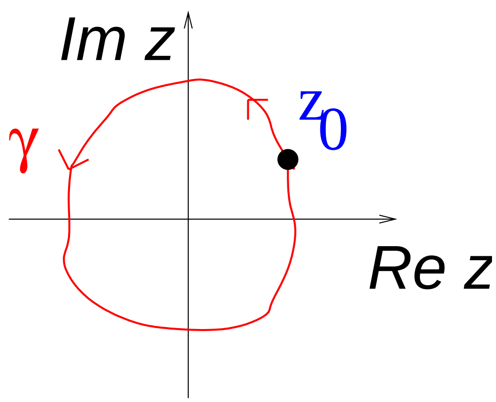
# III. MULTI-VALUED FUNCTIONS

Ex. : $\ln z = \ln |z| + i(\theta + 2\pi n)$ , $n = 0, \pm 1, \pm 2, \ldots$

- $\gamma$ closed path encircling $z = 0$. Start at $z_0$ and let $z$ vary along $\gamma$.



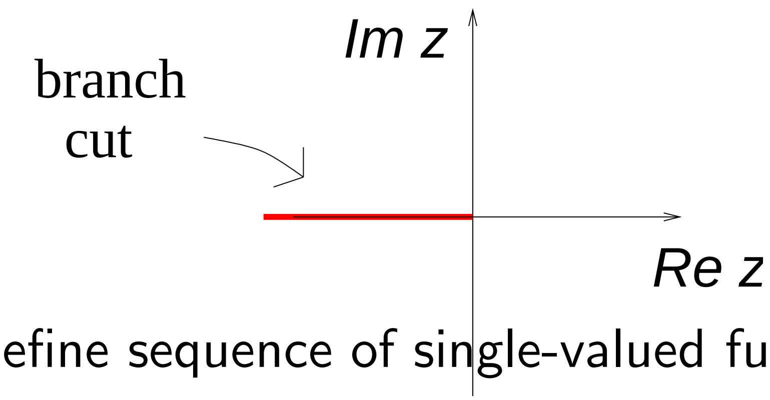

after full cycle : $(\ln z_0)_{final} = (\ln z_0)_{initial} + 2\pi i \neq (\ln z_0)_{initial}$

$\Uparrow$

$z = 0$ <u>branch point</u>

Note: $z = \infty$ is also branch point for $\ln z$.

- Imagine "cutting" the complex plane along a line joining the branch points:




- Define sequence of single-valued functions in "cut" plane:

$$f_n(z) = f_n(r, \theta) = \ln r + i(\theta + 2\pi n) \quad , \quad -\pi \le \theta \le \pi \quad (n = 0, \pm 1, \pm 2, \dots)$$

- Each is discontinuous across the cut:

$$\lim_{\varepsilon \longrightarrow 0} [f_n(r, \pi - \varepsilon) - f_n(r, -\pi + \varepsilon)] = 2\pi i$$

- On the other hand, $f_n$ above the cut $= f_{n+1}$ below the cut:

$$\lim_{\varepsilon \longrightarrow 0} f_n(r, \pi - \varepsilon) = \lim_{\varepsilon \longrightarrow 0} f_{n+1}(r, -\pi + \varepsilon)$$

- Construct Riemann surface $=$ stack of cut complex planes, joined along the cut.

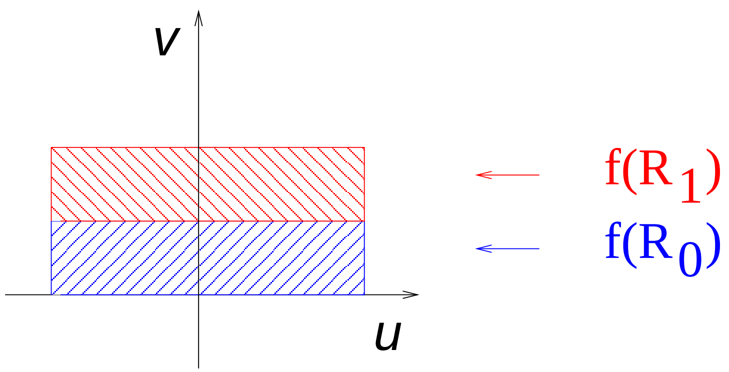
$\Rightarrow$ single-valued $f$ defined <u>on Riemann surface</u>

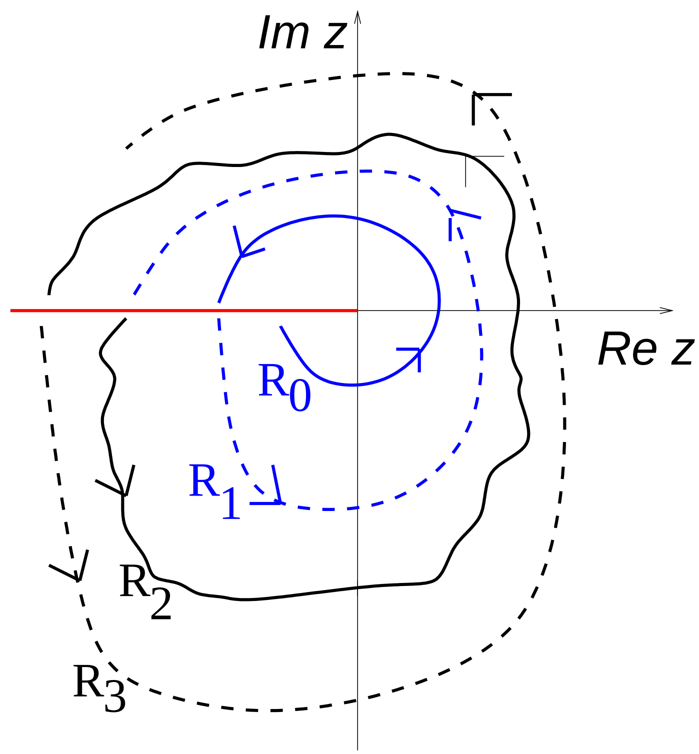
# RIEMANN SURFACE

• Helix-like superposition of "cut" planes, with upper edge of cut in $n$-th plane joined with lower edge of cut in $(n+1)$-th plane. Each plane is "Riemann sheet".

▷ $\ln z$ single-valued and holomorphic, except at branch point, on Riemann surface.

*w=ln z maps Riemann sheets $R_0$, $R_1$, ... onto horizontal strips:*




*A cycle around a branch point makes us move to another Riemann sheet.*

# Classification of branch points

$\Diamond$ branch point of order $n$: if the function is restored to the starting value by taking $n+1$ cycles around it

Ex.: $\sqrt{z}$ branch point of order 1; $\sqrt[3]{z}$ branch point of order 2

$\Diamond$ branch point of order $\infty$: if successive cycles around it bring us further and further away from initial Riemann sheet

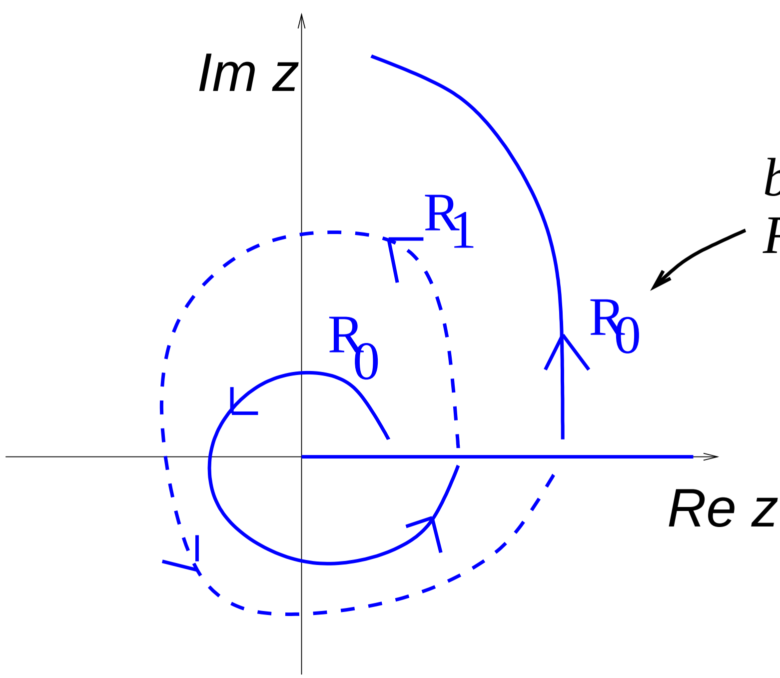
Ex.: $\ln z$ branch point of order $\infty$

## Note 3 distinct representations:

- multi-valued function on complex plane
- sequence of single-valued, discontinuous functions on "cut" plane
- one single-valued, continuous function on Riemann surface

# ROOT FUNCTIONS AND THEIR RIEMANN SURFACES

$f(z) = z^{1/2}$ : z=0, z=infinity are branch points.




*back to first Riemann sheet*

⇓

*Riemann surface is two−sheeted and closed.*

$f(R_0)$=upper half plane
$f(R_1)$=lower half plane

Note : Riemann surface for $f(z)=z^{1/n}$ is a closed, n−sheeted surface.
The n−th sheet is reconnected to the first sheet.

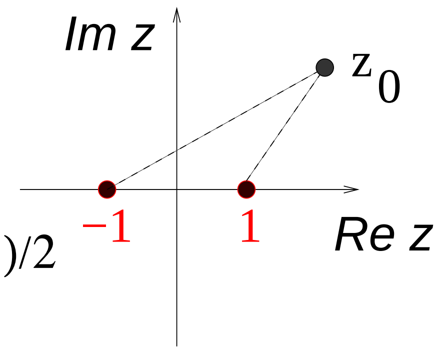
# RIEMANN SURFACE OF $f(z) = \sqrt{z^2 - 1}$

$$z-1 = r_1 e^{i\theta_1}$$

$$z+1 = r_2 e^{i\theta_2}$$

$$\Downarrow$$

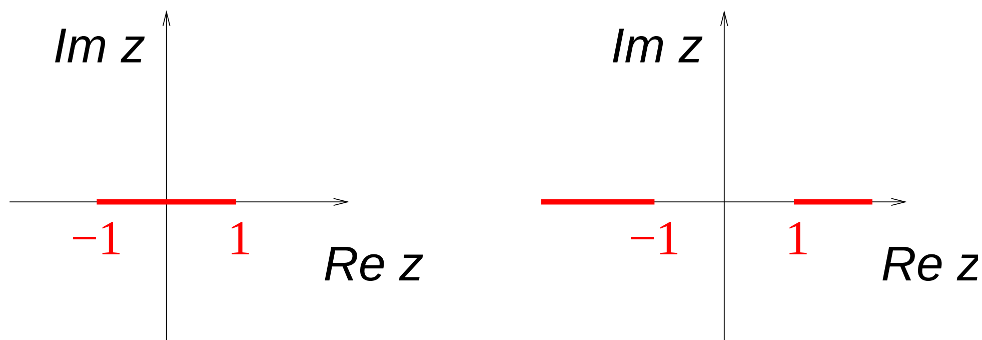
$$f(z) = (r_1 r_2)^{1/2} e^{i(\theta_1 + \theta_2)/2}$$




*z=1 and z=–1 are branch points.*

*z=infinity is not branch point.*

Valid branch cuts:

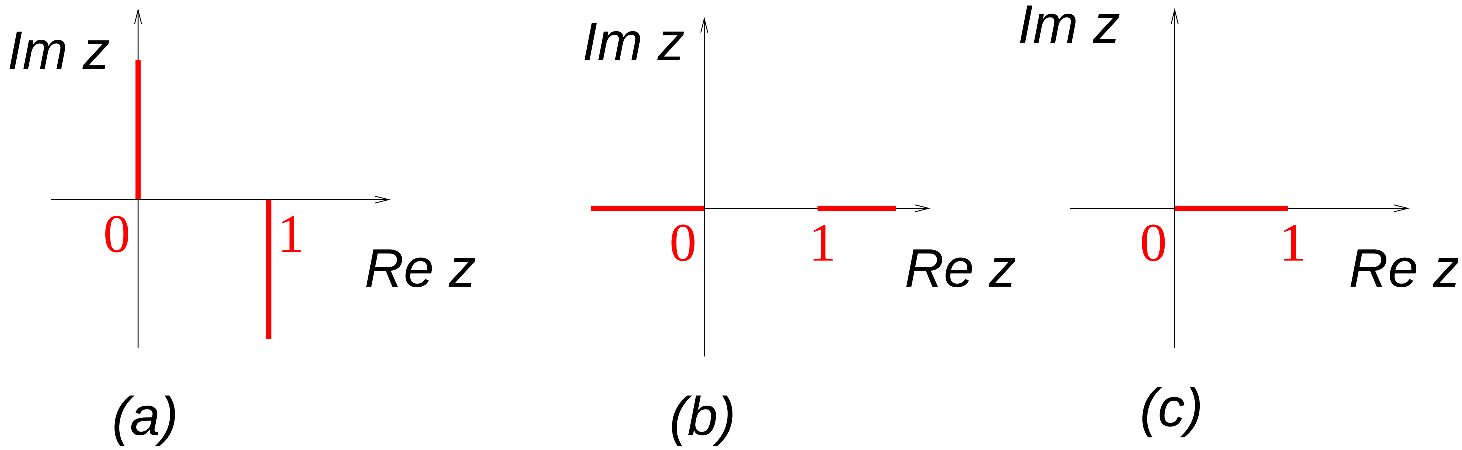



- Riemann surface is a closed, 2-sheeted surface such that any cycle surrounding one of the branch points brings us to a new sheet, whereas any cycle surrounding both branch points restores $f$ to initial value.

$$f(z) = z^{1/2}(z-1)^{1/2}$$

$z = 0$, $z = 1$ are 1st-order branch points

Valid branch cuts:




$$\text{Note} : f(z) = z^{1/3}(z-1)^{1/3}$$

has 2nd-order branch points at $z = 0$, $z = 1$, $z = \infty$;

cuts *(a)* and *(b)* above are valid cuts for this function as well, while *(c)* is not.

## NOTE

• In practical applications, you may be able to tell on which Riemann sheet you are sitting by knowing the value of the function at one point.

Example

Suppose that a branch of the function

$$f(z) = (z-1)^{2/3}$$

is defined by means of the branch cut in Fig. 2 and that it takes the value 1 when $z=0$. Determine the value of $f(z)$ and of its derivative $f'(z)$ at the point $z=-i$.

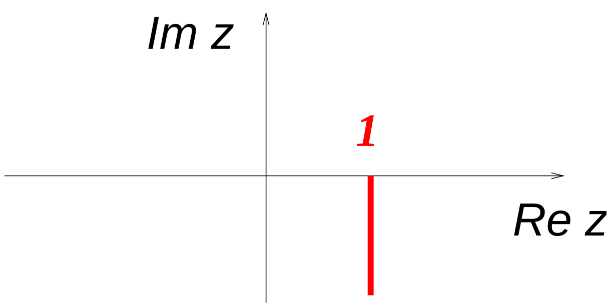
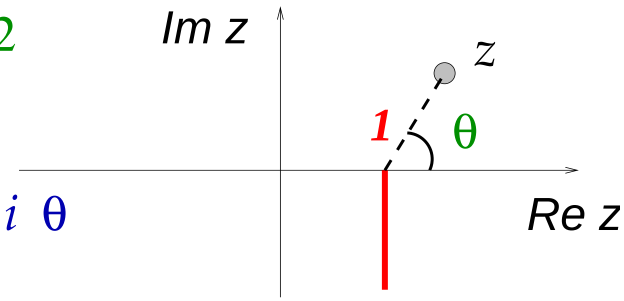



Fig. 2

$$-\pi/2 \leq \theta < 3\pi/2$$

$$z - 1 = |z - 1|\, e^{i\theta}$$




$$f(z) = (z-1)^{2/3} = |z-1|^{2/3} e^{i2\theta/3} e^{i4\pi n/3} \quad , \qquad n = 0, 1, 2$$

$$f(z=0) = 1 \implies f(z=0) = e^{i\pi 2/3} e^{i4\pi n/3} = 1 \implies n = 1$$

$$\text{Then} \qquad f(z=-i) = |-i-1|^{2/3} e^{i(5\pi/4)2/3} e^{i4\pi/3} = 2^{1/3}\left(\sqrt{3}/2 + i/2\right)$$

$$f'(z) = \frac{2}{3}\,(z-1)^{-1/3} \implies f'(z=-i) = \frac{2}{3}\,|-i-1|^{-1/3} e^{-i(5\pi/4)/3} e^{-i2\pi/3}$$

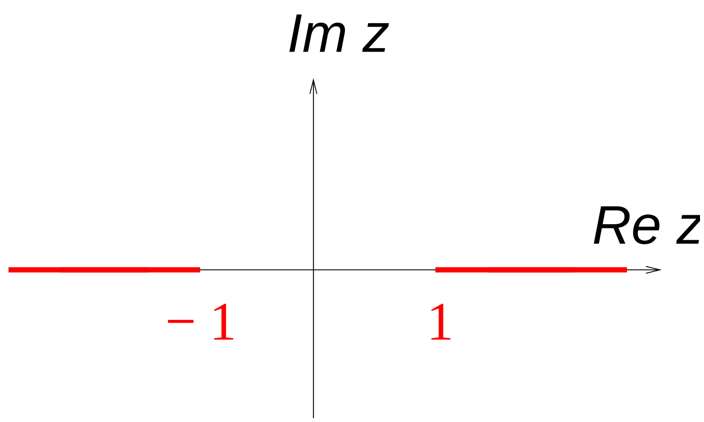
$$= -2^{5/6}\,(\cos\pi/12 - i\sin\pi/12)\,/\,3$$

## INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

$$\arccos z = \frac{1}{i} \ \ln(z + \sqrt{z^2 - 1})$$

has branch points of order 1 at $z = 1$, $z = -1$ and
branch point of order $\infty$ at $z = \infty$.
Valid cuts are semi-infinite lines stretching from $\pm 1$ out to $\infty$.

For example:




• arccos $z$ has two infinite sets of values for each value of $z \neq \pm 1$:
two possible values of square root and, for each, infinitely many values of log.

♢ Similarly for the other trigonometric and hyperbolic inverse functions.

# Summary

◊ Multi-valued functions can be characterized through their

*branch points*

by specifying a valid set of

*branch cuts*.

order $n$ — $f$ restored to starting value by taking $n+1$ cycles around it

order $\infty$ — $f$ never restored to starting value by successive cycles

◊ *Riemann surfaces* provide a setting for defining complex functions — more natural than complex plane itself