Functions of a complex variable (S1)

Answers for Problem Sheet 2

1. (a) 1 and ∞ are 2nd-order branch points; 1 to $+\infty$ on real axis is valid branch cut.
   (b) 3-sheeted, closed surface; three sheets R_0, R_1, R_2 joined along cut $(1, +\infty)$;
   lower edge of cut in R_2 joined back to upper edge of cut in R_0;
   images of 3 sheets are $0 \leq \arg w \leq 2\pi/3$; $2\pi/3 \leq \arg w \leq 4\pi/3$; $4\pi/3 \leq \arg w \leq 2\pi$.

2. (a) 1 and -1 are ∞-order branch points; 1 to -1 on real axis is valid branch cut.
   (b) $-i$ and ∞ are ∞-order branch points; $-i$ to $-i\infty$ on imaginary axis is valid branch cut.
   (c) 1, -1 and ∞ are ∞-order branch points; $-\infty$ to -1 and 1 to $+\infty$ on real axis is valid branch cut.

3. (a) i and $-i$ are 1st-order branch points; (b) f restored to initial value;
   (c) $z = \infty$ simple pole (no branch point); (d) The segment $-i$ to i on imaginary axis is valid branch cut. The Riemann surface is closed, made of two sheets joined along the cut; edges on opposite sides of cut from the two sheets are joined together.
   $-i\infty$ to $-i$ and i to $+i\infty$ is also a valid branch cut.

4. (b) 1 and -1 are 1st-order branch points; ∞ is ∞-order branch point;
   (c) $f(3) = \pi/2 - i\ln(3 + 2\sqrt{2})$; $f'(3) = -i/\sqrt{8}$.

5. 1, -1, 0, ∞ are 1st-order branch points; -1 to 0 and 1 to $+\infty$ on real axis is valid branch cut.

6. $f(-i) = 2^{1/3}(\sqrt{3}/2 + i/2)$, $f'(-i) = -2^{5/6}e^{-i\pi/12}/3$.

7. (a) $I = (2 + 11i)/3$ (b) $I_1 = 8/3$, $I_2 = -2 + 11i/3$
   (c) $I - I_1 - I_2 = 0$, embodying Cauchy theorem (z^2 holomorphic). $\Rightarrow$ I obtainable from primitive function $(z^3/3)|_0^{2+i}$.

8. (a) $I = -i\pi$ (b) $I' = i\pi$
   (c) $I' - I = 2\pi i \neq 0$ ($\overline{z}$ not holomorphic). On circle $|z| = 1$, $\overline{z} = 1/z \Rightarrow I' - I$ must equal $\int_{|z|=1} dz/z = 2\pi i$.

10. (a) 0 (b) $-e^{i\pi/4}\sqrt{\pi}/2$ (d) $\sqrt{\pi}/(2\sqrt{2})$, $\sqrt{\pi}/(2\sqrt{2})$

11. (a) $i\pi/4$ (b) $-i\pi/2$

12. (a) $i\pi$ (b) 0

14. $(2/\pi)\arctan(x/y)$

15. (a) 2π (b) 2π (c) 0

16. (a) 0 (b) 4π

17. (a) $-4/3$