Lecture 9
RESIDUE CALCULUS, PART II

Applications:

$\Diamond$ Contour integrals in the presence of branch cuts

$\Diamond$ Summation of series by residue calculus

CONTOUR INTEGRALS IN THE PRESENCE OF BRANCH CUTS

- require combining techniques for isolated singular points,

  e.g. residue theorem,

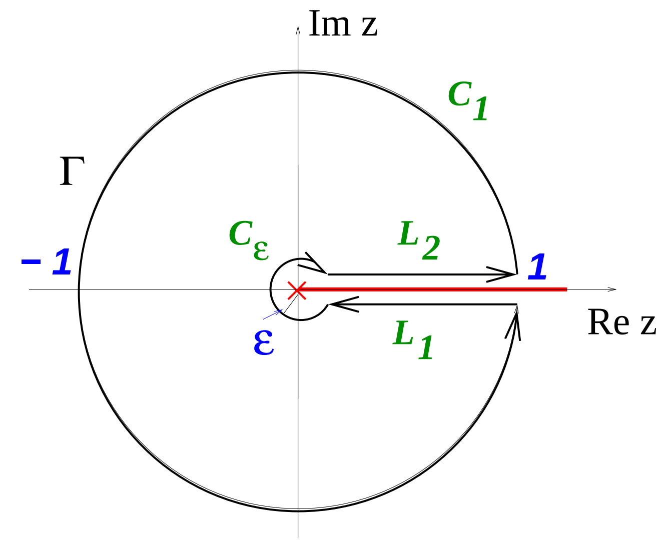
  with techniques for branch points

Integral of the square root round the unit circle

Take principal branch : $f(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2}$, $0 \le \theta < 2\pi$. Branch cut along $\mathbb{R}^+$.

- can't apply Cauchy theorem to $|z| = 1$ but can apply it to contour Γ:

$$\oint_\Gamma f(z)\,dz = 0$$



- Then write $\oint_\Gamma = \int_{C_1} + \int_{L_1} + \int_{C_\varepsilon} + \int_{L_2}$

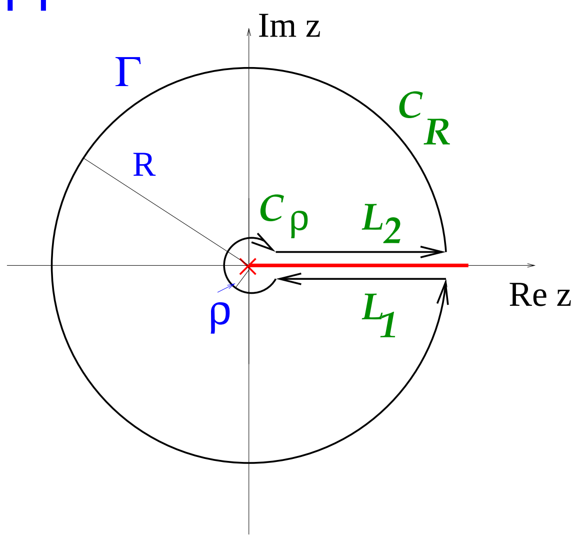
Let $\varepsilon \to 0$. By Darboux inequality $\left| \oint_{C_\varepsilon} \sqrt{z}\,dz \right| \le 2\pi\varepsilon\sqrt{\varepsilon} \overset{\varepsilon \to 0}{\longrightarrow} 0$.

Thus $\oint_{C_1} \sqrt{z}\,dz = -\int_1^0 dx\,\sqrt{x}\,e^{i2\pi/2} - \int_0^1 dx\,\sqrt{x} = -2\int_0^1 dx\,\sqrt{x} = -4/3$.

CALCULATION OF $\mathcal{I} = \int_0^\infty dx \, \frac{x^{p-1}}{x^2+1} \quad , \qquad 0 < p < 2$

Principal branch : $z^{p-1} = |z|^{p-1} e^{i(p-1)\theta}$, $0 \leq \theta < 2\pi$. Branch cut along $\mathbb{R}^+$.

- Residue theorem applied to contour Γ $\Rightarrow$



$$\Rightarrow \oint_\Gamma \frac{z^{p-1}}{z^2+1} \, dz = 2\pi i [\mathrm{Res}_{z=+i} f + \mathrm{Res}_{z=-i} f] = 2\pi i \cos(\frac{\pi p}{2}) e^{i\pi(p-1)}$$

- Next write $\oint_\Gamma = \int_{C_R} + \int_{L_1} + \int_{C_\rho} + \int_{L_2}$

Let $R \to \infty$, $\rho \to 0$. $\int_{C_R} \longrightarrow 0$ for $R \to \infty$, $\int_{C_\rho} \longrightarrow 0$ for $\rho \to 0 \Rightarrow$

$$\Rightarrow \oint_\Gamma \frac{z^{p-1}}{z^2+1}\, dz = \int_0^\infty dx\, \frac{1}{x^2+1}\left[x^{p-1} - x^{p-1}\, e^{2\pi i(p-1)}\right]$$

$$= \int_0^\infty dx\, \frac{x^{p-1}}{x^2+1}[1 - e^{2\pi i(p-1)}] = 2ie^{i\pi(p-1)}\sin(\pi p)\, \mathcal{I}$$

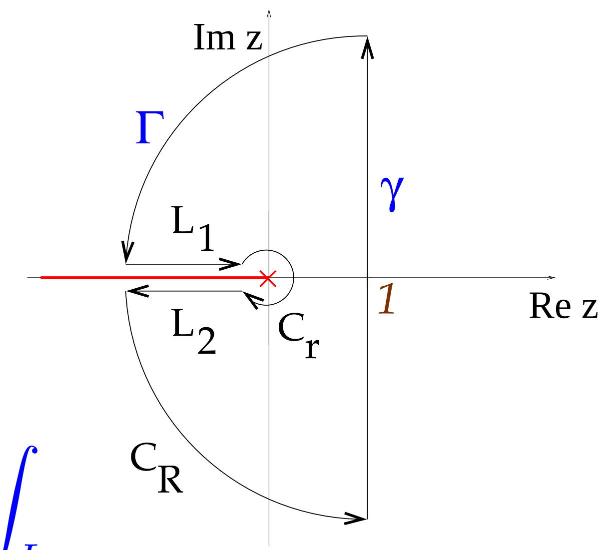
Therefore $\mathcal{I} = 2\pi i \cos(\frac{\pi p}{2}) e^{i\pi(p-1)}/[2ie^{i\pi(p-1)}\sin(\pi p)]$

$$= \pi\cos(\frac{\pi p}{2})/\sin(\pi p) = \pi/[2\sin(\pi p/2)]$$

CALCULATION OF $\mathcal{I} = \int_\gamma e^z \, \frac{1}{\sqrt{z}} \, dz$ where $\gamma = (1 - i\infty, 1 + i\infty)$

Principal branch : $f(z) = \frac{1}{\sqrt{z}} = \frac{1}{\sqrt{|z|}} e^{-i\theta/2}$, $-\pi \leq \theta < \pi$. Branch cut along $\mathbb{R}^-$.

- Apply Cauchy theorem to contour Γ:



$$\oint_\Gamma e^z \, \frac{1}{\sqrt{z}} \, dz = 0$$

- Then write $\oint_\Gamma = \int_\gamma + \int_{C_R} + \int_{L_1} + \int_{C_r} + \int_{L_2}$

Let $R \to \infty$, $r \to 0$. Apply Jordan lemma $\Rightarrow \int_{C_R} \to 0$, $\int_{C_r} \to 0$.

$$\text{Thus } \mathcal{I} = \int_\gamma e^z \frac{1}{\sqrt{z}}\, dz = \int_{-\infty}^{0} dx\ \underbrace{\left[\frac{e^x}{\sqrt{-x}\ e^{-i\pi/2}} - \frac{e^x}{\sqrt{-x}\ e^{i\pi/2}} \right]}_{\textit{discontinuity across the cut}} .$$

Change integration variable $x \to -x$. Then

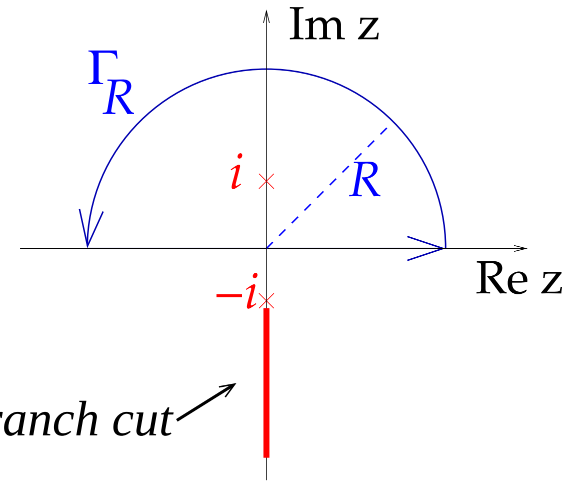
$$\mathcal{I} = \int_0^\infty dx\ \left[\frac{e^{-x}}{\sqrt{x}\ (-i)} - \frac{e^{-x}}{\sqrt{x}\ i} \right]$$

$$= -\frac{2}{i} \int_0^\infty dx\ \frac{e^{-x}}{\sqrt{x}} = 2i\sqrt{\pi}$$

CALCULATION OF $\mathcal{I} = \int_0^\infty dx \, \frac{\ln(x^2+1)}{x^2+1}$

Consider

$$\oint_{\Gamma_R} \frac{\ln(z+i)}{(z^2+1)} \, dz$$



Take principal branch of log. Branch cut

- By residue theorem $\oint_{\Gamma_R} \frac{\ln(z+i)}{z^2+1} \, dz = 2\pi i \, \mathsf{Res}_{z=i} f = 2\pi i \, \frac{\ln 2i}{2i} = \pi \left(\ln 2 + \frac{i\pi}{2} \right)$.

- By Jordan lemma $\int_{S_R} \to 0$ for $R \to \infty$.

- $\int_{-R}^{R} \frac{\ln(x+i)}{x^2+1} \, dx = \int_0^R \frac{\ln(-x+i)}{x^2+1} \, dx + \int_0^R \frac{\ln(x+i)}{x^2+1} \, dx = \int_0^R \frac{\ln(x^2+1) + i\pi}{x^2+1} \, dx$

- Equating real parts for $R \to \infty \implies \int_0^\infty dx \, \frac{\ln(x^2+1)}{x^2+1} = \pi \, \ln 2$

SUMMATION OF SERIES BY RESIDUE CALCULUS

$$\sum_{n=1}^{\infty} g(n) \quad , \qquad \sum_{n=1}^{\infty} (-1)^n \, g(n) \quad \text{with } g \text{ given function}$$

In certain cases the sum of these series can be calculated by exploiting the structure of poles and residues of complex-valued functions.

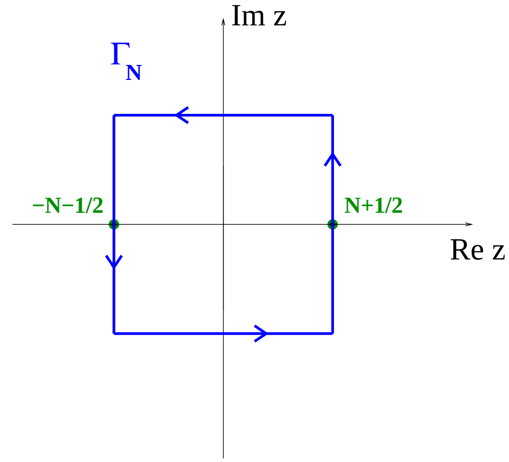
$$\sum_{n=1}^{\infty} g(n)$$

♠ $\pi\cot(\pi z)$ has poles of order 1 at $z = n$, $n \in \mathbb{Z}$, with residue 1
$\Rightarrow$ Consider $\varphi(z) = \pi\cot(\pi z) g(z)$. If g has no poles at n,

$$\text{Res}_{z=n}\varphi = \lim_{z \longrightarrow n} \pi(z-n)\cot(\pi z) g(z) = g(n)$$

$$\oint_{\Gamma_N} \varphi(z)\, dz$$

cot(z) bounded on Γ_N



On vertical sides : $|\cot\pi z| = |\cot\pi(N+1/2+iy)| = |\tan i\pi y| = |\tanh\pi y| \leq 1$

On horizontal sides : $|\cot\pi z| = |\dfrac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}| \leq \dfrac{e^{-\pi y}+e^{\pi y}}{|e^{-\pi y}-e^{\pi y}|} \leq \coth\pi(N+1/2) \leq \coth 3\pi/2$

♢ If g vanishes sufficiently fast at $N \to \infty$ for $\oint_{\Gamma_N} \to 0$, then
$\sum_n g(n) =$ - sum of residues of φ at the poles of g

$$\text{Example}: \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

- $g(z) = 1/z^2$. This has a pole at $z = 0$.

- Then $\displaystyle \sum_{n=\pm 1, \pm 2, \ldots} \frac{1}{n^2} = -\mathrm{Res}_{z=0}\, \varphi$, where $\displaystyle \varphi(z) = \frac{\pi \cot(\pi z)}{z^2}$.

- Laurent expansion :

$$\frac{\pi \cot(\pi z)}{z^2} = \frac{\pi}{z^2} \, \frac{1 - (\pi z)^2/2! + (\pi z)^4/4! + \ldots}{\pi z - (\pi z)^3/3! + (\pi z)^5/5! + \ldots}$$

$$= \frac{1}{z^3} \underbrace{- \frac{\pi^2}{3}}_{\text{residue}} \frac{1}{z} + \ldots$$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n=\pm 1, \pm 2, \ldots} \frac{1}{n^2} = -\frac{1}{2}\, \mathrm{Res}_{z=0}\, \varphi = -\frac{1}{2} \left(-\frac{\pi^2}{3} \right) = \frac{\pi^2}{6}$$

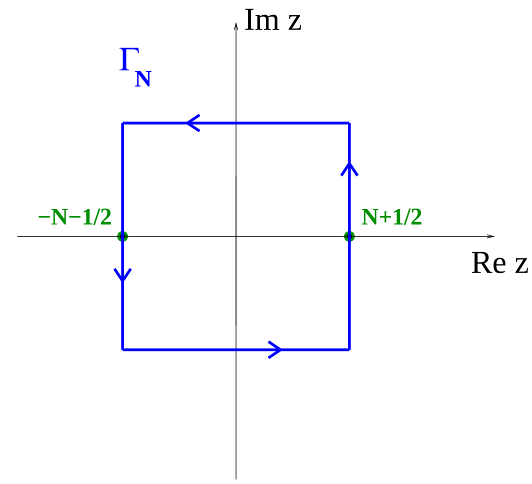
Note. The Riemann zeta function is defined as $\displaystyle \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ for Re z > 1. So $\displaystyle \zeta(2) = \frac{\pi^2}{6}$.

The ζ can be continued to any z in $\mathbb{C}$, with a pole of order 1 at $z = 1$.

Example : $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$

$$\varphi(z) = \frac{\pi \cot \pi z}{z^2 + 3}$$

$$\oint_{\Gamma_N} \varphi(z)\, dz$$



$$\text{Res}_{z=n}\varphi = 1/(n^2+3)\,, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\text{Res}_{z=i\sqrt{3}}\varphi = \pi \cot(i\pi\sqrt{3})/(2i\sqrt{3}) = -\pi \coth(\pi\sqrt{3})/(2\sqrt{3})$$

$$\text{Res}_{z=-i\sqrt{3}}\varphi = \pi \cot(-i\pi\sqrt{3})/(-2i\sqrt{3}) = -\pi \coth(\pi\sqrt{3})/(2\sqrt{3})$$

$$\text{Then } \oint_{\Gamma_N} \varphi(z)\; dz = 2\pi i \left[-\frac{\pi}{\sqrt{3}} \coth(\pi\sqrt{3}) + 2\sum_{n=1}^{N} \frac{1}{n^2+3} + \frac{1}{3} \right]$$

$$\oint_{\Gamma_N} \to 0 \ \text{ for } \ N \to \infty \ \text{ because } \ |\varphi(z)| \to 0 \ \text{ like } \ |z|^{-2}\ .$$

$$\text{Thus } \sum_{n=1}^{N} \frac{1}{n^2+3} = \frac{\pi}{2\sqrt{3}} \coth(\pi\sqrt{3}) - \frac{1}{6}\ .$$

♠ Alternating-sign series $\sum_{n=1}^{\infty}(-1)^n \, g(n)$ can be summed using

$$\varphi(z) = \frac{\pi}{\sin(\pi z)} \, g(z)$$

$$\Rightarrow \ \text{Res}_{z=n}\varphi = \lim_{z \longrightarrow n} \pi(z-n)\frac{1}{\sin(\pi z)} g(z) = (-1)^n g(n)$$

♢ Under the same hypotheses as in the previous discussion
$\sum_n (-1)^n \, g(n) =$ - sum of residues of φ at the poles of g

♠ Series of the form $\sum_{n=1}^{\infty}(-1)^n \, g(2n+1)$ can be summed using

$$\varphi(z) = \frac{\pi}{\cos(\pi z)} \, g(z)$$

- φ has poles at $z = n + 1/2$, $n \in \mathbb{Z}$

$$\text{Example :} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

- With $\varphi(z) = \dfrac{\pi}{z^2 \sin(\pi z)}$, we have $\displaystyle\sum_{n=\pm1,\pm2,\ldots} \frac{(-1)^n}{n^2} = \mathsf{Res}_{z=0}\ \varphi$.

- Laurent expansion : $\dfrac{\pi}{z^2 \sin(\pi z)} = \dfrac{\pi}{z^2} \dfrac{1}{\pi z - (\pi z)^3/3! + (\pi z)^5/5! + \ldots}$

$$= \frac{1}{z^3} + \underbrace{\frac{\pi^2}{6}}_{\text{residue}} \frac{1}{z} + \ldots$$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{1}{2} \sum_{n=\pm1,\pm2,\ldots} \frac{(-1)^n}{n^2} = -\frac{1}{2}\ \mathsf{Res}_{z=0}\ \varphi = -\frac{1}{2}\frac{\pi^2}{6} = -\frac{\pi^2}{12}$$