Functions of a complex variable (S1)

Trinity Term 2012

Lecturer: F Hautmann

• Why complex variables?

▷ calculational techniques

to compute integrals, solve equations, make series expansions, ...

▷ functions that are "nice and smooth"

derive their properties from behaviour in the complex plane

Lecture times: Wed, Thu, Fri at 12:00 in the
Martin Wood Lecture Theatre

- Weekly problem sheets posted on lecture webpage: http://www-thphys.physics.ox.ac.uk/people/FrancescoHautmann/ComplexVariable/

References

- Mathematical methods textbooks containing useful presentations of complex variable:

[1] P. Dennery and A. Krzywicki: Mathematics for Physicists, Dover

[2] M. Boas: Mathematical Methods in the Physical Sciences

- Textbooks on complex variables with emphasis on applications:

[3] R. Churchill, Complex variables and applications, McGraw-Hill

[4] G. Carrier, M. Krook and C. Pearson, Functions of a complex variable

- Classic reference books on complex analysis:

[5] L. Ahlfors: Complex analysis

[6] W. Rudin: Real and complex analysis

- Further references:

H. Priestley: Introduction to complex analysis, Oxford University Press

M. Ablowitz and A. Fokas: Complex variables: introduction and applications

Y. Kwok: Applied complex variables, Cambridge University Press

- A good source for worked problems and examples:

M. Spiegel: Complex variables, Schaum's Outline Series, McGraw-Hill

- Lecture notes by J. Binney on Complex Variable:

http://www-thphys.physics.ox.ac.uk/people/JamesBinney/advcalc.pdf

Synopsis

I The complex plane

II Complex differentiation

III Multi-valued functions

IV Complex integration

V Power series expansions

VI Residue calculus

VII Conformal mapping

VIII Integral transforms

LECTURE 1: OUTLINE

Introduction to complex variables

1.1 The complex plane

- Algebra of complex numbers
- “Extended” complex plane

1.2 Functions on $\mathbb{C}$

- Elementary complex functions
- Functions as “mappings”

1.3 Point sets in the complex plane

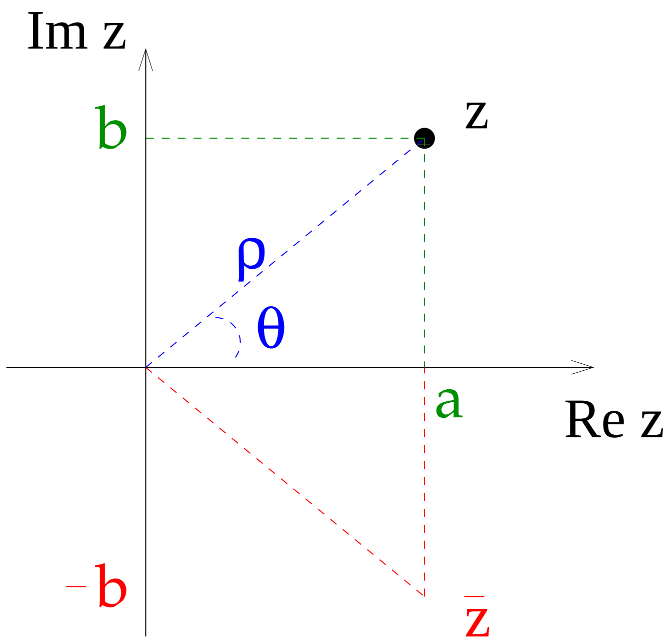
- open, closed sets
- compact
- connected

1.1 The complex plane

◊ $\mathbb{C}$ = set of complex nos with +, × operations

$$z = a + ib \quad , \qquad i^2 = -1$$

$$a = \mathrm{Re} z \quad , \qquad b = \mathrm{Im} z$$



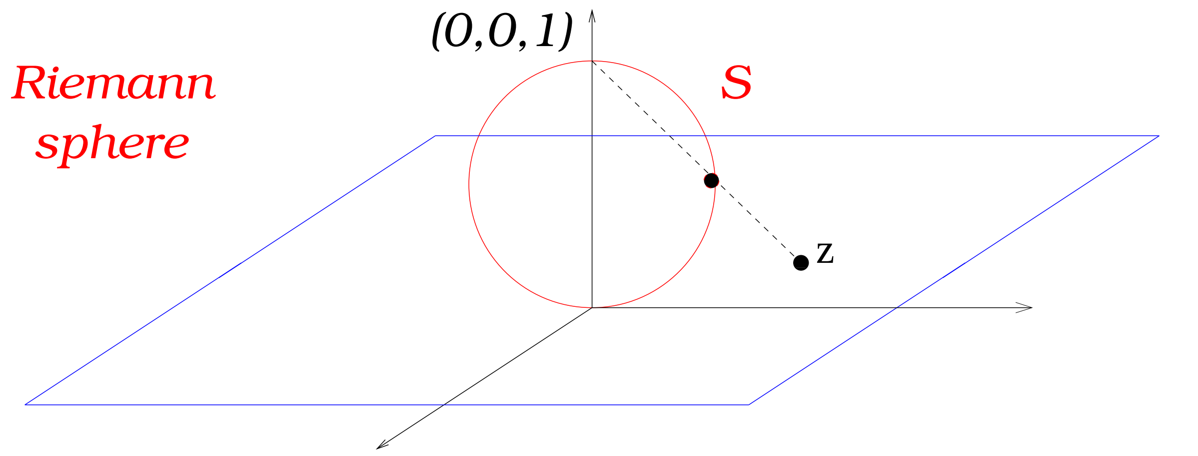
ρ = mod(z) = | z |

θ = arg(z)

$$z = a + ib = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta)$$

$$\overline{z} = a - ib = \rho e^{-i\theta} = \rho(\cos\theta - i\sin\theta)$$

EXTENDED COMPLEX PLANE



- 1-to-1 correspondence between $\mathbb{C}$ and S:

$$\varphi : z = re^{i\theta} \to \left(\frac{r\cos\theta}{1+r^2}, \frac{r\sin\theta}{1+r^2}, \frac{r^2}{1+r^2} \right)$$

$\hookrightarrow$ stereographic projection

- $z = \infty$ defined as the point associated to $(0, 0, 1)$ on S by φ projection

- extended complex plane is $\mathbb{C} \cup \{z = \infty\}$

$\mathbf{1.2}$ Elementary functions on $\mathbb{C}$

- Complex polynomials and rational functions defined by algebraic operations in $\mathbb{C}$

- Complex exponential: $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$

$\longrightarrow$ complex trigon. and hyperb. fctns in terms of exp.
e.g. $\cos z = (e^{iz} + e^{-iz})/2$

- Complex logarithm $\ln z$: $e^{\ln z} = z$

$\Rightarrow \ln z = \ln |z| + i(\theta + 2n\pi)$, $n = 0, \pm 1, \ldots$ ($\leftarrow$ multi-valued)

$\longrightarrow$ complex powers: $z^\alpha = e^{\alpha \ln z}$ (α complex)

FUNCTIONS AS MAPPINGS

$$f : S \subset \mathbb{C} \to \mathbb{C}$$

$$f : z \in S \mapsto w = f(z)$$

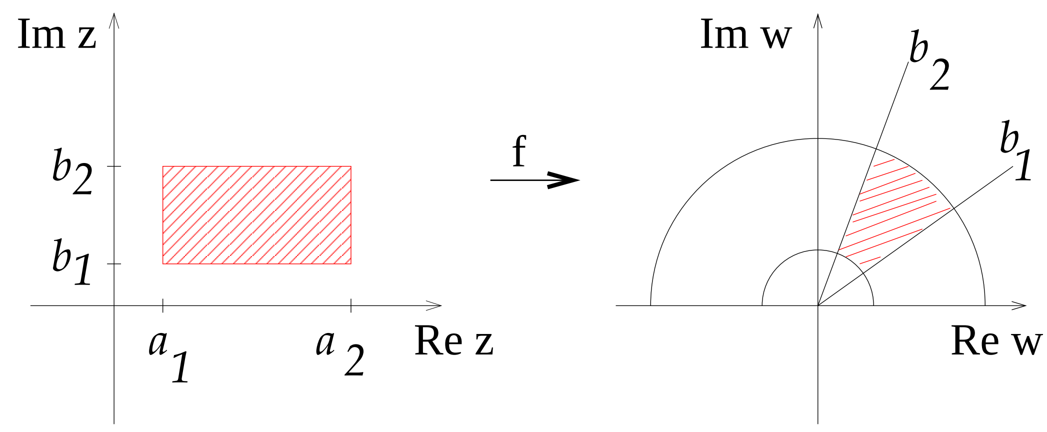
- For any subset A of S, image of A through $f = f(A)$ is the set of points w such that $w = f(z)$ for some z belonging to A.
- f maps A on to $f(A)$

Example

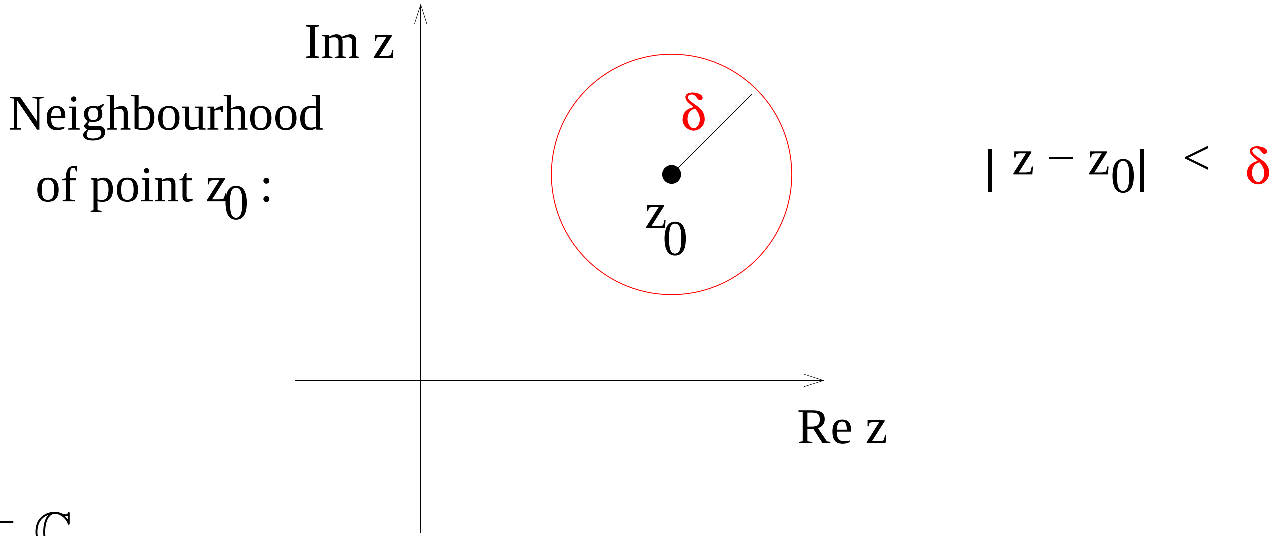
$w = f(z) = e^z$. Set $z = x + iy$; $w = \rho e^{i\phi}$. Then $\rho = e^x$; $\phi = y$.

lines $x = a \xrightarrow{f}$ circles $\rho = e^a$

lines $y = b \xrightarrow{f}$ rays $\phi = b$



1.3 Point sets in the complex plane



$S \subset \mathbb{C}$.

- $z_0 \in S$ isolated point of S if there exists a neighbourhood of z_0 which does not contain any other point belonging to S
- z_0 limit point of S if every neighbourhood of z_0 contains at least one element of S, other than z_0 itself
- z_0 interior point of S if there exists a neighbourhood of z_0 all points of which belong to S
- z_0 boundary point of S if every neighbourhood of z_0 contains points of S and points not belonging to S

▷ Every limit point that is not interior is boundary point.

EXAMPLE. $S_0 = \{z \in \mathbb{C} : 0 < |z| < 1\}$

- $z = 0$ is boundary point; each point on circle $|z| = 1$ is boundary point.

♢ A set is closed if it contains all its limit points.

Ex.: $\{z : |z| \leq 1\}$

♢ A set is open if all its points are interior points.

Ex.: $\{z : |z| < 1\}$

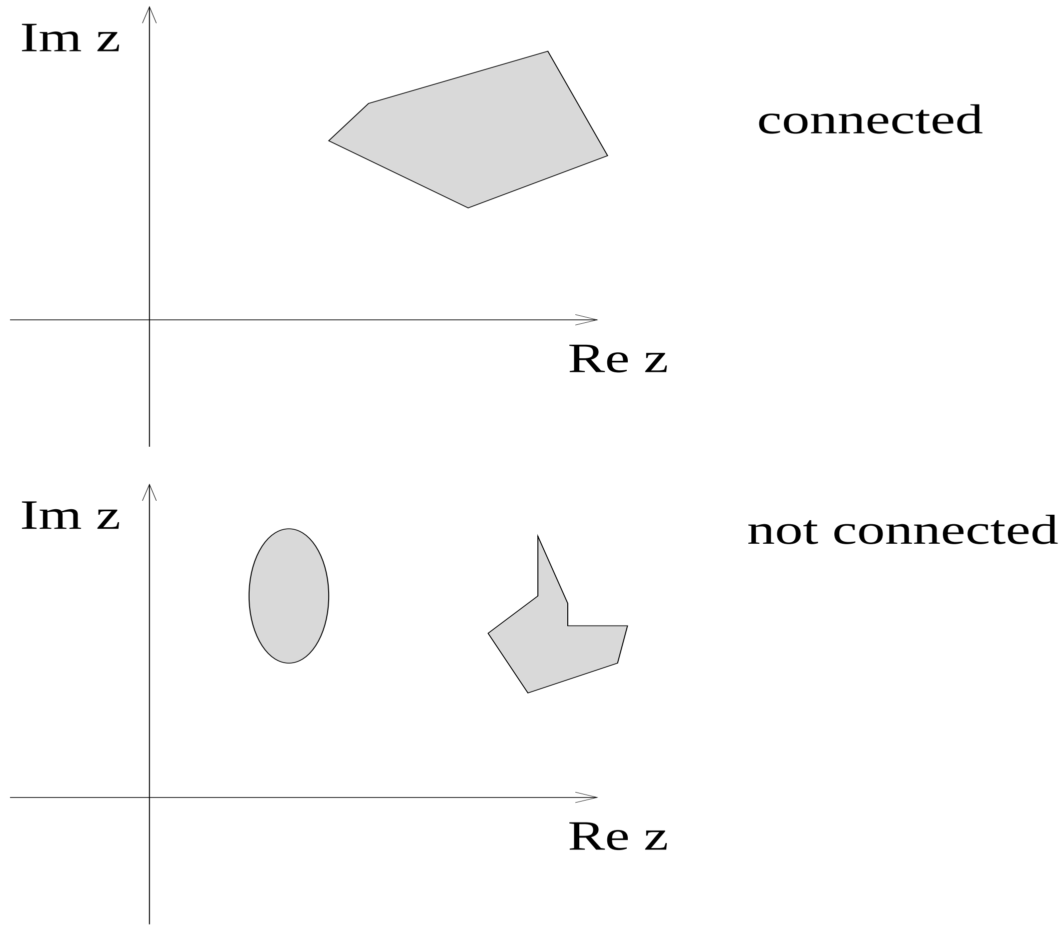
EXAMPLE. $S_1 = \{z = i/n : n \in \mathbb{N}\}$

- S_1 neither open nor closed; $z = 0$ limit point; all points in S_1 are isolated.

♢ A set S is bounded if for some constant M, $|z| < M$ for every $z \in S$.

- compact = closed and bounded

◊ A set $S \subset \mathbb{C}$ is connected if every two points in S can be joined by a path all points of which belong to S.



- connected open set = domain (or region)