\omega_0) \\
&\simeq \frac{\omega_0}{2\pi\gamma} \quad \text{(for small } \gamma/\omega_0).
\end{aligned} \tag{4.51}$$

Q is the inverse of the fraction of the oscillator's energy that is dissipated in one period. It is approximately equal to the number of oscillations conducted before the energy decays by a factor of e.

5 Systems of linear differential equations

Many physical systems require more than one variable to quantify their configuration: for instance an electric circuit might have two connected current loops, so one needs to know what current is flowing in each loop at each moment. A set of differential equations – one for each variable – will determine the dynamics of such a system.

For example, a system of first-order differential equations in the n variables

$$y_1(x), y_2(x), \ldots, y_n(x) \tag{5.1}$$

will have the general form

$$y_1' = F_1(x, y_1, y_2, \ldots, y_n)$$

$$y_2' = F_2(x, y_1, y_2, \ldots, y_n)$$

$$\ldots$$

$$y_n' = F_n(x, y_1, y_2, \ldots, y_n) \tag{5.2}$$

for given functions F_1, …, F_n.

Observe also that in general an nth-order differential equation

$$y^{(n)} = G(x, y, y', y'', \ldots, y^{(n-1)}) \tag{5.3}$$

can be thought of as a system of n first-order equations. To see this, set new variables $y_1 = y$; $y_2 = y'$; … ; $y_n = y^{(n-1)}$. Then the system of first-order equations

$$y_1' = y_2$$

$$\ldots$$

$$y_{n-1}' = y_n$$

$$y_n' = G(x, y_1, y_2, \ldots, y_n) \tag{5.4}$$

is equivalent to the starting nth-order equation.

If we have a system of differential equations which are linear and have constant coefficients, the procedure for solving them is an extension of the procedure for solving a single linear differential equation with constant coefficients. In Subsec. 5.1 we discuss this case and illustrate the solution methods. In Subsec. 5.2 we consider applications to linear electrical circuits.

5.1 Systems of linear ODE's with constant coefficients

Systems of linear ODEs with constant coefficients can be solved by a generalization of the method seen for single ODE by writing the general solution as

$$\text{General solution} \;=\; \text{PI} \;+\; \text{CF}\,,$$

where the complementary function CF will be obtained by solving a *system of auxiliary equations*, and the particular integral PI will be obtained from a *set of trial functions* with functional form as the inhomogeneous terms.

The steps for treating a system of linear differential equations with constant coefficients are:

1. Arrange the equations so that terms on the left are all proportional to an unknown variable, and already known terms are on the right.
2. Find the general solution of the equations that are obtained by setting the right sides to zero. The result of this operation is the CF. It is usually found by replacing the unknown variables by multiples of $\mathrm{e}^{\alpha t}$ (if t is the independent variable) and solving the resulting algebraic equations.
3. Find a particular integral by putting in a trial solution for each term – polynomial, exponential, etc. – on the right hand side.

This is best illustrated by some examples.

Example 5.1

Solve

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}y}{\mathrm{d}t}+y&=t,\\ -\frac{\mathrm{d}y}{\mathrm{d}t}+3x+7y&=\mathrm{e}^{2t}-1.\end{aligned}$$

It is helpful to introduce the shorthand

$$\mathbf{L}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\dfrac{\mathrm{d}x}{\mathrm{d}t} & +\dfrac{\mathrm{d}y}{\mathrm{d}t}+y\\[2ex] 3x & -\dfrac{\mathrm{d}y}{\mathrm{d}t}+7y\end{pmatrix}.$$

To find CF

Set $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}X\mathrm{e}^{\alpha t}\\Y\mathrm{e}^{\alpha t}\end{pmatrix}$ $\quad\alpha, X, Y$ complex nos to be determined

Plug into $\mathbf{L}\begin{pmatrix}x\\y\end{pmatrix}=0$ and cancel the factor $\mathrm{e}^{\alpha t}$

$$\begin{aligned}\alpha X+(\alpha+1)Y&=0,\\ 3X+(7-\alpha)Y&=0.\end{aligned}\tag{5.5}$$

The theory of equations, to be discussed early next term, shows that these equations allow X, Y to be non-zero only if the determinant

$$\begin{vmatrix} \alpha & \alpha+1 \\ 3 & 7-\alpha \end{vmatrix} = 0,$$

which in turn implies that $\alpha(7-\alpha) - 3(\alpha+1) = 0 \quad \Rightarrow \quad \alpha = 3,\ \alpha = 1$. We can arrive at the same conclusion less quickly by using the second equation to eliminate Y from the first equation. So the bottom line is that $\alpha = 3, 1$ are the only two viable values of α. For each value of α either of equations (5.5) imposes a ratio* X/Y

$\alpha = 3 \quad \Rightarrow \quad 3X + 4Y = 0 \quad \Rightarrow \quad Y = -\frac{3}{4}X,$

$\alpha = 1 \quad \Rightarrow \quad X + 2Y = 0 \quad \Rightarrow \quad Y = -\frac{1}{2}X.$

Hence the CF made up of

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} e^{3t} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} = X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{t}.$$

The two arbitrary constants in this CF reflect the fact that the original equations were first-order in two variables.

To find PI

(i) *Polynomial part*

$$\text{Try} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X_0 + X_1 t \\ Y_0 + Y_1 t \end{pmatrix}$$

Plug into $\mathbf{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -1 \end{pmatrix}$

$$\begin{array}{cc} X_1 + Y_1 + Y_1 t + Y_0 = t & 3(X_0 + X_1 t) - Y_1 + 7(Y_0 + Y_1 t) = -1 \\ \Downarrow & \Downarrow \\ Y_1 = 1;\ X_1 + Y_1 + Y_0 = 0 & 3X_0 + 7Y_0 = 0;\ 3X_1 + 7Y_1 = 0 \\ \Downarrow & \Downarrow \\ X_1 + Y_0 = -1 & X_1 = -\frac{7}{3} \end{array}$$

Consequently, $Y_0 = -1 + \frac{7}{3} = \frac{4}{3}$ and $X_0 = -\frac{7}{3}Y_0 = -\frac{28}{9}$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{28}{9} - \frac{7}{3}t \\ \frac{4}{3} + t \end{pmatrix}$$

(ii) *Exponential part*

$$\text{Try} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{2t}$$

* The allowed values of α are precisely those for which you get the same value of X/Y from both of equations (5.5).

Plug into $\mathbf{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{e}^{2t} \end{pmatrix}$ and find

$$\begin{aligned} 2X + (2+1)Y &= 0 \quad \Rightarrow \quad X = -\tfrac{3}{2}Y \\ 3X + (-2+7)Y &= 1 \quad \Rightarrow \quad (-\tfrac{9}{2}+5)Y = 1 \end{aligned}$$

Hence $Y = 2$, $X = -3$.

Putting everything together the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} \mathrm{e}^{3t} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \mathrm{e}^{t} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} \mathrm{e}^{2t} + \begin{pmatrix} -\frac{28}{9} - \frac{7}{3}t \\ \frac{4}{3} + t \end{pmatrix} \tag{5.6}$$

We can use the arbitrary constants in the above solution to obtain a solution in which x and y or $\dot{x}$ and $\dot{y}$ take on any prescribed values at $t = 0$:

Example 5.2

For the differential equations of Example 5.1, find the solution in which

$$\begin{aligned} \dot{x}(0) &= -\tfrac{19}{3} \\ \dot{y}(0) &= 3 \end{aligned}$$

Solution: Evaluate the time derivative of the GS at $t = 0$ and set the result equal to the given data:

$$\begin{pmatrix} -\frac{19}{3} \\ 3 \end{pmatrix} = 3X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + 2\begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{7}{3} \\ 1 \end{pmatrix}$$

Hence

$$\begin{aligned} 3X_a + X_b &= 2 \\ -\tfrac{9}{4}X_a - \tfrac{1}{2}X_b &= -2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X_a &= \frac{-2}{-3/2} = \tfrac{4}{3} \\ X_b &= 2 - 3X_a = -2 \end{aligned}$$

Here's another, more complicated example.

Example 5.3

Solve

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + 2x &= 2\sin t + 3\cos t + 5\mathrm{e}^{-t} \\ \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - y &= 3\cos t - 5\sin t - \mathrm{e}^{-t} \end{aligned} \quad \text{given} \quad \begin{aligned} x(0) = 2; &\quad y(0) = -3 \\ \dot{x}(0) = 0; &\quad \dot{y}(0) = 4 \end{aligned}$$

To find CF

Set $x = X\mathrm{e}^{\alpha t}$, $y = Y\mathrm{e}^{\alpha t}$

$$\Rightarrow \quad \begin{matrix} (\alpha^2+2)X & + & \alpha Y \\ \alpha X & + & (\alpha^2-1)Y \end{matrix} = 0 \quad \Rightarrow \quad \alpha^4 = 2$$

$$\Rightarrow \quad \alpha^2 = \pm\sqrt{2} \quad \Rightarrow \quad \alpha = \pm\beta, \pm\mathrm{i}\beta \quad (\beta \equiv 2^{1/4})$$

and $Y/X = -(\alpha^2+2)/\alpha$ so the CF is

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} \beta \\ 2+\sqrt{2} \end{pmatrix} \mathrm{e}^{-\beta t} + X_b \begin{pmatrix} -\beta \\ 2+\sqrt{2} \end{pmatrix} \mathrm{e}^{\beta t} \\ + X_c \begin{pmatrix} \mathrm{i}\beta \\ 2-\sqrt{2} \end{pmatrix} \mathrm{e}^{-\mathrm{i}\beta t} + X_d \begin{pmatrix} -\mathrm{i}\beta \\ 2-\sqrt{2} \end{pmatrix} \mathrm{e}^{\mathrm{i}\beta t}$$

Notice that the functions multiplying X_c and X_d are complex conjugates of one another. So if the solution is to be real X_d has to be the complex conjugate of X_c and these two complex coefficients contain only two real arbitrary constants between them. There are four arbitrary constants in the CF because we are solving second-order equations in two dependent variables.

To Find PI

Set $(x, y) = (X, Y)\mathrm{e}^{-t} \quad \Rightarrow$

$$\begin{aligned} X - Y + 2X &= 5 \\ -X + Y - Y &= -1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X &= 1 \\ Y &= -2 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-t}$$

Have $2\sin t + 3\cos t = \Re\mathrm{e}(\sqrt{13}\mathrm{e}^{\mathrm{i}(t+\phi)})$, where $\cos\phi = 3/\sqrt{13}$, $\sin\phi = -2/\sqrt{13}$. Similarly $3\cos t - 5\sin t = \Re\mathrm{e}(\sqrt{34}\mathrm{e}^{\mathrm{i}(t+\psi)})$, where $\cos\psi = 3/\sqrt{34}$, $\sin\psi = 5/\sqrt{34}$ Set $(x, y) = \Re\mathrm{e}[(X, Y)\mathrm{e}^{\mathrm{i}t}]$ and require

$$\begin{aligned} -X + \mathrm{i}Y + 2X = X + \mathrm{i}Y &= \sqrt{13}\mathrm{e}^{\mathrm{i}\phi} \\ \mathrm{i}X - Y - Y = \mathrm{i}X - 2Y &= \sqrt{34}\mathrm{e}^{\mathrm{i}\psi} \end{aligned} \quad \Rightarrow \quad \begin{aligned} -\mathrm{i}Y &= \sqrt{13}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{i}\sqrt{34}\mathrm{e}^{\mathrm{i}\psi} \\ \mathrm{i}X &= 2\mathrm{i}\sqrt{13}\mathrm{e}^{\mathrm{i}\phi} - \sqrt{34}\mathrm{e}^{\mathrm{i}\psi} \end{aligned}$$

so

$$\begin{aligned} x &= \Re\mathrm{e}(2\sqrt{13}e^{\mathrm{i}(t+\phi)} + \mathrm{i}\sqrt{34}\mathrm{e}^{\mathrm{i}(t+\psi)}) \\ &= 2\sqrt{13}(\cos\phi\cos t - \sin\phi\sin t) - \sqrt{34}(\sin\psi\cos t + \cos\psi\sin t) \\ &= 2[3\cos t + 2\sin t] - 5\cos t - 3\sin t \\ &= \cos t + \sin t \end{aligned}$$

Similarly

$$\begin{aligned} y &= \Re\mathrm{e}(\sqrt{13}\mathrm{i}\mathrm{e}^{\mathrm{i}(t+\phi)} - \sqrt{34}\mathrm{e}^{\mathrm{i}(t+\psi)}) \\ &= \sqrt{13}(-\sin\phi\cos t - \cos\phi\sin t) - \sqrt{34}(\cos\psi\cos t - \sin\psi\sin t) \\ &= 2\cos t - 3\sin t - 3\cos t + 5\sin t \\ &= -\cos t + 2\sin t. \end{aligned}$$

Thus the complete PI is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t + \sin t \\ -\cos t + 2\sin t \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-t}.$$

For the initial-value problem

$$\mathrm{PI}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad ; \quad \dot{\mathrm{PI}}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$
$$\mathrm{CF}(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad ; \quad \dot{\mathrm{CF}}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So the PI satisfies the initial data and $X_a = X_b = X_c = X_d = 0$.

In general the number of arbitrary constants in the general solution should be the sum of the orders of the highest derivative in each variable. There are exceptions to this rule, however, as the following example shows. This example also illustrates another general point: that before solving the given equations, one should always try to simplify them by adding a multiple of one equation or its derivative to the other.

Example 5.4

Solve

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t} + y &= t, \\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 3x + 7y &= \mathrm{e}^{2t}. \end{aligned} \tag{5.7}$$

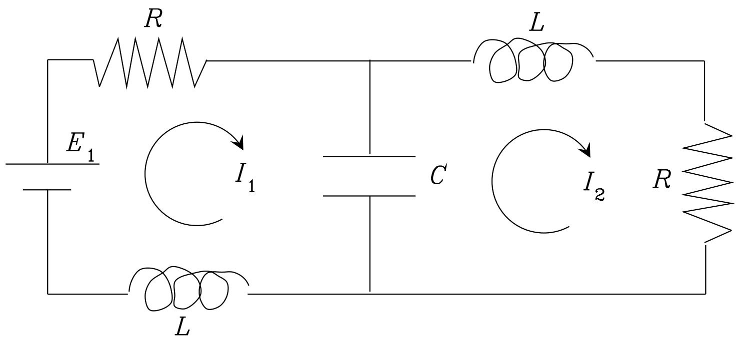
We differentiate the first equation and substract the result from the second. Then the system becomes first-order – in fact the system solved in Example 5.1. From (5.6) we see that the general solution contains only two arbitrary constants rather than the four we might have expected from a cursory glance at (5.7). To understand this phenomenon better, rewrite the equations in terms of $z \equiv x + y$ as $\dot{z} + z - x = t$ and $\ddot{z} + 7z - 4x = \mathrm{e}^{2t}$. The first equation can be used to make x a function $x(z, \dot{z}, t)$. Using this to eliminate x from the second equation we obtain an expression for $\ddot{z}(z, \dot{z}, t)$. From this expression and its derivatives w.r.t. t we can construct a Taylor series for z once we are told $z(t_0)$ and $\dot{z}(t_0)$. Hence the general solution should have just two arbitrary constants.

5.2 *LCR* circuits

The dynamics of a linear electrical circuit is governed by a system of linear equations with constant coefficients. These may be solved by the general technique described in the previous section. In many cases they may be more easily solved by judicious addition and subtraction along the lines illustrated in Example 5.4.

Example 5.5

Consider the electrical circuit pictured in the figure.



Using Kirchhoff's laws

$$\begin{aligned} RI_1 + \frac{Q}{C} + L\frac{\mathrm{d}I_1}{\mathrm{d}t} &= E_1 \\ L\frac{\mathrm{d}I_2}{\mathrm{d}t} + RI_2 - \frac{Q}{C} &= 0. \end{aligned} \tag{5.8}$$

We first differentiate to eliminate Q

$$\begin{aligned} \frac{\mathrm{d}^2I_1}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}I_1}{\mathrm{d}t} + \frac{1}{LC}(I_1 - I_2) &= 0 \\ \frac{\mathrm{d}^2I_2}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}I_2}{\mathrm{d}t} - \frac{1}{LC}(I_1 - I_2) &= 0. \end{aligned} \tag{5.9}$$

We now add the equations to obtain

$$\frac{\mathrm{d}^2S}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}S}{\mathrm{d}t} = 0 \quad \text{where} \quad S \equiv I_1 + I_2. \tag{5.10}$$

Subtracting the equations we find

$$\frac{\mathrm{d}^2D}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}D}{\mathrm{d}t} + \frac{2}{LC}D = 0 \quad \text{where} \quad D \equiv I_1 - I_2. \tag{5.11}$$

We now have two uncoupled second-order equations, one for S and one for D. We can solve each in the standard way (Section 4.2).

Example 5.6

Determine the time evolution of the LCR circuit in Example 5.5.

Solution.

The auxiliary equation for (5.10) is $\alpha^2 + R\alpha/L = 0$, and its roots are

$$\alpha = 0 \quad \Rightarrow \quad S = \text{constant} \quad \text{and} \quad \alpha = -R/L \quad \Rightarrow \quad S \propto \mathrm{e}^{-Rt/L}. \tag{5.12}$$

Since the right side of (5.10) is zero, no PI is required.

The auxiliary equation for (5.11) is

$$\alpha^2 + \frac{R}{L}\alpha + \frac{2}{LC} = 0 \quad \Rightarrow \quad \alpha = -\tfrac{1}{2}\frac{R}{L} \pm \frac{\mathrm{i}}{\sqrt{LC}}\sqrt{2 - \tfrac{1}{4}CR^2/L} = -\tfrac{1}{2}\frac{R}{L} \pm \mathrm{i}\omega_R. \quad (5.13)$$

Again no PI is required.

Adding the results of (5.12) and (5.13), the general solutions to (5.10) and (5.11) are

$$I_1 + I_2 = S = S_0 + S_1\mathrm{e}^{-Rt/L} \quad ; \quad I_1 - I_2 = D = D_0\mathrm{e}^{-Rt/2L}\sin(\omega_R t + \phi).$$

From the original equations (5.9) it is easy to see that the steady-state currents are $I_1 = I_2 = \frac{1}{2}S_0 = \frac{1}{2}E_1/R$. Hence, the final general solution is

$$\begin{aligned} I_1 + I_2 &= S(t) = K\mathrm{e}^{-Rt/L} + \frac{E_1}{R} \\ I_1 - I_2 &= D(t) = D_0\mathrm{e}^{-Rt/2L}\sin(\omega_R t + \phi). \end{aligned} \quad (5.14)$$

Example 5.7

Suppose that the battery in the LCR circuit above is first connected up at $t = 0$. Determine I_1, I_2 for $t > 0$.

Solution: We have $I_1(0) = I_2(0) = 0$ and from the diagram we see that $\dot{I}_1(0) = E_1/L$ and $\dot{I}_2 = 0$. Looking at equations (5.14) we set $K = -E_1/R$ to ensure that $I_1(0) + I_2(0) = 0$, and $\phi = 0$ to ensure that $I_1(0) = I_2(0)$. Finally we set $D_0 = \dfrac{E_1}{L\omega_R}$ to ensure that $\dot{D}(0) = \dfrac{E_1}{L}$.

6 Green Functions*

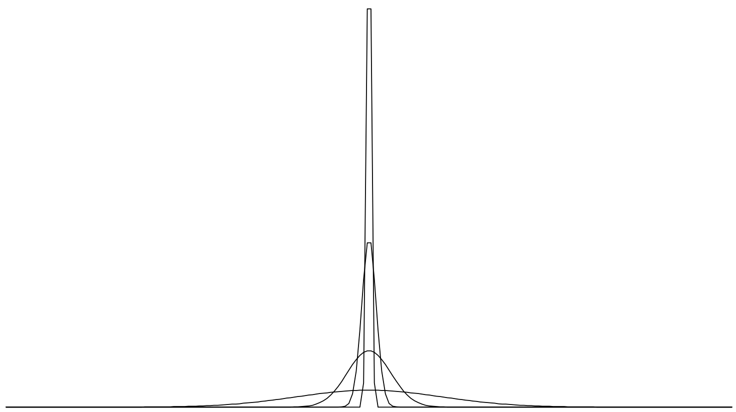
In this section we describe a powerful technique for generating particular integrals. We illustrate it by considering the general second-order linear equation

$$L_x(y) \equiv a_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = h(x). \tag{6.1}$$

On dividing through by a_2 one sees that without loss of generality we can set $a_2 = 1$.

6.1 The Dirac δ-function

Consider series of ever bumpier functions such that $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$, e.g.



Define $\delta(x)$ as limit of such functions. ($\delta(x)$ itself isn't a function really.) Then

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$$

δ's really important property is that

$$\int_a^b f(x)\delta(x - x_0)\,\mathrm{d}x = f(x_0) \quad \forall \begin{cases} a < x_0 < b \\ f(x) \end{cases}$$

Exercises (1):

(i) Prove that $\delta(ax) = \delta(x)/|a|$. If x has units of length, what dimensions has δ?

(ii) Prove that $\delta(f(x)) = \sum_{x_k} \delta(x - x_k)/|f'(x_k)|$, where the x_k are all points satisfying $f(x_k) = 0$.

* Lies beyond the syllabus

6.2 Defining the Green's function

Now suppose for each fixed x_0 we had the function $G_{x_0}(x)$ such that

$$L_x G_{x_0} = \delta(x - x_0). \tag{6.2}$$

Then we could easily obtain the desired PI:

$$y(x) \equiv \int_{-\infty}^{\infty} G_{x_0}(x) h(x_0) \, \mathrm{d}x_0. \tag{6.3}$$

y is the PI because

$$\begin{aligned} L_x(y) &= \int_{-\infty}^{\infty} L_x G_{x_0}(x) h(x_0) \, \mathrm{d}x_0 \\ &= \int \delta(x - x_0) h(x_0) \, \mathrm{d}x_0 \\ &= h(x). \end{aligned}$$

Hence, once you have the **Green's function** G_{x_0} you can easily find solutions for various h.

6.3 Finding G_{x_0}

Let $y = v_1(x)$ and $y = v_2(x)$ be two linearly independent solutions of $L_x y = 0$ – i.e. let the CF of our equation be $y = Av_1(x) + Bv_2(x)$. At $x \neq x_0$, $L_x G_{x_0} = 0$, so $G_{x_0} = A(x_0)v_1(x) + B(x_0)v_2(x)$. But in general we will have different expressions for G_{x_0} in terms of the v_i for $x < x_0$ and $x > x_0$:

$$G_{x_0} = \begin{cases} A_-(x_0)v_1(x) + B_-(x_0)v_2(x) & x < x_0 \\ A_+(x_0)v_1(x) + B_+(x_0)v_2(x) & x > x_0. \end{cases} \tag{6.4}$$

We need to choose the four functions $A_\pm(x_0)$ and $B_\pm(x_0)$. We do this by:

(i) obliging G_{x_0} to satisfy boundary conditions at $x = x_{\min} < x_0$ and $x = x_{\max} > x_0$ (e.g. $\lim_{x\to\pm\infty} G_{x_0} = 0$);

(ii) ensuring $L_x G_{x_0} = \delta(x - x_0)$.

We deal with (i) by defining $u_\pm \equiv P_\pm v_1 + Q_\pm v_2$ with $P_\pm, Q_\pm$ chosen s.t. u_- satisfies given boundary condition at $x = x_{\min}$ and u_+ satisfies condition at $x_{\max}$. Then

$$G_{x_0}(x) = \begin{cases} C_-(x_0)u_-(x) & x < x_0, \\ C_+(x_0)u_+(x) & x > x_0. \end{cases} \tag{6.5}$$

We get $C_\pm$ by integrating the differential equation from $x_0 - \epsilon$ to $x_0 + \epsilon$:

$$\begin{aligned} 1 &= \int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x - x_0) \, \mathrm{d}x = \int_{x_0-\epsilon}^{x_0+\epsilon} L_x G_{x_0} \, \mathrm{d}x \\ &= \int_{x_0-\epsilon}^{x_0+\epsilon} \left(\frac{\mathrm{d}^2 G_{x_0}}{\mathrm{d}x^2} + a_1(x) \frac{\mathrm{d}G_{x_0}}{\mathrm{d}x} + a_0(x) G_{x_0}(x) \right) \mathrm{d}x \\ &= \left[\frac{\mathrm{d}G_{x_0}}{\mathrm{d}x} + a_1(x_0) G_{x_0}(x) \right]_{x_0-\epsilon}^{x_0+\epsilon} + \int_{x_0-\epsilon}^{x_0+\epsilon} \left(a_0 - \frac{\mathrm{d}a_1}{\mathrm{d}x} \right) G_{x_0}(x) \, \mathrm{d}x. \end{aligned} \tag{6.6}$$

We assume that $G_{x_0}(x)$ is finite and continuous at x_0, so the second term in $[\ldots]$ vanishes and the remaining integral vanishes as $\epsilon \to 0$. Then we have two equations for $C_\pm$:

$$\begin{aligned} 1 &= C_+(x_0)\left.\frac{\mathrm{d}u_+}{\mathrm{d}x}\right|_{x_0} - C_-(x_0)\left.\frac{\mathrm{d}u_-}{\mathrm{d}x}\right|_{x_0} \\ 0 &= C_+(x_0)u_+(x_0) - C_-(x_0)u_-(x_0). \end{aligned} \tag{6.7}$$

Solving for $C_\pm$ we obtain

$$C_\pm(x_0) = \left.\frac{u_\mp}{\Delta}\right|_{x_0} \quad \text{where} \quad \Delta(x_0) \equiv \left(\frac{\mathrm{d}u_+}{\mathrm{d}x}u_- - u_+\frac{\mathrm{d}u_-}{\mathrm{d}x}\right)_{x_0}. \tag{6.8}$$

Substing these solutions back into (6.5) we have finally

$$G_{x_0}(x) = \begin{cases} \dfrac{u_+(x_0)u_-(x)}{\Delta(x_0)} & x < x_0 \\ \dfrac{u_-(x_0)u_+(x)}{\Delta(x_0)} & x > x_0. \end{cases} \tag{6.9}$$

Example 6.1

Solve

$$L_x = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - k^2y = h(x) \quad \text{subject to} \quad \lim_{x\to\pm\infty} y = 0.$$

The required complementary functions are $u_- = \mathrm{e}^{kx}$, $u_+ = \mathrm{e}^{-kx}$, so

$$\Delta(x_0) = -k\mathrm{e}^{-kx}\mathrm{e}^{kx} - \mathrm{e}^{-kx}k\mathrm{e}^{kx} = -2k.$$

Hence

$$\begin{aligned} G_{x_0}(x) &= -\frac{1}{2k}\begin{cases} \mathrm{e}^{-k(x_0-x)} & x < x_0 \\ \mathrm{e}^{k(x_0-x)} & x > x_0 \end{cases} \\ &= -\frac{1}{2k}\mathrm{e}^{-k|x_0-x|} \end{aligned}$$

and

$$y(x) = -\frac{1}{2k}\left[\mathrm{e}^{-kx}\int_{-\infty}^{x} \mathrm{e}^{kx_0}h(x_0)\,\mathrm{d}x_0 + \mathrm{e}^{kx}\int_{x}^{\infty} \mathrm{e}^{-kx_0}h(x_0)\right]\mathrm{d}x_0.$$

Suppose $h(x) = \cos x = \Re\mathrm{e}(\mathrm{e}^{\mathrm{i}x})$. Then

$$-2ky(x) = \Re\mathrm{e}\left(\mathrm{e}^{-kx}\left[\frac{\mathrm{e}^{x_0(i+k)}}{i+k}\right]_{-\infty}^{x} + \mathrm{e}^{kx}\left[\frac{\mathrm{e}^{x_0(i-k)}}{i-k}\right]_{x}^{\infty}\right)$$

So

$$y = -\frac{\cos x}{1+k^2}$$

as expected.