University of Oxford Trinity Term 2012

Functions of a complex variable (S1)

Problem sheet 3

I. Power series expansions; singular points

1. (a) Represent the function $f(z) = (z+1)/(z-1)$ in Taylor series about $z = 0$ and determine the region of convergence. (b) Represent f in Laurent series about $z = 0$ for $|z| > 1$.

2. Find the Laurent series expansion for the function $f(z) = 1/[(z+1)(z+3)]$
(a) in the disk $|z| < 1$; (b) in the annulus $1 < |z| < 3$; (c) in the region $|z| > 3$.
(d) Write the Laurent series for $f(z)$ about the point $z = -1$ and give its region of convergence.

3. Find the Laurent series for $f(z) = z^{-2}(1-z)^{-1}$ in the regions (a) $0 < |z| < 1$; (b) $|z| > 1$.

4. Expand the function $z/(1+z^3)$ in power series of z valid in the regions (a) $|z| < 1$; (b) $|z| > 1$.

5. Determine the first four terms of the Laurent series expansion of $f(z) = e^z/[z(z^2+1)]$ valid for $0 < |z| < 1$.

6. Determine the first four terms of the Laurent series expansion of $f(z) = (z-3)\sin[1/(z+2)]$ about the point $z = -2$, and give the region of convergence of the series.

7. Determine the first four terms of the Laurent series expansion of $f(z) = [z(z-3)]^{-2}$ about the point $z = 3$, and give the region of convergence of the series.

8. (a) Locate and classify the singular points of $f(z) = 1/[z^2(1+e^{1/z})]$. (b) Does f have a Laurent series expansion about $z = 0$?

9. Determine the behavior at $z = \infty$ for the functions (a) z^2; (b) e^{-z}; (c) e^{-1/z^2}; (d) $\tan z$.

10. For each of the following functions,

$$\text{i)}\ \frac{1}{z^2\ \sinh z}\ , \qquad \text{ii)}\ z e^{1/z^2}\ , \qquad \text{iii)}\ \frac{\cos z}{z^5}\ , \qquad \text{iv)}\ \cosh\frac{1}{z}\ ,$$

(a) obtain the Laurent series about $z = 0$ and give the region of convergence; (b) classify the singularity at $z = 0$; (c) evaluate the integral of the function round the circle $|z| = 1$.

II. Residue calculus

11. Locate and classify the singular points in the complex z plane for each of the following functions,

$$\text{(a)}\ \frac{1-z}{(1-2z)^2}\ , \qquad \text{(b)}\ e^{1/z^2}\ , \qquad \text{(c)}\ \cot z\ ,$$

and determine the residue of the function at the singularity.

12. Calculate the following contour integrals in the complex plane:

$$(a)\ \oint_{|z|=2} \frac{3z+1}{z(z-1)^3}\, dz\ , \qquad (b)\ \oint_{|z|=3/2} \frac{1-z^2}{1+z^2}\,\frac{dz}{z}\ , \qquad (c)\ \oint_{|z-1|=3/2} \frac{e^{1/z}}{z^2-1}\, dz\ .$$

13. Calculate the following real integrals

$$(a)\ \int_0^\infty \frac{x^2}{(x^2+1)\ (x^2+4)}\, dx\ , \qquad (b)\ \int_0^\infty \frac{\cos 3x}{1+x^2}\, dx\ , \qquad (c)\ \int_0^{2\pi} \frac{1}{1+8\cos^2\theta}\, d\theta$$

by complex contour integration methods.

14. Calculate the integral along the real axis

$$\int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{1+x^2}\, dx \qquad (\lambda \in \mathbb{R})$$

by complex contour integration. [Refer to Fig. 1. Discuss the contour in the complex z plane with relation to the sign of λ, based on the behavior of the exponential on the semicircular arc.]

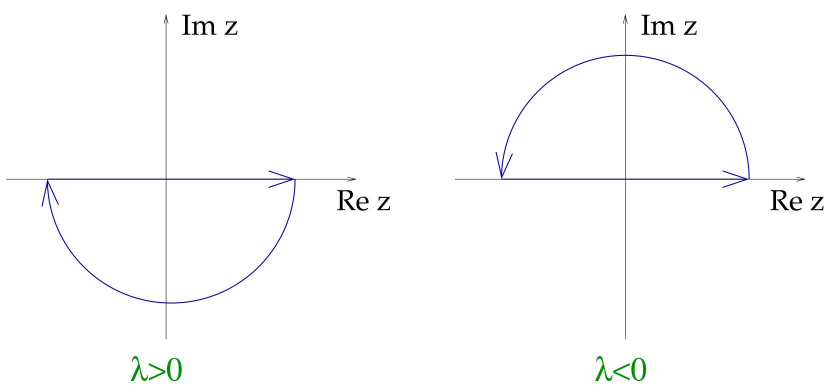

Fig.1

15. Calculate the following real integrals

$$(a)\ \int_0^{2\pi} e^{\cos\theta}\ \cos(\theta - \sin\theta)\ d\theta\ , \quad (b)\ \int_0^{2\pi} e^{\cos\theta}\ \sin(\theta - \sin\theta)\ d\theta\ ,$$

$$(c)\ \int_0^{2\pi} e^{-\cos\theta}\ \cos(\theta + \sin\theta)\ d\theta\ , \quad (d)\ \int_0^{2\pi} e^{-\cos\theta}\ \sin(\theta + \sin\theta)\ d\theta\ ,$$

by complex contour integration methods. [Suggestion. Consider the integral of $e^{1/z}$ and of $e^{-1/z}$ on the unit circle centered at the origin in the complex z plane. Evaluate these integrals by residue theorem. Relate their real and imaginary parts to the given integrals.]

16. Apply complex contour integration methods to compute

$$I = \int_{-\infty}^{+\infty} \frac{e^{x/2}}{\cosh x}\, dx\ .$$

[Suggestion. Evaluate the integral of the complex-valued function $f(z) = \exp(z/2)/\cosh z$ along the rectangular contour R in the complex z plane depicted in Fig. 2. Relate this result to the given integral I for $L \to \infty$.]

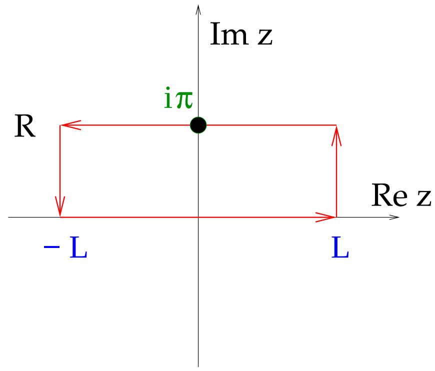

Fig.2

17. (a) Apply complex integration methods to compute the sum of the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad .$$

[Suggestion. Consider the integral of the complex-valued function $f(z) = \pi \cot(\pi z)/z^2$ along the square contour Q_N in the complex z plane depicted in Fig. 3, where N is a natural number ≥ 1. Evaluate this integral using residue theorem. Use the result to compute the sum of the given series, by examining the limit $N \to \infty$.]

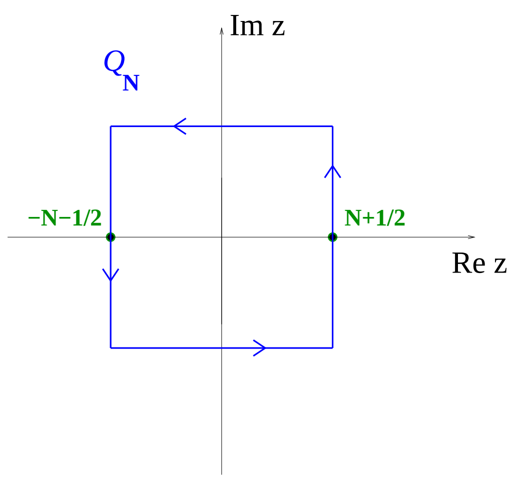

Fig.3

(b) Extend the above calculation to compute the sum of the series

$$S(a) = \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} \qquad (a \in \mathbb{R}) \quad .$$

18. (a) Take the principal branch of the logarithm function $\ln z$ and evaluate the integral

$$\oint_{\Gamma} dz \, \frac{(\ln z)^2}{z^2 + 1} \quad ,$$

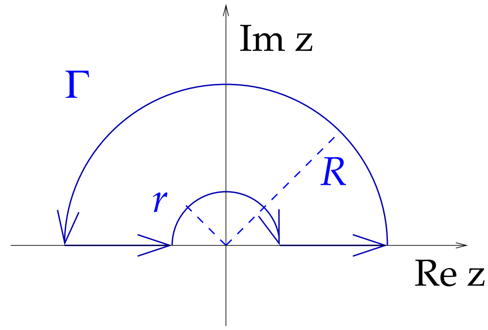

Fig.4

where Γ is the closed contour in Fig. 4, consisting of two semicircles in the upper half plane with centre at the origin and radii r and R respectively ($r < 1$, $R > 1$), and intervals $(-R, -r)$ and (r, R) on the real axis.

(b) Use the result in (a) to calculate the real-axis integrals

$$i) \int_0^{\infty} dx \; \frac{\ln x}{x^2+1} \quad , \qquad ii) \int_0^{\infty} dx \; \frac{(\ln x)^2}{x^2+1} \quad .$$

19. Take the principal branch of the function

$$f(z) = \frac{1}{\sqrt{z}}$$

defined by setting the branch cut along the negative real semiaxis. Calculate the integral in the complex plane

$$\int_{\gamma} e^z \, \frac{1}{\sqrt{z}} \, dz \; ,$$

where γ is the straight line parallel to the imaginary axis with real part equal to 1.
[Suggestion. Consider the integral round the closed contour Γ in Fig. 5. Apply Cauchy theorem to this. Let the radii of the small circle and of the large circle in Fig. 5 tend to 0 and ∞ respectively, and apply Jordan lemma. Obtain the result by evaluating the integral along the branch cut.]

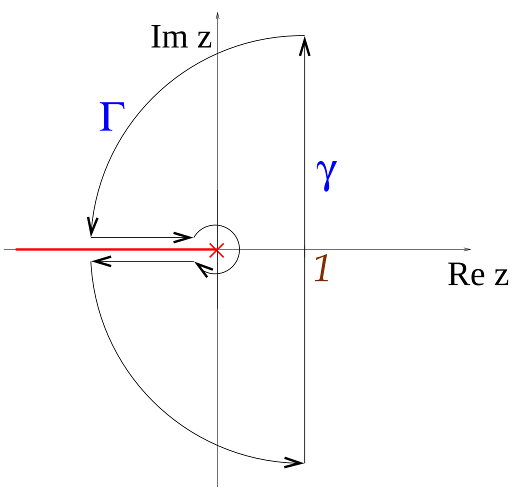

Fig.5