Functions of a Complex Variable (S1)

VI. RESIDUE CALCULUS

▷ Definition: residue of a function $f$ at point $z_0$

▷ Residue theorem

▷ Relationship between complex integration and power series expansion

▷ Techniques and applications of complex contour integration
RESIDUE CALCULUS

• Complex differentiation, complex integration and power series expansions provide three approaches to the theory of holomorphic functions.

  • Cauchy integral formulas can be seen as providing the relationship between the first two.

  • Residues serve to formulate the relationship between complex integration and power series expansions.
DEFINITION OF RESIDUE

◊ Let $f$ be holomorphic everywhere within and on a closed curve $C$ except possibly at a point $z_0$ in the interior of $C$ where $f$ may have an isolated singularity.

★ Define residue of $f$ at $z_0$:

$$\text{Res}_{z_0} f = \frac{1}{2\pi i} \oint_C f(z) \, dz$$

If $z_0$ is a non-singular point, $\text{Res}_{z_0} f = 0$. Otherwise, $\text{Res}_{z_0} f$ may be $\neq 0$. 
RESIDUE THEOREM

◊ Let $C$ be a closed path within and on which $f$ is holomorphic except for $m$ isolated singularities. Then

\[
\oint_C f(z) \, dz = 2\pi i \sum_{j=1}^{m} \text{Res}_{z_j} f
\]

▷ reformulation of Cauchy theorem via arguments similar to those used for deformation theorem
Relationship with Laurent expansion

Consider Laurent series expansion of \( f \) about \( z_0 \):

\[
f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,
\]

where \( c_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \)

\( \triangleright \) \( \text{Res}_{z_0} f \) is nothing but the coefficient of \( (z - z_0)^{-1} \)
in the Laurent expansion of \( f \) about \( z_0 \)

\[
c_{-1} = \frac{1}{2\pi i} \oint_{\Gamma} d\xi \ f(\xi) = \text{Res}_{z_0} f
\]

• Laurent expansion thus provides a general method to compute residues.
EXAMPLES

• Compute the residue at the singularity of the function

\[ f(z) = \frac{1 - z}{(1 - 2z)^2} \]

\[ z = \frac{1}{2} \quad \text{pole of order 2} \]

\[ \frac{1 - z}{(1 - 2z)^2} = \frac{1}{8} \frac{1}{(z - 1/2)^2} - \frac{1}{4} \frac{1}{z - 1/2} \quad \Rightarrow \quad \text{Res}_{z=1/2} f = -\frac{1}{4} \]

• Compute the residue at the singularity of the function

\[ f(z) = e^{1/z^2} \]

\[ z = 0 \quad \text{essential singularity} \]

\[ e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2!} \frac{1}{z^4} + \ldots \quad \Rightarrow \quad \text{Res}_{z=0} f = 0 \]
Calculation of residues in the case of poles

If \( z_0 \) is pole of order \( n \) for \( f \), then

\[
f(z) = \frac{h(z)}{(z - z_0)^n}, \quad h \text{ holomorphic and } h(z_0) \neq 0
\]

Substituting this into the definition of residue gives

\[
\text{Res}_{z_0} f = \frac{1}{2\pi i} \oint_C \frac{h(z)}{(z - z_0)^n} \, dz
\]

\[
= \frac{1}{(n-1)!} \left[ \frac{d^{n-1} h(z)}{dz^{n-1}} \right]_{z=z_0} \quad \text{by Cauchy formula}
\]

\[
= \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)]
\]

Ex. (previous slide) \( f(z) = \frac{1 - z}{(1 - 2z)^2} \)

\[
\text{Res}_{z=1/2} f = \lim_{z \to 1/2} \frac{d}{dz} [(z - 1/2)^2 f(z)] = \lim_{z \to 1/2} \frac{d}{dz} \left[ \frac{1 - z}{4} \right] = -\frac{1}{4}
\]
METHODS TO CALCULATE RESIDUES

♦ General method: from Laurent expansion

\[ f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n, \text{ where } c_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \]

\[ \text{Res}_{z_0} f = c_{-1} = \frac{1}{2\pi i} \oint_{\Gamma} d\xi \ f(\xi) \]

♦ Method for \( z_0 \) pole of order \( n \):

\[ \text{Res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n - 1)!} \frac{d^{n-1}}{dz^{n-1}}[(z - z_0)^n f(z)] \]

For \( n = 1 \): \[ \text{Res}_{z_0} f = \lim_{z \to z_0} [(z - z_0) f(z)] \]
\[ I = \oint_C \frac{\sin z}{z^6} \, dz \]

where \( C \) is the circle of centre \( z = 0 \) and radius 1

\[ z = 0 \text{ pole of order 5} \]

\[ \frac{\sin z}{z^6} = \frac{1}{z^6} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots \right) = \frac{1}{z^5} - \frac{1}{6} \frac{1}{z^3} + \frac{1}{120} \frac{1}{z} + \text{analytic part} \]

\[ \Rightarrow I = \oint_C \frac{\sin z}{z^6} \, dz = 2\pi i \text{Res}_{z=0}(\text{Integrand}) = \frac{i\pi}{60} \]

Note : \[ \oint_C \frac{\cos z}{z^6} \, dz = 0 \]

\( z = 0 \) pole of order 6 with zero residue
CALCULATION OF CONTOUR INTEGRALS BY RESIDUE THEOREM

Let $C$ be the circle of centre $z = 0$ and radius 3.

$$\oint_C dz \frac{5z - 2}{z(z - 1)} = 2\pi i \left[ \text{Res}_{z=0}(\text{Integrand}) + \text{Res}_{z=1}(\text{Integrand}) \right] = 2\pi i(2 + 3) = 10\pi i$$

$$\oint_C dz \ e^{-1/z} = 2\pi i \ \text{Res}_{z=0}(\text{Integrand}) = 2\pi i(-1) = -2\pi i$$

$$\oint_C dz \ \frac{1}{z^2 + 1} e^{\pi z/4} = 2\pi i \left[ \text{Res}_{z=+i}(\text{Integrand}) + \text{Res}_{z=-i}(\text{Integrand}) \right]$$

$$= 2\pi i \left[ \frac{e^{i\pi/4}}{2i} + \frac{e^{-i\pi/4}}{-2i} \right] = 2\pi i \sin \frac{\pi}{4} = i\pi \sqrt{2}$$
EVALUATION OF REAL INTEGRALS BY COMPLEX CONTOUR INTEGRATION METHODS

\[ I = \int_{-\infty}^{\infty} dx \frac{x^2}{(x^2 + 1)(x^2 + 4)} \]

\[ \oint_{\Gamma_R} \frac{z^2}{(z^2 + 1)(z^2 + 4)} \, dz \]

\[ = \int_{-R}^{R} dx \frac{x^2}{(x^2 + 1)(x^2 + 4)} + \int_{S_R} dz \frac{z^2}{(z^2 + 1)(z^2 + 4)} \]

• By residue theorem

\[ \oint_{\Gamma_R} = 2\pi i \left[ \text{Res}_{z=+i} f + \text{Res}_{z=+2i} f \right] = 2\pi i \left( -\frac{1}{6i} + \frac{1}{3i} \right) = \frac{\pi}{3} \cdot \]

• By Jordan lemma

\[ \int_{S_R} \to 0 \text{ for } R \to \infty \text{ because } |zf(z)| \to 0. \]

Thus \( I = \pi/3 \).
\[ I = \int_0^{2\pi} d\theta \frac{1}{2 + \cos \theta} \]

\[ z = e^{i\theta} ; \quad dz = izd\theta ; \quad \cos \theta = \frac{z + 1/z}{2} \]

So
\[ I = \oint_{C_1} dz \frac{1}{iz} \frac{1}{2 + (z + 1/z)/2} = \oint_{C_1} dz \frac{2}{i} \frac{1}{z^2 + 4z + 1} \]

\[ z_{+} = -2 \pm \sqrt{3} \]

- By residue theorem
\[ I = 2\pi i \left[ \text{Res}_{z=z_+} f \right] = \frac{2\pi}{\sqrt{3}}. \]
\[ I_1 = \int_0^{2\pi} d\theta \, e^{-\cos \theta} \cos(\theta + \sin \theta) , \quad I_2 = \int_0^{2\pi} d\theta \, e^{-\cos \theta} \sin(\theta + \sin \theta) \]

\[ z = e^{i\theta} ; \quad dz = ie^{i\theta} \, d\theta ; \quad e^{-1/z} = e^{-e^{-i\theta}} = e^{-\cos \theta + i \sin \theta} \]

So \[ \oint_{C_1} dz \, e^{-1/z} = i \int_0^{2\pi} d\theta \, e^{-\cos \theta} \, e^{i(\theta + \sin \theta)} = i(I_1 + iI_2) \]

- By residue theorem

\[ \oint_{C_1} dz \, e^{-1/z} = 2\pi i \, \text{Res}(z=0) \]

Thus \[ -2\pi i = i(I_1 + iI_2) \Rightarrow I_1 = -2\pi , \quad I_2 = 0 . \]
TECHNIQUES OF CONTOUR INTEGRATION:
Choice of integrand in the complex $z$ plane

Example: Consider $I = \int_0^\infty dx \frac{\cos x}{x^2 + 1}$.

$$\int_{S_R} dz \frac{\cos z}{z^2 + 1}$$ does not vanish on semicircle $S_R$ for $R \to \infty$.

Take instead $f(z) = \frac{e^{iz}}{z^2 + 1}$.

• By Jordan lemma $\int_{S_R} \frac{e^{iz}}{z^2 + 1} dz \to 0$ for $R \to \infty$.

• By residue theorem $\oint_{\Gamma_R} \frac{e^{iz}}{z^2 + 1} dz = 2\pi i \text{ Res}_{z=i}f = 2\pi i \frac{e^{-1}}{2i} = \frac{\pi}{e}$.

Thus $\int_{-\infty}^{+\infty} dx \frac{\cos x}{x^2 + 1} + i \int_{-\infty}^{+\infty} dx \frac{\sin x}{x^2 + 1} = \frac{\pi}{e} \implies I = \frac{\pi}{2e}$. 
TECHNIQUES OF CONTOUR INTEGRATION:
Choice of contour in the complex $z$ plane

Example: Consider $I = \int_{-\infty}^{\infty} dx \frac{e^{x/2}}{\cosh x}$.

- $f(z) = \frac{e^{z/2}}{\cosh z}$ has infinitely many poles, $z = i(\pi/2 + n\pi), \; n \in \mathbb{Z}$.

Choose contour so as to enclose only a finite number of poles:
- Rectangular contour $R$ encircles one only, $z = i\pi/2$, for any $L$.

By residue theorem
$$\oint_{R} dz \frac{e^{z/2}}{\cosh z} = 2\pi i \ \text{Res}_{z=i\pi/2}[f] = 2\pi e^{i\pi/4}$$
Let $L \to \infty$. Integrals along vertical sides vanish because

$$|\cosh(L+iy)| = |e^{L+iy} + e^{-L-iy}|/2 \geq ||e^{L+iy}| - |e^{-L-iy}||/2 = (e^L - e^{-L})/2 \geq e^L/4,$$

and so, by Darboux inequality,

$$\left| \int_0^\pi i dy \frac{e^{(L+iy)/2}}{\cosh(L+iy)} \right| \leq \int_0^\pi dy \frac{e^{L/2}}{e^L/4} = 4\pi e^{-L/2} \to 0 \text{ for } L \to \infty.$$

Similarly for the other vertical side.

Because $\cosh(x + i\pi) = -\cosh x$, integrals along horizontal sides are related by

$$\int_{-L}^L dx \frac{e^{(x+i\pi)/2}}{\cosh(x+i\pi)} = e^{i\pi/2} \int_{-L}^L dx \frac{e^{x/2}}{\cosh x}.$$

Taking $L \to \infty \implies I = \int_{-\infty}^{+\infty} \frac{e^{x/2}}{\cosh x} dx = \frac{2\pi}{1 + e^{i\pi/2}} = \frac{\pi}{\cos(\pi/4)} = \pi \sqrt{2}$. 