Functions of a Complex Variable (S1)

Lecture 11

VII. Integral Transforms

An introduction to Fourier and Laplace transformations

- Integral transforms from application of complex calculus
- Properties of Fourier and Laplace transforms
- Applications to differential equations

Fourier transformation

$$\widetilde{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} \; f(t) \; dt \equiv \mathrm{FT}\,[f]$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \; \widetilde{f}(\omega) \; d\omega \equiv \mathrm{FT}^{-1}\,[\widetilde{f}]$$

- We see next how to obtain this transformation starting from a complex integral representation of step functions.

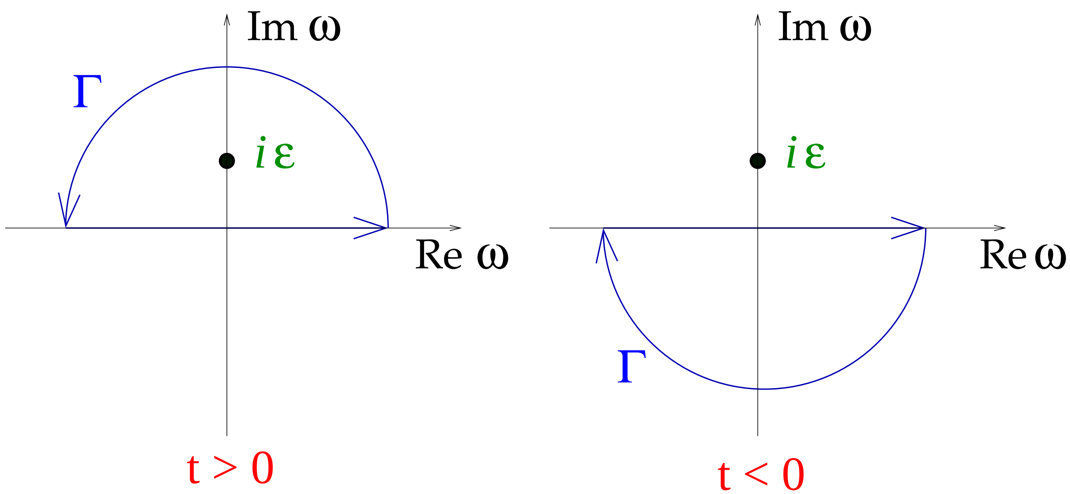
The step function $\Theta(t)$

- Consider $$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{i\omega t}\,\frac{1}{\omega - i\varepsilon}\,d\omega \qquad (\varepsilon \in \mathbb{R}^+)$$

- Evaluate this integral by complex contour integration:

for $t > 0$ Jordan lemma applies to semicircular arc in UHP $\Longrightarrow$ integral $= \text{Res}_{i\varepsilon} = 1$;

for $t < 0$ Jordan lemma applies to semicircular arc in LHP $\Longrightarrow$ integral $= 0$.

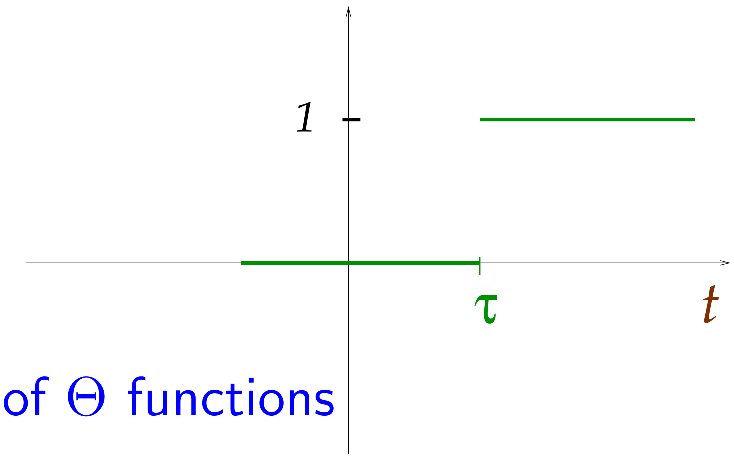


- Thus $$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{i\omega t}\,\frac{1}{\omega - i\varepsilon}\,d\omega = \Theta(t)$$

where $\Theta(t)$ is the step function: $\Theta(t) = 1$ for $t > 0$, $\Theta(t) = 0$ for $t < 0$.

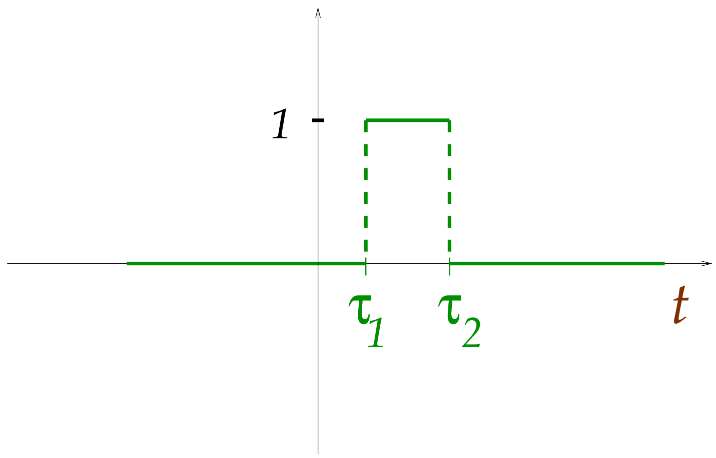
step function centred at $t = \tau$:

$$\Theta(t-\tau) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{i\omega(t-\tau)} \, \frac{1}{\omega - i\varepsilon} \, d\omega$$



"finite pulse": combination of Θ functions

$$\Theta(t-\tau_1) - \Theta(t-\tau_2) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{i\omega t} \, \frac{e^{-i\omega\tau_1} - e^{-i\omega\tau_2}}{\omega - i\varepsilon} \, d\omega$$

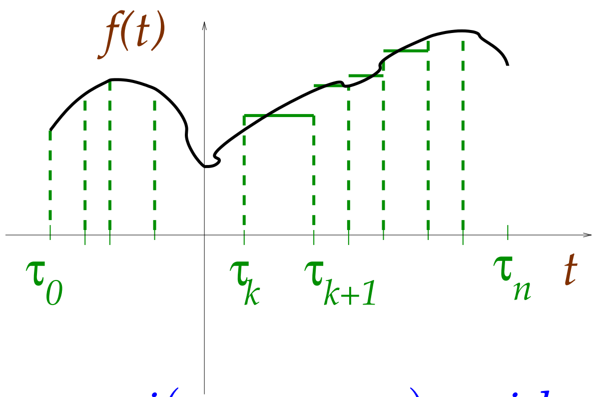


Construction of Fourier transform

- First construct approximation to $f(t)$ by sum of step functions

$$f(t) \simeq \sum_{k=0}^{n-1} f(\tau_k)\,[\Theta(t-\tau_k) - \Theta(t-\tau_{k+1})]$$

$$= \frac{1}{2\pi i}\,\sum_{k=0}^{n-1} f(\tau_k) \int_{-\infty}^{+\infty} e^{i\omega t}\,\frac{e^{-i\omega\tau_k} - e^{-i\omega\tau_{k+1}}}{\omega - i\varepsilon}\,d\omega$$



- Then let each $\tau_{k+1} - \tau_k \to 0$ and expand in $\Delta\tau_k = i(\tau_{k+1} - \tau_k) = i d\tau$:

$$\Longrightarrow \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t}\,\underbrace{\int_{-\infty}^{+\infty} f(\tau)\,e^{-i\omega\tau}\,d\tau}_{\widetilde{f}(\omega)}\,d\omega$$

- Thus we obtain representation of f as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \, \widetilde{f}(\omega) \, d\omega \; \equiv \mathrm{FT}^{-1} \, [f]$$

where $\widetilde{f}(\omega)$ is the *Fourier transform* defined by

$$\widetilde{f}(\omega) = \int_{-\infty}^{+\infty} f(t) \, e^{-i\omega t} \, dt \; \equiv \mathrm{FT} \, [f]$$

◊ We will apply this to functions f in $L^2 \cap L^1$, i.e., such that

$$\int_{-\infty}^{+\infty} |f|^2 < \infty \; , \qquad \int_{-\infty}^{+\infty} |f| < \infty$$

PROPERTIES OF FOURIER TRANSFORMATION

linearity : $\mathrm{FT}\,[\alpha f + \beta g] = \alpha\ \mathrm{FT}\,[f] + \beta\ \mathrm{FT}\,[g]$

scaling : $\mathrm{FT}\,[f(\lambda t)] = \frac{1}{\lambda}\ \widetilde{f}(\frac{\omega}{\lambda})$

shift : $\mathrm{FT}\,[f(t-a)] = e^{-i\omega a}\ \widetilde{f}$

FT of derivative : $\mathrm{FT}\,[f^{(n)}] = (i\omega)^n\ \widetilde{f}$

derivative of FT : $i^n\ \widetilde{f}^{(n)} = \mathrm{FT}\,[t^n f]$

convolution theorem : $\mathrm{FT}\,[f * g] = \widetilde{f}\ \widetilde{g}$

where $(f * g)(t) \stackrel{\mathrm{df}}{=} \int_{-\infty}^{+\infty} dy\ f(y)\ g(t-y)$

FT of product : $\mathrm{FT}\,[fg] = \frac{1}{2\pi}\ \widetilde{f} * \widetilde{g}$

PARSEVAL IDENTITY

- Take convolution theorem for the particular case $g(t) = \overline{f}(-t)$.

$$\Longrightarrow \quad \widetilde{g}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} \; \overline{f}(-t) \; dt = \int_{-\infty}^{+\infty} e^{i\omega t} \; \overline{f}(t) \; dt = \overline{\widetilde{f}}(\omega)$$

Then $f * g = \mathrm{FT}^{-1} \, [\widetilde{f} \; \widetilde{g}]$ gives

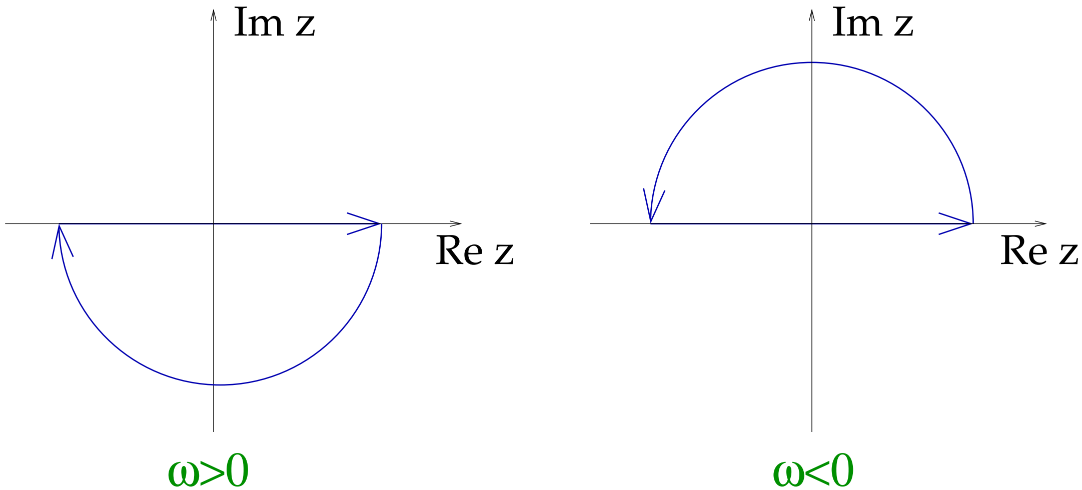
$$\int_{-\infty}^{+\infty} dy \; f(y) \; \overline{f}(y-t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \; e^{i\omega t} \; \widetilde{f}(\omega) \; \overline{\widetilde{f}}(\omega)$$

$$t = 0 \quad \Rightarrow \quad \int_{-\infty}^{+\infty} dy \; |f(y)|^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \; |\widetilde{f}(\omega)|^2$$

- Example: Compute the FT of $f(t) = 1/(1+t^2)$.

$$\widetilde{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} \, \frac{1}{1+t^2} \, dt$$

This integral can be computed by complex contour integration methods, closing the contour as in the figure below depending on the sign of ω.



$$\omega > 0 \ : \quad \widetilde{f}(\omega) = -2\pi i \ \mathrm{Res}_{-i} = \pi e^{-\omega}$$

$$\omega < 0 \ : \quad \widetilde{f}(\omega) = 2\pi i \ \mathrm{Res}_{i} = \pi e^{\omega}$$

$$\text{Thus} \quad \widetilde{f}(\omega) = \pi e^{-|\omega|} \quad \text{for any} \quad \omega \ .$$

Application of FT to boundary value problems

Example : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad , \quad y > 0$

$u(x,0) = G(x) \qquad \text{(Dirichlet boundary condition)}$

- Taking FT in x at fixed y gives ODE for $\widetilde{u}(\omega, y)$:

$$-\omega^2 \widetilde{u} + \frac{\partial^2 \widetilde{u}}{\partial y^2} = 0 \;\; , \quad \widetilde{u}(\omega, 0) = \widetilde{G}(\omega) \quad \Longrightarrow \quad \widetilde{u}(\omega, y) = \widetilde{G}(\omega) e^{-|\omega| y}$$

- Now use convolution theorem : $\quad u(x,y) = G * \underbrace{\mathrm{FT}^{-1}\left[e^{-|\omega| y}\right]}_{[\pi y (1 + x^2/y^2)]^{-1}}$

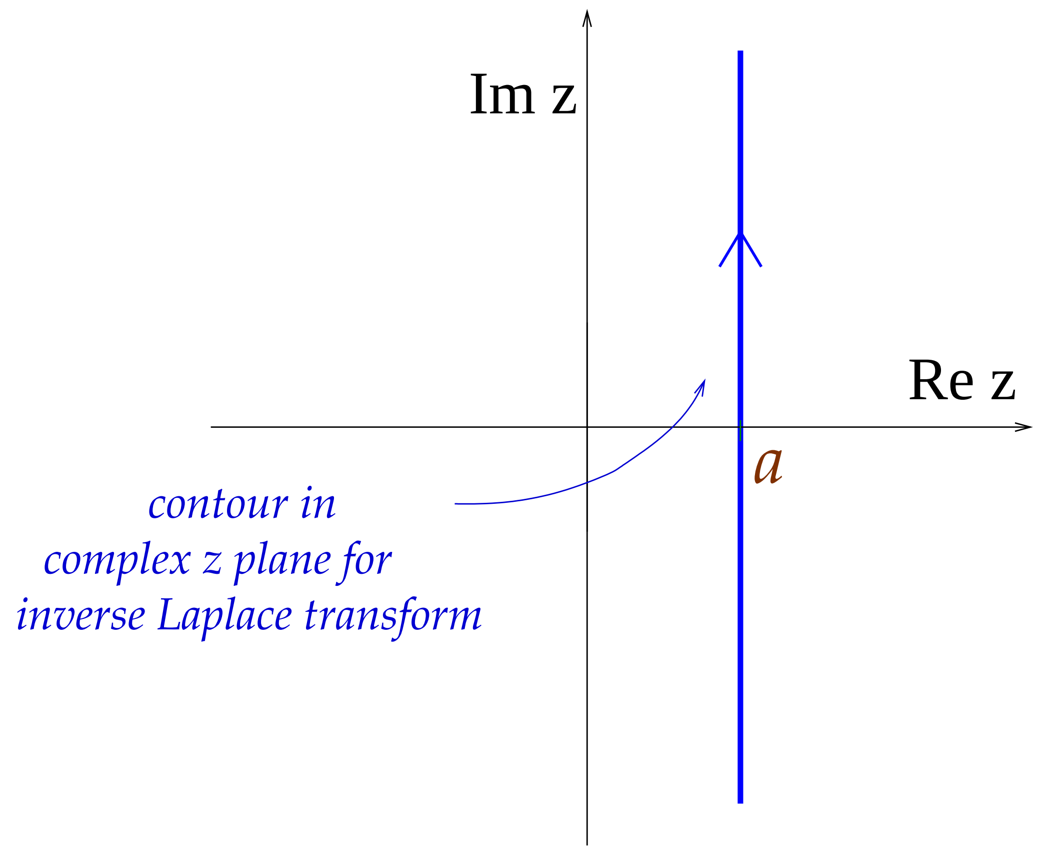
- Evaluating the convolution product gives

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\zeta \; \frac{y}{(\zeta - x)^2 + y^2} \; G(\zeta) \qquad \text{Poisson integral formula}$$

Laplace transformation

$$\widetilde{f}(z) = \int_0^{+\infty} e^{-zt}\ f(t)\ dt \equiv \mathrm{LT}\left[f\right]$$

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{zt}\ \widetilde{f}(z)\ dz \equiv \mathrm{LT}^{-1}\left[\widetilde{f}\right]$$



$\diamondsuit$ f exponentially bounded , i.e. $|f(t)| < Me^{\mu t}$, M, μ real positive constants

$\diamondsuit$ f defined for $t \geq 0$ (set $f = 0$ for $t < 0$)

- Consider $g(t) = f(t)e^{-\mu t}$. Define LT $[f]$ from FT $[g]$:

$$\text{FT } [g] = \int_{-\infty}^{+\infty} g(t)\ e^{-i\omega t}\ dt = \int_{0}^{+\infty} f(t)\ e^{-(\mu+i\omega)t}\ dt$$

$$= \int_{0}^{+\infty} f(t)\ e^{-zt} dt \equiv \text{LT } [f] = \widetilde{f}(z) \quad (z = \mu + i\omega)$$

- Construct inverse Laplace transform from FT^{-1} $[\widetilde{g}]$:

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\ e^{i\omega t}\ \text{FT } [g]$$

$$\text{i.e.,} \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\ e^{(\mu+i\omega)t}\ \widetilde{f}(z) = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} dz\ e^{zt}\ \widetilde{f}(z)$$

PROPERTIES OF LAPLACE TRANSFORMATION

linearity : $\mathrm{LT}\,[\alpha f + \beta g] = \alpha\ \mathrm{LT}\,[f] + \beta\ \mathrm{LT}\,[g]$

scaling : $\mathrm{LT}\,[f(\lambda t)] = \frac{1}{\lambda}\ \widetilde{f}(\frac{z}{\lambda})$

derivative of LT : $(-1)^n \widetilde{f}^{(n)} = \mathrm{LT}\,[t^n f]$

LT of derivative : $\mathrm{LT}\,[f'] = z\ \widetilde{f} - f(0)$

$$\mathrm{LT}\,[f^{(n)}] = z^n\ \widetilde{f} - z^{n-1} f(0) - \ldots - f^{(n-1)}(0)$$

convolution theorem : $\mathrm{LT}\,[f * g] = \widetilde{f}\ \widetilde{g}$

where $(f * g)(t) \overset{\mathrm{df}}{=} \int_0^t dy\ f(y)\ g(t-y)$

EXAMPLE: Laplace transforms of powers

$\lozenge$ For integer powers $f(t) = t^n \ , \quad n \in \mathbb{N}$

$$\mathrm{LT}\,[f] = \widetilde{f}(z) = \int_0^\infty dt \ e^{-zt} \ t^n = \frac{n!}{z^{n+1}} \quad \Rightarrow \quad \widetilde{f}(1) = n!$$

$\lozenge$ For non − integer powers $f(t) = t^\alpha \ , \quad \alpha \in \mathbb{C} \implies$ generalization of the factorial to complex variable : Euler gamma function defined for $\mathrm{Re}\,\alpha > 0$ as

$$\Gamma(\alpha) = \int_0^\infty dt \ e^{-t} \ t^{\alpha-1}$$

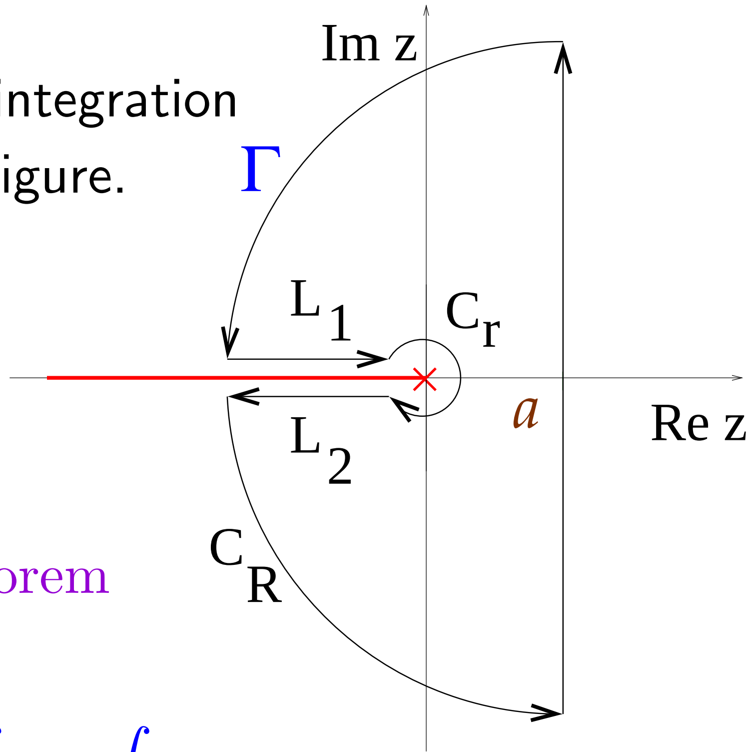
- For $\alpha = n \,, \quad \Gamma(n) = (n-1)!$
- can be continued to any α in $\mathbb{C}$; poles of order 1 at $\alpha = 0, -1, -2, \ldots$
- $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ (integrate by parts)
- $\Gamma(1/2) = \sqrt{\pi}$ (change variable $t \to \rho = \sqrt{t}$)

Thus $\quad \mathrm{LT}\,[t^\alpha](z) = \dfrac{\Gamma(\alpha+1)}{z^{\alpha+1}}$

Example: Compute the LT^{-1} of $F(z) = 1/\sqrt{z}$.

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{zt} \, \frac{1}{\sqrt{z}} \, dz \qquad (a > 0)$$

This integral can be computed by complex integration methods, closing the contour as in the figure.



- branch cut along $\mathbb{R}^-$
- $\oint_\Gamma e^{zt} \, \frac{1}{\sqrt{z}} \, dz = 0$ by Cauchy theorem
- Then write $\oint_\Gamma = \int_{a-i\infty}^{a+i\infty} + \int_{C_R} + \int_{L_1} + \int_{C_r} + \int_{L_2}$

Let $R \to \infty$, $r \to 0$. Apply Jordan lemma $\Rightarrow \int_{C_R} \to 0$, $\int_{C_r} \to 0$.

- Thus

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{zt} \, \frac{1}{\sqrt{z}} \, dz$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{0} dx \; \underbrace{\left[\frac{e^{xt}}{\sqrt{-x} \; e^{-i\pi/2}} - \frac{e^{xt}}{\sqrt{-x} \; e^{i\pi/2}} \right]}_{discontinuity \; across \; the \; branch \; cut}$$

- Change integration variable $x \to -x$. Then

$$f(t) = \frac{1}{2\pi i} \int_{0}^{\infty} dx \; \left[\frac{e^{-xt}}{\sqrt{x} \; (-i)} - \frac{e^{-xt}}{\sqrt{x} \; i} \right]$$

$$= \frac{1}{\pi} \int_{0}^{\infty} dx \; \frac{e^{-xt}}{\sqrt{x}} = \frac{1}{\pi} \int_{0}^{\infty} d\rho \; \frac{2}{\sqrt{t}} \; e^{-\rho^2} = \frac{1}{\sqrt{\pi t}}$$

Linear integral equations

$\diamondsuit$ Solve the equation for $y(t)$

$$y(t) = h(t) - \int_0^t K(t-\tau)\ y(\tau)\ d\tau \quad \text{where } h \text{ and } K \text{ are assigned functions}$$

$\diamondsuit$ Apply LT to the equation and use convolution theorem :

$$\widetilde{y}(z) = \widetilde{h}(z) - \widetilde{K}(z)\widetilde{y}(z) \quad \Rightarrow \quad \widetilde{y}(z) = \widetilde{h}(z)/(1+\widetilde{K}(z))$$

$\diamondsuit$ Then the solution is given by Laplace inverse transform :

$$y(t) = \mathrm{LT}^{-1}[\widetilde{y}] = \mathrm{LT}^{-1}[\widetilde{h}(z)/(1+\widetilde{K}(z))]$$

EXAMPLE

Determine the function $y(t)$ obeying the equation

$$y(t) = t + \int_0^t \sin(t-\tau)\ y(\tau)\ d\tau$$

$$h(t) = t \ \Rightarrow \ \widetilde{h}(z) = 1/z^2 \ \ ; \quad K(t) = -\sin t \ \Rightarrow \ \widetilde{K}(z) = -1/(1+z^2)$$

$$\text{Then} \quad \widetilde{y}(z) = \widetilde{h}(z)/(1+\widetilde{K}(z)) = (1+z^2)/z^4$$

$$\Rightarrow \quad y(t) = \mathrm{LT}^{-1}[\widetilde{y}] = \mathrm{LT}^{-1}[1/z^4 + 1/z^2] = t^3/3! + t = t^3/6 + t$$