Functions of a complex variable (S1)
Lecture 3

CAUCHY-RIEMANN EQUATIONS

$f = u + iv$ holomorphic $\Leftrightarrow$ u, v contin. differentiable and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ , \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\text{i.e.} \qquad \overline{\partial} f = 0 \qquad [\overline{\partial} \equiv \partial / \partial \overline{z} = (\partial / \partial x + i \partial / \partial y)/2]$$

- f holomorphic $\Rightarrow$ f continuous
- f, g holomorphic $\Rightarrow$ $c_1 f + c_2 g$, fg, $f \circ g$ holomorphic

- For holomorphic f, f real-valued $\Rightarrow$ f constant
- For holomorphic f, $|f|$ constant $\Rightarrow$ f constant

Harmonic functions

$f = u + iv$ holomorphic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

▷ Take derivative of 1st equation wrt x, derivative of 2nd equation wrt y, and subtract:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

▷ Similarly:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Thus: $f = u + iv$ holomorphic $\Rightarrow$ u, v harmonic:

$$\Delta u = 0 \;, \qquad \Delta v = 0$$

♣ A converse applies in the sense that given a harmonic function u, I can use Cauchy-Riemann to find another harmonic function v such that $u + iv$ is holomorphic.

↖ harmonic conjugate

Example

Given $u(x, y) = xy$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = y \Rightarrow v(x, y) = \frac{y^2}{2} + c(x)$$

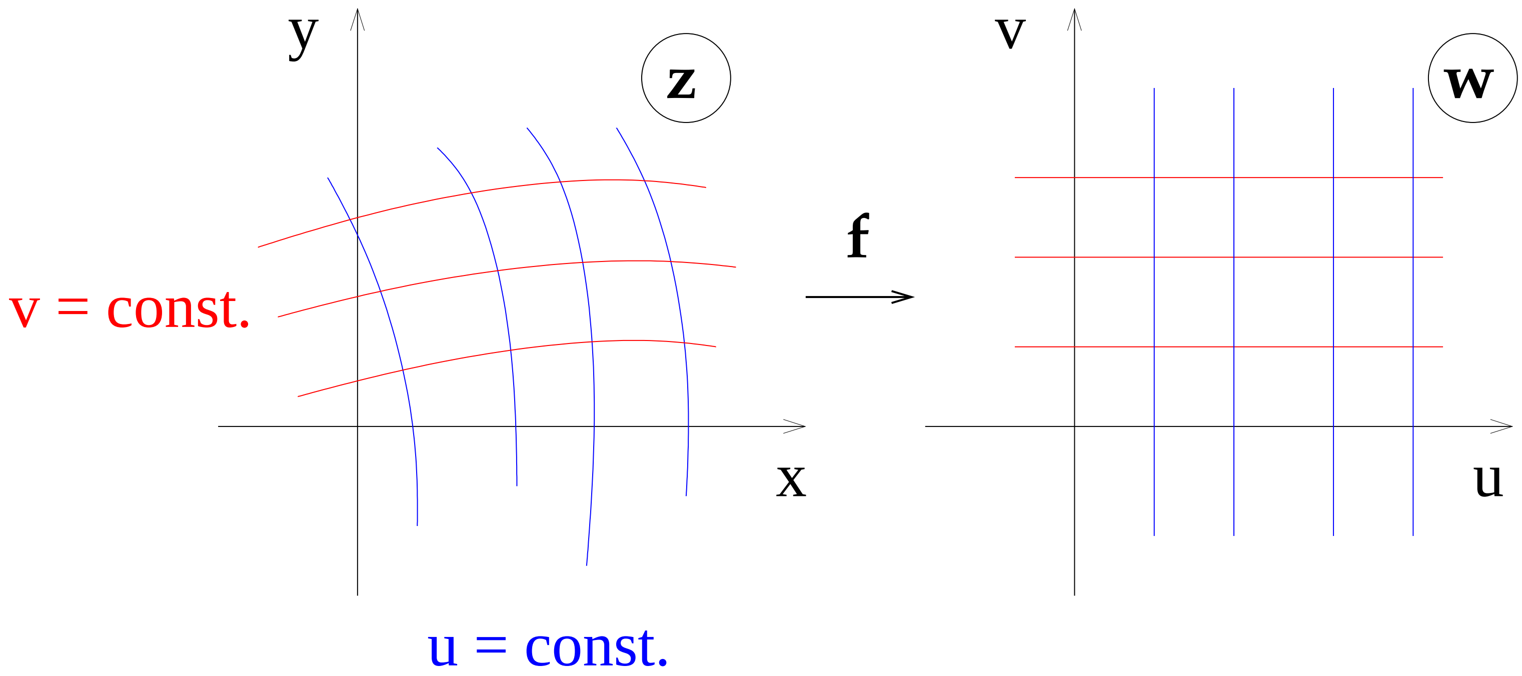
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -x \Rightarrow v(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \text{ const.} \qquad \text{harmonic conjugate}$$

$$\Rightarrow f(z) = u + iv = xy + i\left(\frac{y^2}{2} - \frac{x^2}{2} + \text{ const.}\right) = -\frac{i}{2}\, z^2 + \text{ const.} \quad \text{holomorphic}$$

CONFORMAL MAPPING

• For every point z where f is holomorphic and $f' \neq 0$, the mapping $z \mapsto w = f(z)$ is conformal, i.e., it preserves angles.

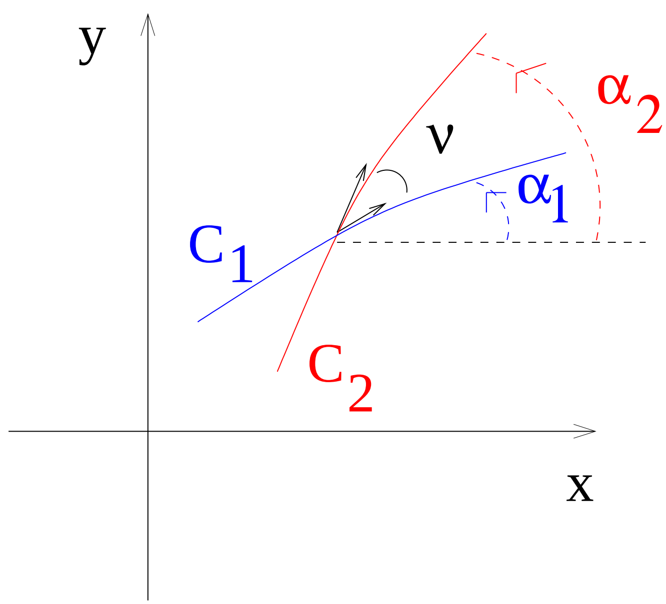
♢ A particular case of this:



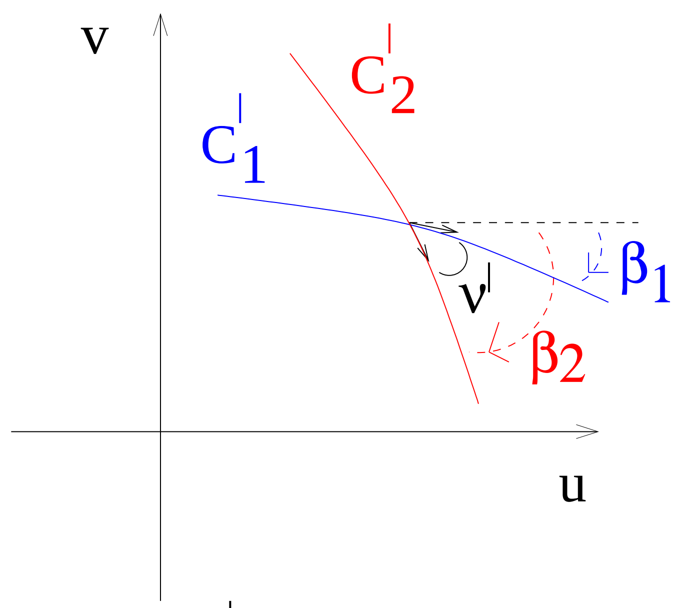
CONFORMALITY = CONSERVATION OF ANGLES

◇ If C_1 and C_2 are two curves in the z plane intersecting at z_0 with angle ν, f holomorphic with $f' \neq 0$ $\Rightarrow$ the images through f, C_1' and C_2', of the curves C_1 and C_2 intersect with angle

$$\nu' = \nu$$

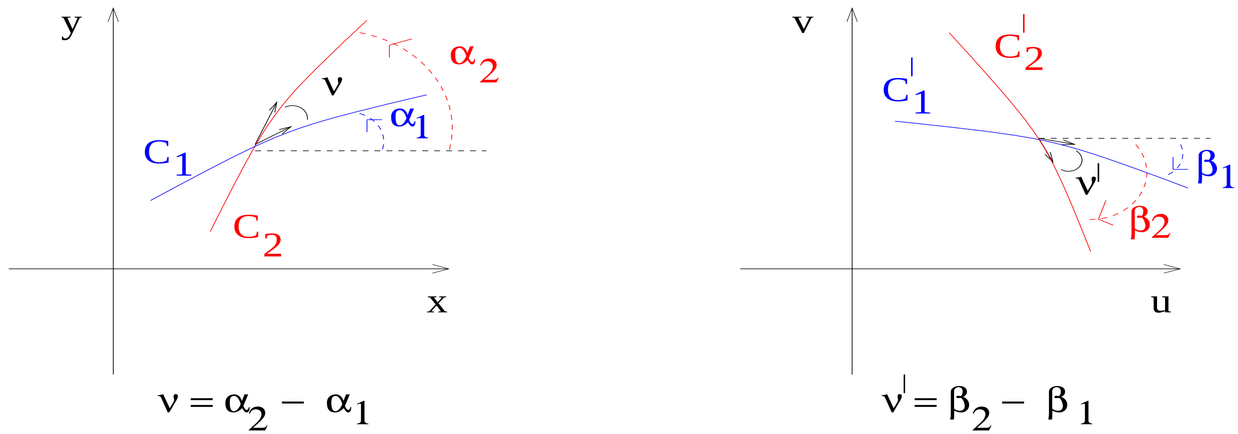


$\nu = \alpha_2 - \alpha_1$



$\nu' = \beta_2 - \beta_1$

Proof



Let z be generic point on C_1; w its image on C_1'.

- Set $z - z_0 = re^{i\theta}$, $w - w_0 = r'e^{i\theta'}$, $f'(z_0) = |f'(z_0)|e^{i\psi_0}$,

and consider $\dfrac{w - w_0}{z - z_0} = \dfrac{r'}{r} \; e^{i(\theta' - \theta)}$.

- For $z \to z_0$ along C_1 , $\theta \to \alpha_1$, $\theta' \to \beta_1 \Rightarrow \psi_0 = \beta_1 - \alpha_1$.

By the same reasoning for $z \to z_0$ along C_2, $\psi_0 = \beta_2 - \alpha_2$.

Hence $\beta_1 - \alpha_1 = \beta_2 - \alpha_2$.

$\Rightarrow \nu = \alpha_2 - \alpha_1 = \beta_2 - \beta_1 = \nu'$

$f : z \mapsto w = f(z)$ holomorphic with $f'(z_0) \neq 0$:

♠ Tangent vectors dz to each curve at z_0 are transformed into vectors dw at $w_0 = f(z_0)$ which are

- magnified by factor $|f'(z_0)|$
- rotated through angle $\psi_0 = \arg\, f'(z_0)$

$\Rightarrow$ angles between curves remain the same (conformal mapping)

Behaviour at critical points $f'(z_0) = 0$:

- $f(z_0) = w_0$; $f'(z_0) = \ldots = f^{(m-1)}(z_0) = 0$, $f^{(m)}(z_0) \neq 0$

$\Rightarrow$ angle between any two curves at z_0 is multiplied by m under mapping $w = f(z)$

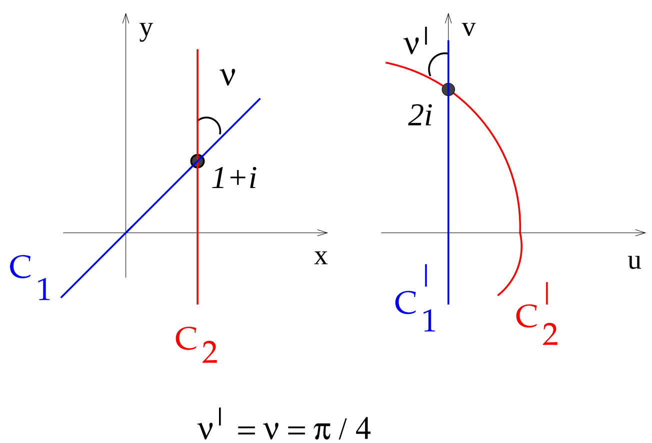
Example: quadratic map

$$z \mapsto w = f(z) = z^2 = x^2 - y^2 + 2ixy$$

- Consider curves C_1 and C_2

$$C_1 = \{z : y = x\} \longrightarrow \{w : u = 0\}$$

$$C_2 = \{z : x = 1\} \longrightarrow \{w : v^2 + 4(u-1) = 0\}$$



$$z_0 = 1 + i \; ; f'(z_0) = 2z_0 \neq 0 \;\Rightarrow\; \text{conformal}$$

$$\psi_0 = arg(2z_0) = arg(2+2i) = \pi/4 \quad \text{rotation angle}$$

$$|f'(z_0)| = |2(1+i)| = 2\sqrt{2} \quad \text{magnification factor}$$

- $z = 0$ critical point: $f'(0) = 0, f''(0) \neq 0 \;\Rightarrow$ angles are doubled at $z = 0$

Example: Moebius map

$$z \mapsto w = f(z) = \frac{az+b}{cz+d} \quad , \qquad ad - bc \neq 0$$

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0 \;\Rightarrow\; \text{conformal}$$

♣ 3 independent parameters (may set e.g. $ad - bc = 1$)

$\Rightarrow$ any three distinct points z_1, z_2, z_3 may be mapped

into three distinct points w_1, w_2, w_3

by a Moebius map (one possibly at ∞)

♣ obtained from combining rotation ($ze^{i\alpha}$), dilation (αz),

translation ($z + b$), inversion ($1/z$)

$\Rightarrow$ maps circles and lines into circles and lines

$$\text{Ex. :} \quad z \mapsto w = \frac{i-z}{i+z}$$

maps axis $\text{Im}z = 0$ on to circle $|w| = 1$; half plane $\text{Im}z > 0$ on to disk $|w| < 1$;

$\text{Im}z < 0$ on to $|w| > 1$

TRANSFORMATION OF HARMONIC FUNCTIONS BY CONFORMAL MAPPING

$H(x, y)$ harmonic : $\Delta H(x, y) = 0$.

- Apply mapping $f : z \mapsto w = u + iv$

f holomorphic, $f' \neq 0 \longrightarrow z = f^{-1}(w)$

- Then $H(x, y) \longrightarrow \underbrace{(H \circ f^{-1})}_{H'}(u, v)$

Is H' harmonic?

Yes, because $$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = |f'(z)|^2 \left(\frac{\partial^2 H'}{\partial u^2} + \frac{\partial^2 H'}{\partial v^2} \right)$$ $\leftarrow$ [Verify]

So $f' \neq 0 \implies \Delta H = 0$ iff $\Delta H' = 0$

SUMMARY

Complex differentiation

▷ holomorphic functions = differentiable in an open set

▷ Cauchy-Riemann equations: $\overline{\partial} f = 0 \qquad (\overline{\partial} \equiv \partial / \partial \overline{z})$

▷ $f = u + iv$ holomorphic $\Rightarrow u$, v harmonic: $\Delta u = 0$, $\Delta v = 0$

▷ f holomorphic, $f' \neq 0 \Rightarrow$ mapping $z \mapsto w = f(z)$ is conformal