$\mathbf{II}$. COMPLEX DIFFERENTIATION

$$f : S \to \mathbb{C}$$

$$z = x + iy; \; f(z) = u(x, y) + iv(x, y)$$

- complex differentiability is a far stronger condition than the condition that u and v be differentiable as functions of real variables x and y

OUTLINE

$\Diamond$ Differentiability in complex sense

$\Diamond$ Cauchy-Riemann equations

$\Diamond$ Holomorphic functions

$$f : S \to \mathbb{C}$$

- $\triangleright$ f continuous at $z_0 \in S$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

- $\triangleright$ f differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\Delta f}{\Delta z} \quad \text{exists}$$

$$= f'(z_0) \equiv \frac{df}{dz}(z_0)$$

- $\triangleright$ f holomorphic at z_0 if there exists $\delta > 0$ such that f differentiable whenever $|z - z_0| < \delta$

• holomorphic = differentiable in an open set

Examples: $\mathrm{Re}\, z$ continuous but not differentiable; z^2 holomorphic; $|z|^2$ differentiable at $z = 0$ but not holomorphic

CAUCHY-RIEMANN EQUATIONS

$f = u + iv$ is holomorphic on open set $D \subset \mathbb{C}$ if and only if u, v are continuously differentiable and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ , \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \text{(CR eqs.)}$$

Prove $\Rightarrow$.

- Take first $\Delta z = \Delta x$.

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\Delta x \to 0 \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

- Take next $\Delta z = i\Delta y$. Similarly, you get

$$\Delta y \to 0 \Rightarrow f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

holomorphy $\Rightarrow$ limits must be equal: $\partial u / \partial x = \partial v / \partial y \ , \ \partial v / \partial x = -\partial u / \partial y$

Prove $\Leftarrow$.

- u and v continuously differentiable $\Rightarrow$

$$u(x+\Delta x, y+\Delta y) - u(x,y) = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \eta\Delta x + \eta'\Delta y$$

$$v(x+\Delta x, y+\Delta y) - v(x,y) = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \eta_1\Delta x + \eta_1'\Delta y$$

where η, $\eta_1 \to 0$ as $\Delta x \to 0$ and η', $\eta_1' \to 0$ as $\Delta y \to 0$. Then

$$f(z+\Delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + (\eta + i\eta_1)\Delta x + (\eta' + i\eta_1')\Delta y$$

- Using the Cauchy-Riemann equations gives

$$f(z+\Delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + (\eta + i\eta_1)\Delta x + (\eta' + i\eta_1')\Delta y$$

- Dividing through by Δz and taking the limit $\Delta z \to 0$

$$\Rightarrow f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

EXPRESSING CAUCHY-RIEMANN EQUATIONS IN TERMS OF $\partial/\partial z$, $\partial/\partial \overline{z}$

$$f = u + iv$$

♠ Using $\partial \equiv \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\overline{\partial} \equiv \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$,

Cauchy-Riemann equations can be recast in compact form as

$$\overline{\partial} f = 0$$

because $\overline{\partial} f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{=0 \text{ by CR}} + \frac{i}{2}\underbrace{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}_{=0 \text{ by CR}} = 0$.

♠ The complex derivative is given by

$$f'(z) = \partial f$$

because $\partial f = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{=2(\partial u/\partial x) \text{ by CR}} + \frac{i}{2}\underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{=2(\partial v/\partial x) \text{ by CR}} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

- Example. $\cos z = \cos(x+iy)$ is holomorphic on the entire $\mathbb{C}$, while $\cos \overline{z} = \cos(x-iy)$ is nowhere holomorphic.

Note

Holomorphic functions are independent of $\overline{z}$: functions of z alone.

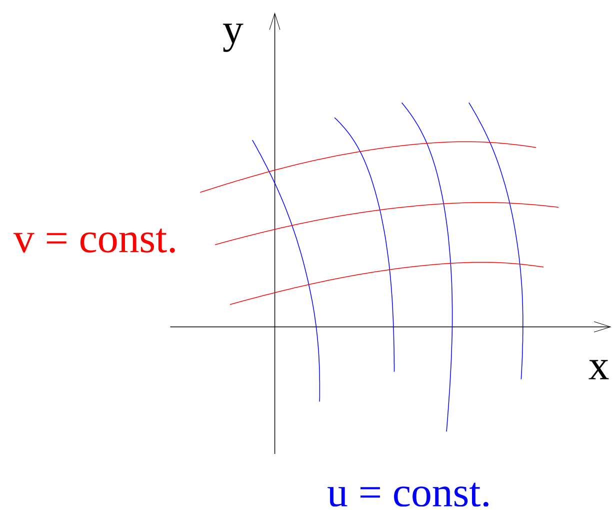
They are true functions of a complex variable, not just complex functions of two real variables.

GEOMETRIC INTERPRETATION OF CAUCHY-RIEMANN EQUATIONS

$$f(z) = u(x,y) + iv(x,y)$$

f holomorphic $\Rightarrow$

$$\Rightarrow \nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0 \quad \text{by CR}$$



♠ Cauchy-Riemann $\Rightarrow$ level curves $u(x,y) =$ const. and $v(x,y) =$ const. form orthogonal families of curves

- We will interpret this in the next lecture as a particular case of a general property of holomorphic f: conformality.

SINGULAR POINTS

$z = a$ is singular point of f if f is not holomorphic in a.

The singular point $z = a$ is

- isolated if there exists a neighbourhood of a with no other singular points.
- a pole if $1/f$ is holomorphic in a neighbourhood of a and a is a zero of $1/f$.
- an essential singularity if neither f nor $1/f$ are bounded in a neighbourhood of a.

Examples

$f(z) = 1/z$ has a pole at $z = 0$;

$f(z) = e^{1/z}$ has an essential singularity at $z = 0$;

both cases above are isolated singular points.

$f(z) = 1/\sin(1/z)$ has a non-isolated singularity at $z = 0$,

poles at $z = 1/n\pi, \; n = \pm 1, \pm 2, \ldots$

entire f = holomorphic in the whole finite complex plane

meromorphic f = holomorphic in an open set except possibly for poles

Behaviour at $z = \infty$

The behaviour of $f(z)$ at $z = \infty$ is by definition the behaviour of $g(\zeta) \equiv f(1/\zeta)$ at $\zeta = 0$.

Example: $f(z) = z^2$ has a pole at $z = \infty$
(because $g(\zeta) = 1/\zeta^2$ has a pole at $\zeta = 0$)

$f(z) = e^{1/z}$ is holomorphic at $z = \infty$
$f(z) = 1/\sin(1/z)$ has a pole at $z = \infty$