IV. COMPLEX INTEGRATION

Lecture 5: outline

▷ Introduction: defining integrals in complex plane

▷ Boundedness formulas

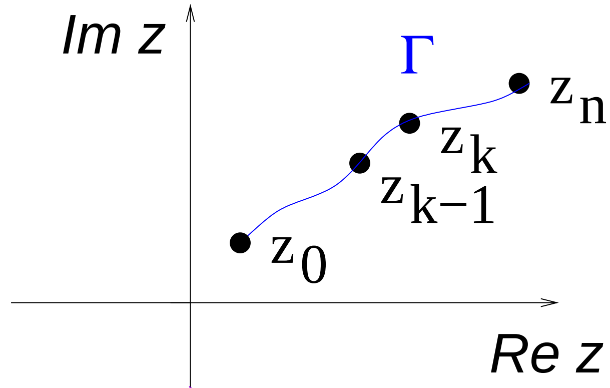
- Darboux inequality
- Jordan lemma

▷ Cauchy theorem

Corollaries:
- deformation theorem
- primitive of holomorphic f

Integral of continuous $f(z) = u + iv$ along path Γ in complex plane

$$I = \int_\Gamma f(z)\, dz$$



can be defined by

- subdividing Γ into partial arcs (z_{k-1}, z_k), $k = 1, \ldots, n$
- constructing the sum

$$S = \sum_{k=1}^{n} (z_k - z_{k-1}) f(\zeta_k) \quad , \quad \zeta_k \in (z_{k-1}, z_k)$$

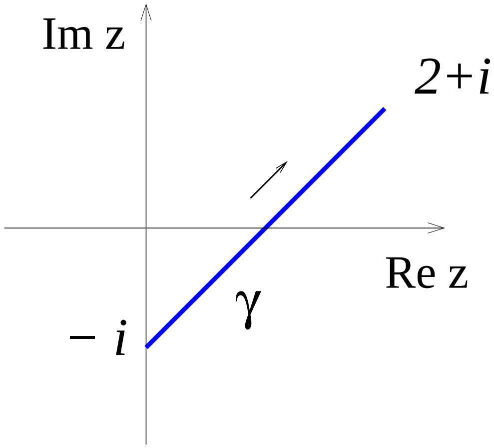
- making subdivision finer and finer and taking the limit

$$I = \int_\Gamma f(z)\; dz \stackrel{df}{=} \lim S \;\; \text{for} \;\; |z_k - z_{k-1}| \to 0 \;, n \to \infty$$

$$\Rightarrow I = \int_\Gamma f(z)\; dz = \int_\Gamma (udx - vdy) + i \int_\Gamma (vdx + udy)$$

in terms of line integrals of real $u(x, y)$, $v(x, y)$

$$\text{Example :} \quad I = \int_\gamma z^2 \, dz$$



- Let $z = x + iy$. The curve γ can be parameterized as

$$x(t) = t \quad , \qquad y(t) = t - 1 \quad , \qquad 0 \le t \le 2$$

$$\Rightarrow \; z = t + i(t-1) \quad , \qquad dz = dx + idy = (1+i)dt$$

- Therefore $I = \int_\gamma z^2 \, dz = \int_\gamma (x^2 - y^2 + 2ixy)(dx + idy)$

$$= \int_0^2 dt \; (1+i)[t^2 - (t-1)^2 + 2it(t-1)]$$

$$= (1+i)[-t + 2(1-i)t^2/2 + 2it^3/3]\Big|_0^2 = \frac{2}{3}(1+5i)$$

DARBOUX INEQUALITY

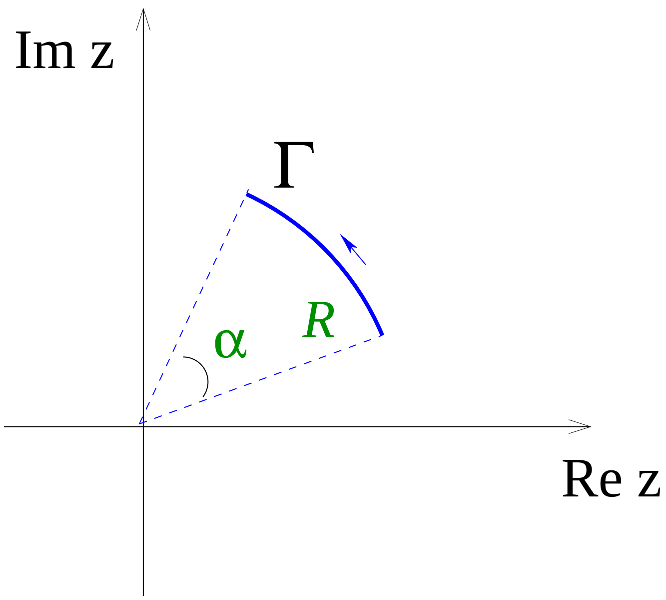
- If $|f(z)| \leq M$ on path Γ for some real constant M, then

$$\left| \int_\Gamma f(z)\, dz \right| \leq M\ L$$

$$\text{where} \quad L = \int_\Gamma |dz| \quad \text{length of } \Gamma$$

Example:

$$I = \int_\Gamma (1/z^2)\, dz$$

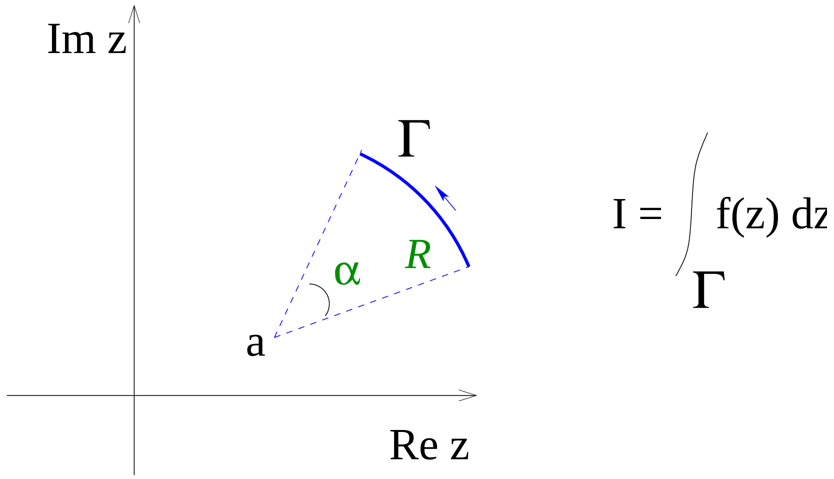


On Γ $\quad |1/z^2| = 1/R^2$

$L = R\ \alpha$

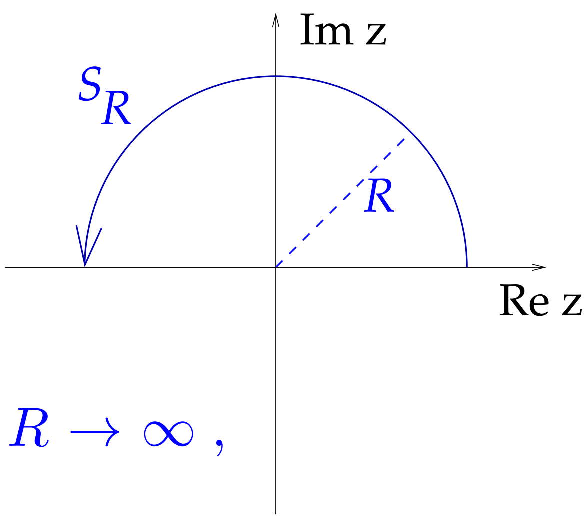
Thus: $I \leq \alpha / R$

JORDAN LEMMA



$$I \to 0 \text{ for } R \to \infty \text{ (resp. } R \to 0) \text{ if } |(z-a)|\,|f(z)| \to 0 \text{ for } R \to \infty \text{ (resp. } R \to 0)$$

$$\text{Example :} \quad I_R = \int_{S_R} \frac{1}{z^2+1}\, dz$$

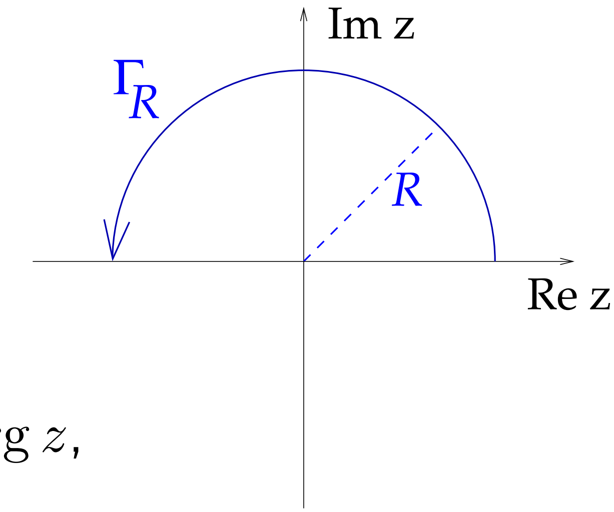


$$\text{Because } |z|\,|f(z)| = \frac{R}{|z^2+1|} \leq \frac{R}{|R^2-1|} \to 0 \text{ for } R \to \infty\ ,$$

$$\text{Jordan lemma} \Rightarrow \lim_{R \to \infty} I_R = 0\ .$$

A VARIANT OF JORDAN LEMMA

$$I_R = \int_{\Gamma_R} f(z)\, e^{i\alpha z}\, dz \quad , \quad \alpha \text{ real and positive}$$



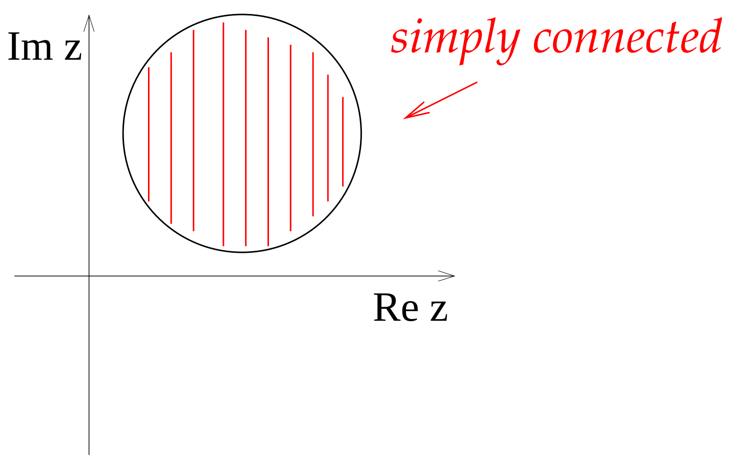
If $|f(z)| \to 0$ for $R \to \infty$ uniformly with respect to $\arg z$,

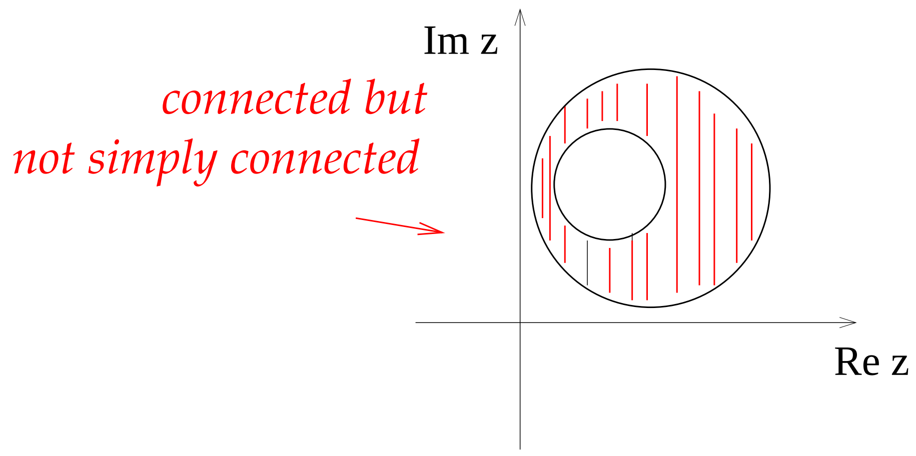
$$\text{then } I_R \to 0 \text{ for } R \to \infty\ .$$

Homotopy

Two closed paths γ and γ' in a domain D are said to be homotopic relatively to D if we can deform one into the other with continuity without getting out of D.

$\diamondsuit$ A domain D is simply connected if every closed path in D is homotopic to the null path.

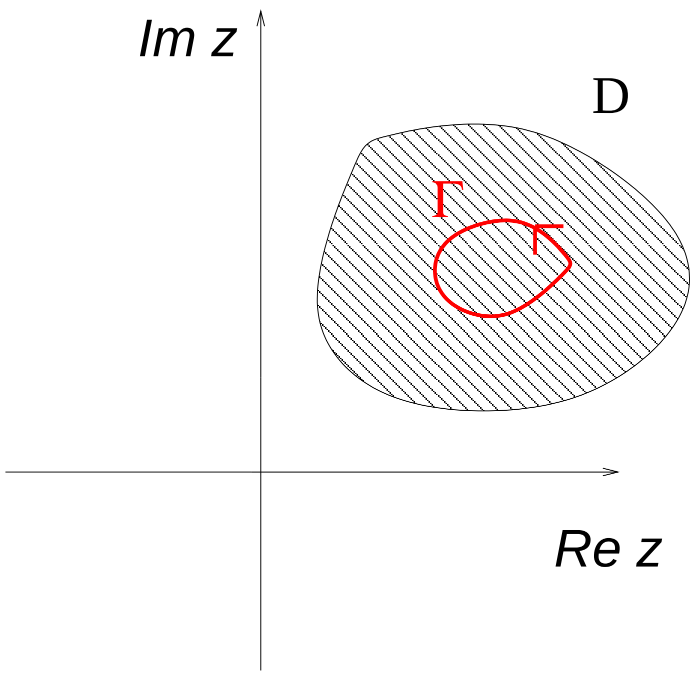




CAUCHY THEOREM

◊ Let f be holomorphic in simply connected domain D.
Let Γ be closed path in D. Then

$$\oint_{\Gamma} f(z)\, dz = 0$$

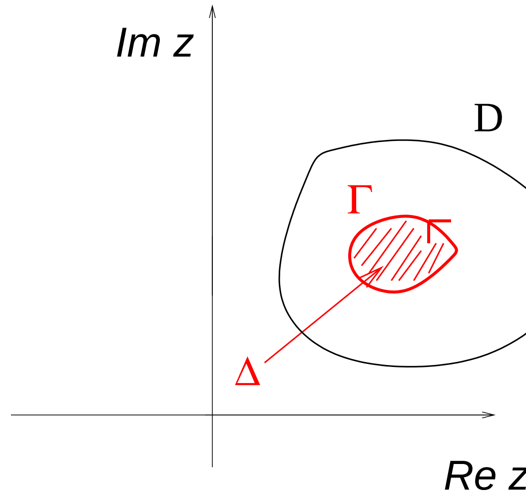


$$\int_\Gamma f(z)\, dz = \int_\Gamma (udx - vdy) + i \int_\Gamma (vdx + udy)$$

- If continuity of f' is assumed, Stokes theorem argument is sufficient

Apply Stokes theorem to the two real integrals.

$\Gamma = \partial\, \Delta$



$$\int_\Gamma (udx - vdy) = \int\int_\Delta \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right)\ dx\ dy$$

$$\int_\Gamma (vdx + udy) = \int\int_\Delta \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)\ dx\ dy$$

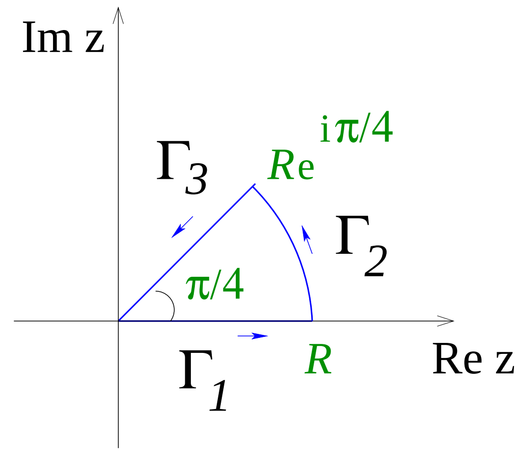
Both integrals over Δ are zero because of Cauchy-Riemann equations.

- Assuming continuous f' is however redundant:

Goursat's proof of Cauchy theorem is stronger and does not need it.

Example: Fresnel integrals $\int_0^\infty dx \ \cos x^2$, $\int_0^\infty dx \ \sin x^2$ from complex integration

- Consider $\oint_\Gamma e^{iz^2} dz$, where $\oint_\Gamma = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3}$



Cauchy theorem $\Rightarrow \oint_\Gamma e^{iz^2} dz = 0$

- Take $R \to \infty$: $\int_{\Gamma_1} e^{iz^2} dz \overset{R\to\infty}{\longrightarrow} \int_0^\infty dx \ \cos x^2 + i \int_0^\infty dx \ \sin x^2$

$$\int_{\Gamma_2} e^{iz^2} dz \overset{R\to\infty}{\longrightarrow} 0 \quad \text{by Darboux inequality}$$

$$\int_{\Gamma_3} e^{iz^2} dz \overset{R\to\infty}{\longrightarrow} \int_\infty^0 dr \ e^{i\pi/4} e^{ir^2 e^{i\pi/2}} = -e^{i\pi/4} \int_0^\infty dr \ e^{-r^2} = -e^{i\pi/4}\sqrt{\pi}/2$$

- Therefore $\int_0^\infty dx \ \cos x^2 + i \int_0^\infty dx \ \sin x^2 = e^{i\pi/4}\sqrt{\pi}/2$

$$\Rightarrow \int_0^\infty dx \ \cos x^2 = \int_0^\infty dx \ \sin x^2 = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

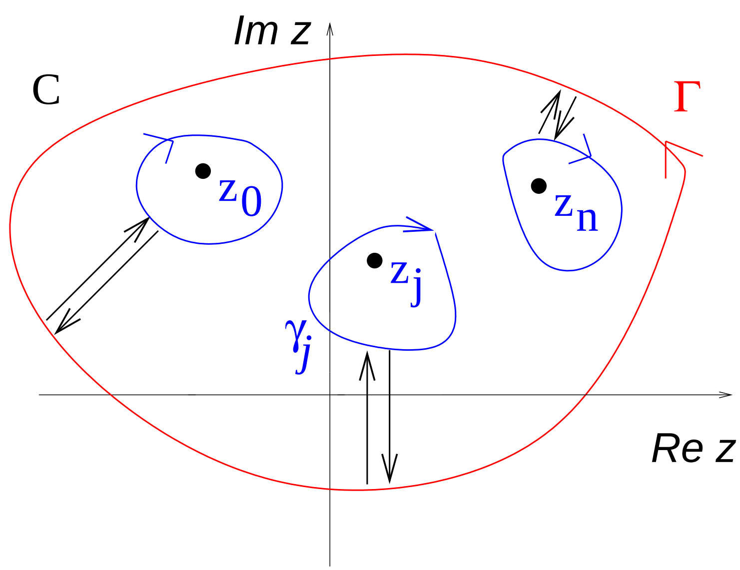
COROLLARY OF CAUCHY: DEFORMATION THEOREM

◊ Let f be holomorphic in simply connected domain D except at points $z_0, z_1, \dots, z_n$.

Let Γ be closed path in D surrounding $z_0, z_1, \dots, z_n$.

If $\gamma_0, \gamma_1, \dots, \gamma_n$ are n closed paths in D each of which encircles one z_j only, then

$$\oint_\Gamma f(z)\, dz = \sum_{j=0}^{n} \oint_{\gamma_j} f(z)\, dz$$



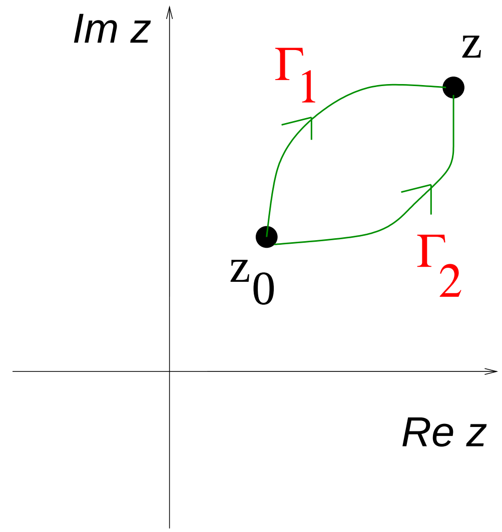
Apply Cauchy to contour C made of Γ, γ_j , and the straight line segments joining each γ to Γ. Each pair of the latter gives vanishing contribution.

PRIMITIVE OF A HOLOMORPHIC f IN A SIMPLY CONNECTED D

f holomorphic on
D simply connected domain.

$$\Rightarrow \oint_{\Gamma_1} f(z)\, dz = \oint_{\Gamma_2} f(z)\, dz$$

by Cauchy theorem

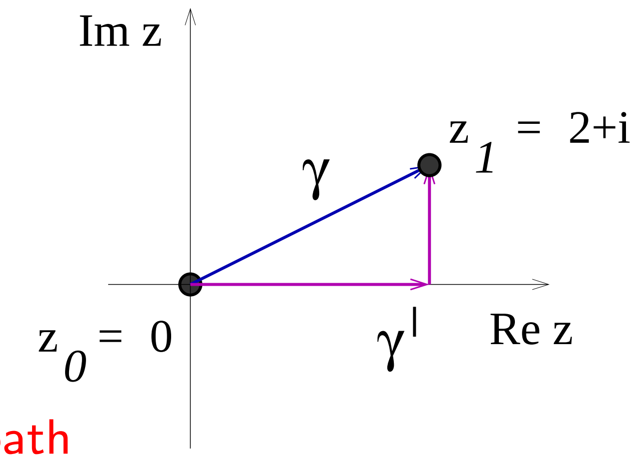


$$\Longrightarrow \; F(z) = \int_{z_0}^{z} f(z')\, dz' \qquad \text{“primitive” of } f$$

is well defined. Furthermore:

- F is holomorphic in D
- $F'(z) = f(z)$ for any z in D

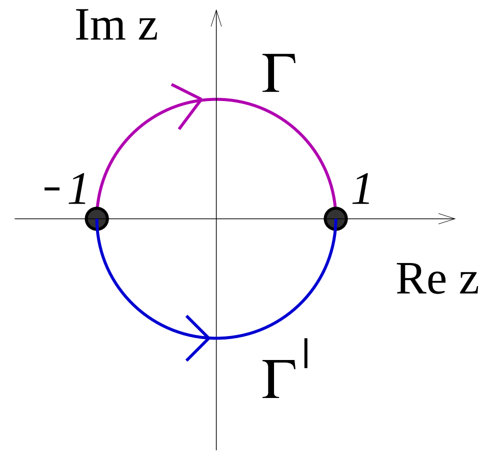
Examples



$$Ex.1: \quad \int_\gamma z^2\, dz = \int_{\gamma'} z^2\, dz$$

♢ z^2 holomorphic $\Rightarrow$ integral independent of path

• Note : $\displaystyle\int_\gamma z^2\, dz = F(z_1) - F(z_0)\ , \quad F(z) = z^3/3$ primitive of z^2



$$Ex.2: \quad \underbrace{\int_\Gamma \overline{z}\, dz}_{-i\pi} \neq \underbrace{\int_{\Gamma'} \overline{z}\, dz}_{i\pi}$$

♢ $\overline{z}$ not holomorphic : integral depends on path

• Note : $\overline{z} = 1/z$ on unit circle $C_1 \quad \Rightarrow \quad \displaystyle\oint_{C_1} \frac{1}{z}\, dz = 2\pi i$