S1 REVISION LECTURE

FUNCTIONS OF A COMPLEX VARIABLE

Complex differentiation

- $\triangleright$ holomorphic functions $=$ differentiable in an open set

- $\triangleright$ Cauchy-Riemann equations: $\overline{\partial} f = 0 \qquad (\overline{\partial} \equiv \partial / \partial \overline{z})$

- $\triangleright$ $f = u + iv$ holomorphic $\Rightarrow u$, v harmonic: $\Delta u = 0$, $\Delta v = 0$

- $\triangleright$ f holomorphic, $f' \neq 0 \Rightarrow$ mapping $z \mapsto w = f(z)$ is conformal

2. Consider the function

$$u(x, y) = e^{-x} \cos y + xy \ .$$

(a) Show that $u(x, y)$ is harmonic over the whole xy plane. [4]

(b) Find a holomorphic function $f(z)$ whose real part is given by $u(x, y)$ and which is real-valued at $z = 0$ ($z = x + \mathrm{i}y$). [7]

(c) Write down the family of curves in the xy plane which are orthogonal to the curves

$$e^{-x} \cos y + xy = \text{constant} \ .$$ [7]

(d) Evaluate the integral of $u(x, y)$ round the circle in the xy plane with centre at the origin and radius 1. [7]

June 2011

(a) $$u(x,y) = e^{-x}\cos y + xy$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = +e^{-x}\cos y - e^{-x}\cos y = 0$$

(b) $f(z) = u(x,y) + iv(x,y)$, $u(x,y) = e^{-x}\cos y + xy$

Cauchy-Riemann equations $\overline{\partial} f = 0$: $(\overline{\partial} \equiv \partial/\partial\overline{z} = [(\partial/\partial x + i\partial/\partial y)/2])$

$$i) \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \Longrightarrow \frac{\partial v}{\partial y} = -e^{-x}\cos y + y$$

$$\Longrightarrow \quad v(x,y) = -e^{-x}\sin y + y^2/2 + \phi(x)$$

$$ii) \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Longrightarrow e^{-x}\sin y + \frac{\partial \phi}{\partial x} = e^{-x}\sin y - x \text{ , i.e. , } \phi = -x^2/2 + C$$

$$\Longrightarrow \quad v(x,y) = -e^{-x}\sin y + y^2/2 - x^2/2 + C$$

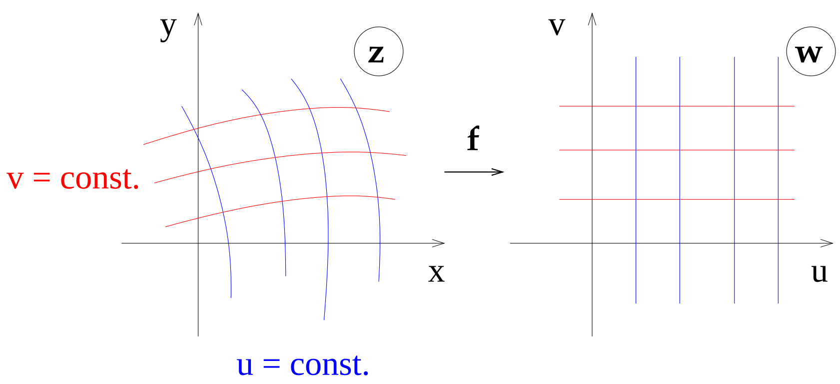
$$\text{Thus } f(z) = u + iv = e^{-x}(\cos y - i\sin y) + xy + i(y^2 - x^2)/2 + iC$$

$$= e^{-z} - iz^2/2 + iC$$

$f(z)$ real-valued at $z = 0$ $\Rightarrow$ $C = 0$. So $f(z) = e^{-z} - iz^2/2$.

(c) The family of curves orthogonal to the curves $u(x, y) =$ const. is given by $v(x, y) =$ const., i.e.

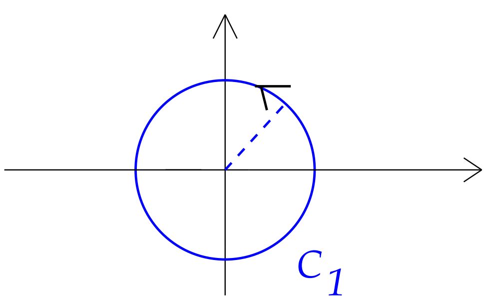
$$(y^2 - x^2)/2 - e^{-x} \sin y = \text{const.}$$



(d) $u(x, y) = e^{-x} \cos y + xy$ harmonic $\Rightarrow$

$\Rightarrow$ mean value of u on the circle $=$ value of u at the centre of the circle

$$\text{i.e.,} \quad \frac{1}{2\pi} \int_{C_1} u = u(0, 0)$$



$$u(0, 0) = 1 \quad \Rightarrow \quad \int_{C_1} u = 2\pi$$

Complex integration and power series

- Cauchy theorem
and Cauchy integral formulas

- Power series expansions:
Taylor and Laurent series

- Residue calculus

2. (a) The function $g(z)$ of the complex variable z has a simple pole at $z = z_0$. Give a formula for the residue of $g(z)$ at $z = z_0$. [3]

(b) State Cauchy's formula expressing the value of an analytic function at a point in terms of an integral along a closed contour surrounding the point. [4]

(c) The function $g(z)$ is analytic everywhere in the z plane cut along the positive real axis, apart from a simple pole of residue r at $z = z_0$ not on the positive real axis. Also $|zg(z)|$ tends to a constant as $|z|$ tends to ∞. The difference $g_+(x) - g_-(x)$ between the values of $g(z)$ just above and just below the positive real axis is $\Delta(x)$. Find a formula for $g(z)$ (z not real positive and $z \neq z_0$) in terms of $\Delta(x)$, z_0 and r. [18]

2(a) *Residue of a simple pole:* $\lim_{z \to z_0} (z - z_0) g(z)$

2(b) *Cauchy integral formula of order 0:* $f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} \, dz'$

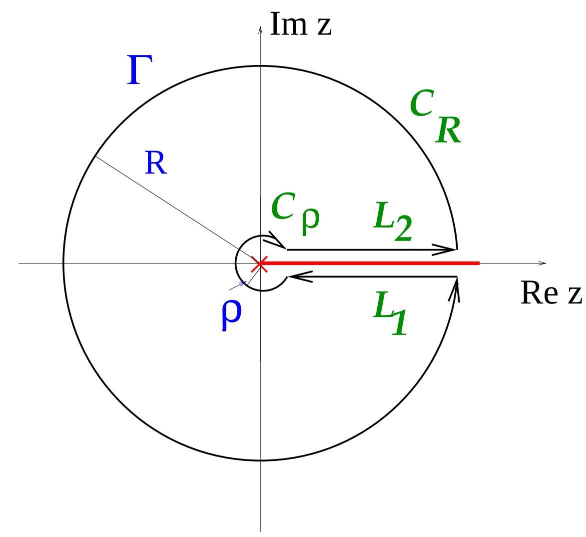
2(c)

- $g(z) = \frac{r}{z - z_0} + h(z)$

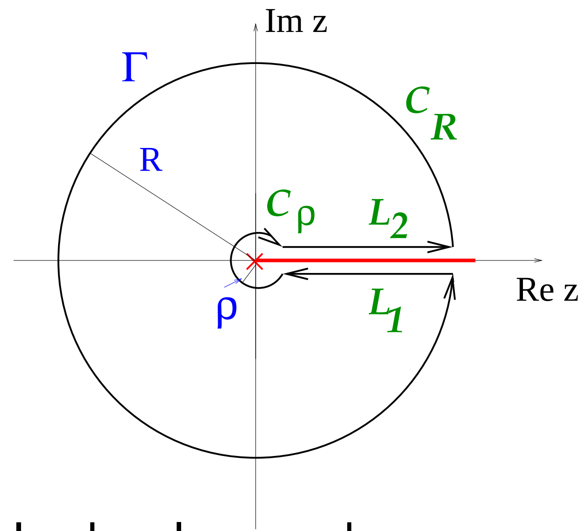
where $h(z)$ is holomorphic in the z plane cut along the positive real axis

- Represent $h(z)$ by Cauchy integral formula on contour Γ

$$h(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{h(z')}{z' - z} \, dz'$$



- $$\oint_\Gamma = \int_{C_R} + \int_{L_1} + \int_{C_\rho} + \int_{L_2}$$



- For $R \to \infty$, integral on C_R goes to 0 by Jordan lemma because

$$|zg(z)| \to \text{const.} \implies |zh(z)| \to \text{const.} \implies |zh(z)|/|z-z'| \to 0 \quad \text{as} \quad |z| \to \infty$$

- Also, for $\rho \to 0$ integral on C_ρ goes to 0 by Jordan lemma because

$$|zh(z)|/|z-z'| \to 0$$

- Then $$h(z) = \int_{L_1} + \int_{L_2} = \frac{1}{2\pi i}\int_0^\infty \frac{\Delta(x)}{x-z}\, dx$$

Therefore $$g(z) = \frac{r}{z-z_0} + \frac{1}{2\pi i}\int_0^\infty \frac{\Delta(x)}{x-z}\, dx$$

June 2011

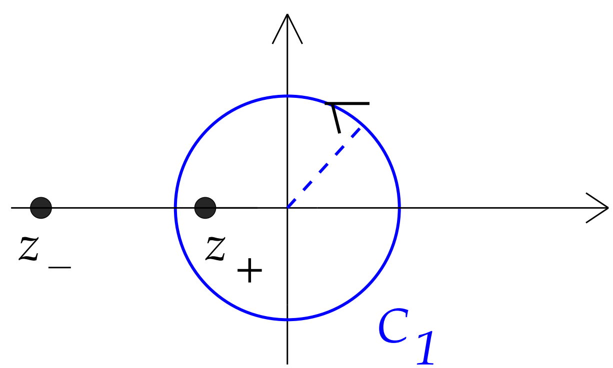
Evaluate the following real integral by complex contour integration methods:

$$\int_0^{2\pi} dt \, \frac{1}{2+\cos t}$$

$$z = e^{it} \quad ; \quad dz = izdt \quad ; \quad \cos t = \frac{z+1/z}{2}$$

$$\text{So } \; I = \int_0^{2\pi} dt \, \frac{1}{2+\cos t} = \oint_{C_1} dz \, \frac{1}{iz} \, \frac{1}{2+(z+1/z)/2} = \oint_{C_1} dz \, \frac{2}{i} \, \frac{1}{z^2+4z+1}$$

$$z_{\pm} = -2 \pm \sqrt{3}$$



- By residue theorem $I = 2\pi i \, [\mathrm{Res}_{z=z_+} f] = \dfrac{2\pi}{\sqrt{3}}$.

3. (a) List the positions of all singularities (if any) in the complex z plane of each of the following:
(i) $z^2 \exp(-z)$; (ii) $z^2 \exp(-1/z)$; (iii) $\ln(z^2+4)$.
For each singularity, state whether it is isolated. If it is isolated, state whether it is an essential singularity or a pole. If it is a pole, give its order. [8]

(b) In case (ii) integrate the function around an anti-clockwise circle of unit radius in the z plane and check your answer by making the change of variable $z = w^{-1}$ and integrating in the w plane. [14]

(c) You are given that a student has integrated around an anti-clockwise circle of unit radius in the z plane in case (iii), and obtained the answer 0. Explain why this result may be correct. [3]

June 2007

$3(a)$ $\quad i)$ $z^2 e^{-z}$: $z=\infty$ isolated essential singularity

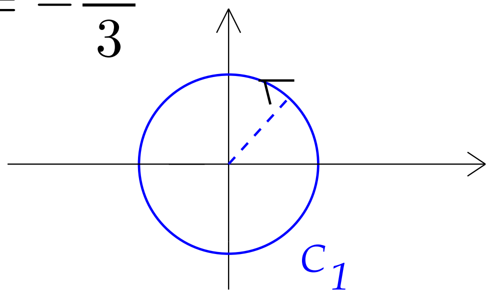
$ii)$ $z^2 e^{-1/z}$: $z=0$ isolated essential singularity; $z=\infty$ pole of order 2

$iii)$ $\ln(z^2+4)$: $z=2i$, $z=-2i$, $z=\infty$ are branch points of order ∞

$3(b)$ $\quad$ Laurent expansion : $\quad z^2 e^{-1/z} = z^2\left[1-\frac{1}{z}-\frac{1}{2!z^2}-\frac{1}{3!z^3}+\dots\right]$

$$= z^2 - z - \frac{1}{2} \underbrace{-\frac{1}{6}}_{residue} \frac{1}{z} + \dots$$

$$\Longrightarrow \quad \oint_{C_1} dz\ z^2 e^{-1/z} = 2\pi i\ \mathrm{Res}_{z=0}\ f = 2\pi i\left(-\frac{1}{6}\right) = -\frac{i\pi}{3}$$



$$z \to w = 1/z$$

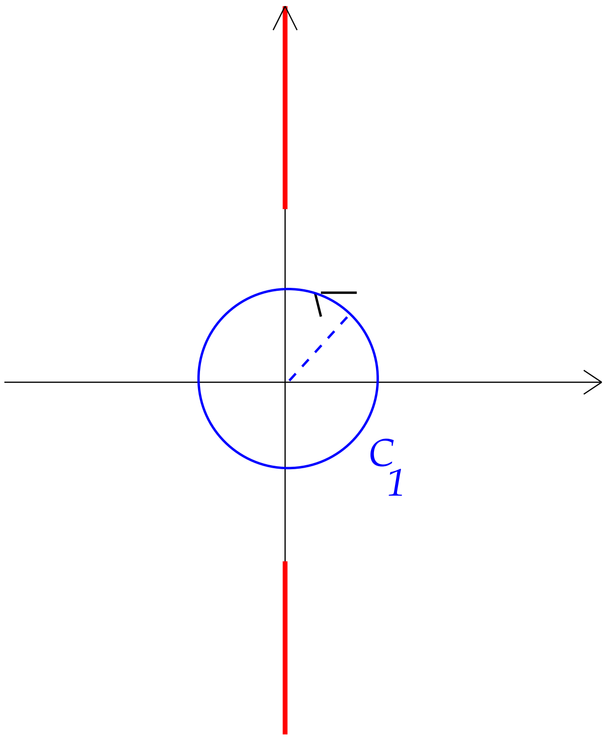
$$\Rightarrow \oint_{C_1} dz\ z^2 e^{-1/z} = \oint_{-C_1} dw\ \left(-\frac{e^{-w}}{w^4} \right)$$

$$= -2\pi i\ \mathrm{Res}_{w=0}\ [\mathrm{Integrand}] = -2\pi i\ \frac{1}{6} = -\frac{i\pi}{3}$$

$3(c)$

Set branch cuts as in figure.

$$\oint_{C_1} \ln(z^2 + 4)\ dz = 0$$



June 2009

2.

(a) Show that $e^{\ln z}$ always equals z, while $\ln e^{z}$ does not always equal z.

(b) Take the principal branch of $\ln z$ and evaluate the integral

$$\oint_{\Gamma} dz \ \frac{(\ln z)^2}{z^2+1} \ ,$$

where Γ is the closed contour consisting of two semicircles in the upper half plane with radii r and R ($r < 1$, $R > 1$), respectively, and centre at the origin, and intervals $(-R, -r)$ and (r, R) on the real axis.

(c) Use the result in (b) to prove that

$$\int_0^{\infty} dx \ \frac{\ln x}{x^2+1} = 0 \ , \qquad \int_0^{\infty} dx \ \frac{(\ln x)^2}{x^2+1} = \frac{\pi^3}{8} \quad .$$

(a)

$z = a + ib = re^{i\theta}$

Then $\ln z = \ln r + i(\theta + 2\pi n)$, n integer.

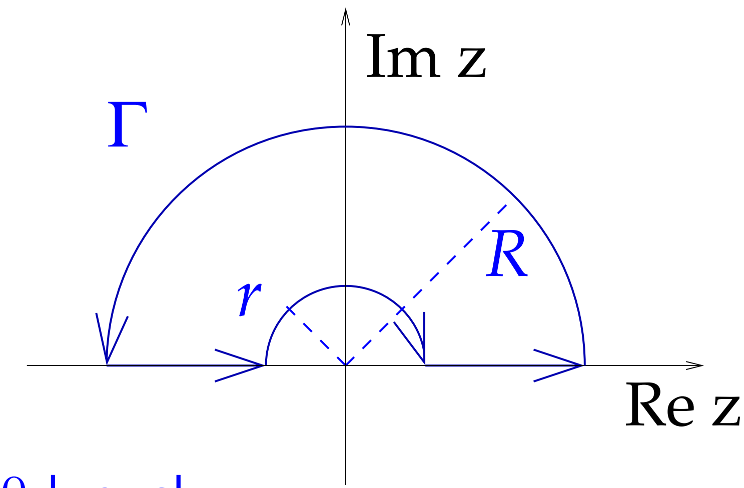
So $e^{\ln z} = e^{\ln r + i(\theta + 2\pi n)} = re^{i\theta} \underbrace{e^{2\pi n i}}_{1} = re^{i\theta} = z$.

On the other hand $e^z = e^{a+ib} = e^a e^{ib}$

So $\ln e^z = \ln e^a + i(b + 2\pi n) = \underbrace{a + ib}_{z} + 2\pi i n = z + 2\pi i n$ which may be $\neq z$.

(b)

$$\oint_\Gamma dz \; \frac{(\ln z)^2}{z^2+1}$$



$\ln z = \ln|z| + i(\theta + 2n\pi)$, $n = 0$ branch

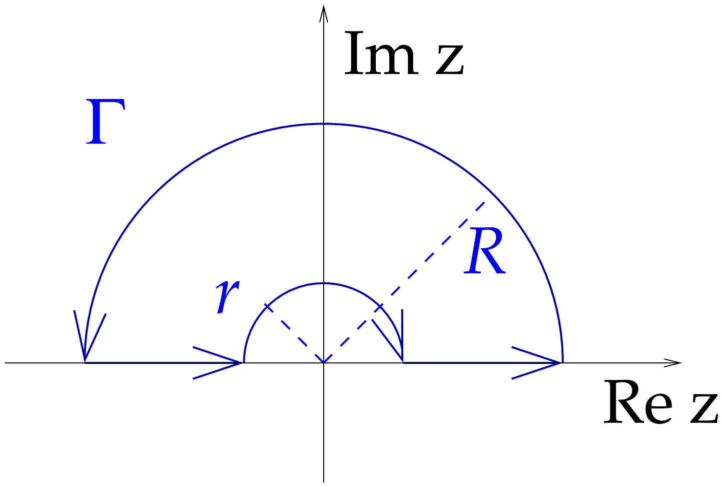
$R > 1$, $r < 1$: $z = i$ is order-1 pole encircled by Γ

$$\Longrightarrow \oint_\Gamma dz \; \frac{(\ln z)^2}{z^2+1} = 2\pi i \mathrm{Res}_i \; f = 2\pi i \frac{(i\pi/2)^2}{2i}$$

$$= -\frac{\pi^3}{4}$$

(c)

$$\oint_\Gamma = \int_{-R}^{-r} + \int_{C_r} + \int_r^R + \int_{C_R}$$



- $$\int_{-R}^{-r} dx \, \frac{(\ln|x| + i\pi)^2}{x^2+1} = \int_r^R dx \, \frac{(\ln x + i\pi)^2}{x^2+1}$$
- For $r \to 0$, $R \to \infty$ integrals on semicircles go to 0 by Jordan lemma.

$$\Longrightarrow \quad -\frac{\pi^3}{4} = \int_0^\infty dx \, \frac{(\ln x + i\pi)^2}{x^2+1} + \int_0^\infty dx \, \frac{(\ln x)^2}{x^2+1}$$

that is $$-\frac{\pi^3}{4} = 2\int_0^\infty dx \, \frac{(\ln x)^2}{x^2+1} + 2\pi i \int_0^\infty dx \, \frac{\ln x}{x^2+1} - \pi^2 \underbrace{\int_0^\infty dx \, \frac{1}{x^2+1}}_{\pi/2}$$

Equating Re and Im $$\Longrightarrow \quad \int_0^\infty dx \, \frac{(\ln x)^2}{x^2+1} = \frac{\pi^3}{8} \; , \quad \int_0^\infty dx \, \frac{\ln x}{x^2+1} = 0$$

2. (a) Give the location and order of the branch points of the function

$$f(z) = \frac{\sqrt{z}}{1+z^2} \ .$$

[5]

(b) Take the principal branch of the square root, $\sqrt{z} = \sqrt{|z|}\exp(i\theta/2)$, $0 \le \theta \le 2\pi$, setting the branch cut in the complex z plane along the positive real semiaxis. Evaluate the contour integral

$$\oint_\Gamma dz \ \frac{\sqrt{z}}{1+z^2} \ ,$$

where Γ is the closed contour consisting of two circles C_R and C_ρ given respectively by $|z| = R$ and $|z| = \rho$ with $R > 1$, $\rho < 1$, and the portions L_1 and L_2 of the positive real semiaxis from ρ to R, taken respectively along the lower edge and the upper edge of the square root branch cut.

[10]

(c) Use the result in (b) to calculate the real integral

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2} \ dx \ .$$

[10]

June 2010

(a) $$f(z) = \sqrt{z}/(1+z^2)$$

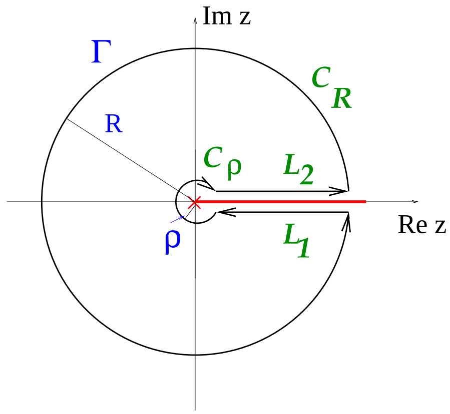
$z = 0$ branch point of first order; $z = \infty$ branch point of first order

(b)

$$\oint_\Gamma dz \ f(z)$$

Principal branch : $z^{1/2} = |z|^{1/2} e^{i\theta/2}$, $0 \le \theta < 2\pi$. Branch cut along $\mathbb{R}^+$.

- Residue theorem applied to contour Γ $\Rightarrow$



$$\Rightarrow \oint_\Gamma \frac{\sqrt{z}}{z^2+1} \, dz = 2\pi i [\text{Res}_{z=+i} f + \text{Res}_{z=-i} f] = 2\pi \cos(\frac{\pi}{4}) = \pi\sqrt{2}$$

(c)

$$\text{Write } \oint_\Gamma = \int_{C_R} + \int_{L_1} + \int_{C_\rho} + \int_{L_2}$$

Let $R \to \infty$, $\rho \to 0$.

$$\int_{C_R} \longrightarrow 0 \text{ for } R \to \infty \, , \; \int_{C_\rho} \longrightarrow 0 \text{ for } \rho \to 0 \Rightarrow$$

$$\Rightarrow \oint_\Gamma \frac{\sqrt{z}}{z^2+1} \, dz = \int_0^\infty dx \, \frac{1}{x^2+1} \left[\sqrt{x} - \sqrt{x} \, e^{2\pi i/2} \right]$$

$$= \int_0^\infty dx \, \frac{\sqrt{x}}{x^2+1} [1 - e^{i\pi}] = 2 \int_0^\infty dx \, \frac{\sqrt{x}}{x^2+1}$$

Using the result in (b) $\Rightarrow$

$$\Rightarrow \int_0^\infty dx \, \frac{\sqrt{x}}{x^2+1} = \frac{\pi}{\sqrt{2}}$$