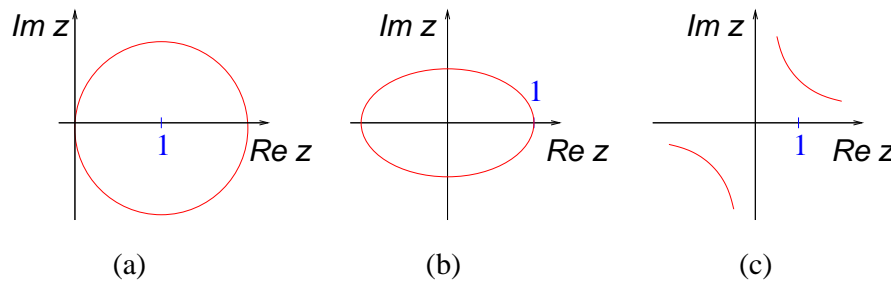


Functions of a complex variable (S1)

Answers for Problem Sheet 1

1. $1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$.
2. $2^{1/6}e^{i\pi/4}, 2^{1/6}e^{i11\pi/12}, 2^{1/6}e^{i19\pi/12}$.
3. $z = \cot[\pi(1 + 2n)/10], n = 0, 1, 2, 3, 4$.
4. (a) $\operatorname{Re} \cos i = (e + 1/e)/2, \operatorname{Im} \cos i = 0$; (b) $\operatorname{Re} \sin i = 0, \operatorname{Im} \sin i = (e - 1/e)/2$.
6. (a) $z = i\pi(2n + 1)$ (b) $z = \ln 2 + i\pi(2n + 1)$
 (c) $z = in\pi$ (d) $z = \pi(2n + 1/2) - i \ln |3 \pm 2\sqrt{2}|$ ($n \in \mathbb{Z}$)

7.



8. (a) $e^{-\pi/2-2\pi n}$ (b) $e^{-\pi/4-2\pi n}[\cos(2^{-1} \ln 2) + i \sin(2^{-1} \ln 2)]$ (c) $e^{i(1+2n)}$ ($n \in \mathbb{Z}$)
9. (a) $2^{-1} \ln 2 + 7\pi i/4$ (b) $2^{-1} \ln 2 + \pi i/4$ (c) $i\pi/2$
10. (a) $z = 0$. (b) no interior points; $z = 0$ and all points in S are boundary points.
 (c) not open; not closed; bounded; not connected; not compact.
12. (a) holomorphic in $\mathbb{C} \setminus \{z = 2n\pi i, n \in \mathbb{Z}\}$; $f'(z) = -e^z/(e^z - 1)^2$
 (b) holomorphic in $\mathbb{C} \setminus \{z = i, z = -i\}$; $f'(z) = -4z/(1 + z^2)^3$
 (c) nowhere holomorphic
 (d) holomorphic in $\mathbb{C} \setminus \{z = \pi(n + 1/2), n \in \mathbb{Z}\}$; $f'(z) = 1/\cos^2 z$
14. (a) (i) harmonic; (ii) harmonic; (iii) harmonic; (iv) not harmonic; (v) harmonic
 (b) (i) $u(x, y) = x^2 - y^2 - x$ real part of $f(z) = z^2 - z$; harmonic conjugate $v(x, y) = 2xy - y$.
 (ii) $u(x, y) = \sin x \cosh y$ real part of $f(z) = \sin z$; harmonic conjugate $v(x, y) = \cos x \sinh y$.
 (iii) $u(x, y) = e^{-x} \cos y + xy$ real part of $f(z) = e^{-z} - iz^2/2$; harmonic conjugate $v(x, y) = (y^2 - x^2)/2 - e^{-x} \sin y$.
 (v) $u(x, y) = x - y$ real part of $f(z) = z(1 + i)$; harmonic conjugate $v(x, y) = x + y$.
16. (a) $f(z) = iz e^{-z}$; (b) $e^{-x}(x \sin y - y \cos y) = \text{const}$.
17. (a) $z = \infty$ essential singularity, $z = -i$ pole (isolated); (b) $z = 1, -1, 2i, -2i$ isolated poles;
 (c) $z = 1/(n\pi), n = \pm 1, \pm 2, \dots$ isolated poles; $z = 0$ non-isolated singular point; $z = \infty$ pole.
18. (a) pole; (b) essential singularity; (c) holomorphic
19. (a) f holomorphic in the complex plane except at 0 and ∞ .
 (b) $f'(z) = 1 - 1/z^2 \Rightarrow$ conformal except at points $z = +1$ and $z = -1$.
 (c) upper and lower unit semicircles mapped onto $[-2, 2]$.
 (d) ellipse in uv plane $u^2/(\rho + 1/\rho)^2 + v^2/(\rho - 1/\rho)^2 = 1$.
20. (a) $f'(z) = -2i/(i + z)^2 \neq 0 \Rightarrow$ conformal; (b) circle $|w| = 1$; (c) disk $|w| < 1$.