Functions of a Complex Variable (S1)
Lecture 10

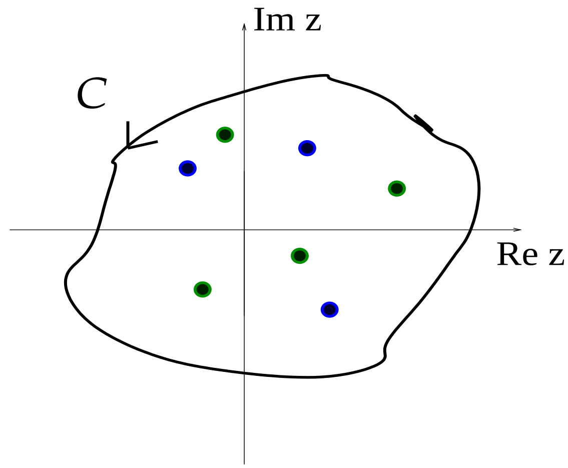
- The argument principle

  ▷ Winding number
  ▷ Counting zeros and poles
  ▷ Rouché theorem

- Applications to

  ▷ expansions in series of fractions
  ▷ infinite product expansions

The argument principle



♠ Let f be meromorphic inside and on a closed contour C, with no zeros or poles on C. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P = \frac{1}{2\pi}\,\Delta_C\,\arg f(z)$$

where N = number of zeros of f inside C,

P = number of poles of f inside C,

counted according to their multiplicity,

$\Delta_C\,\arg f(z)$ = change in the argument of f over C.

Part (a)

$$z_k = \text{ pole of order } n_k \implies \frac{f'(z)}{f(z)} = -\frac{1}{z - z_k}\, n_k + \underbrace{\phi(z)}_{\text{analytic}} \text{ for } z \text{ near } z_k$$

$$z_k = \text{ zero of order } n_k \implies \frac{f'(z)}{f(z)} = \frac{1}{z - z_k}\, n_k + \underbrace{\phi(z)}_{\text{analytic}} \text{ for } z \text{ near } z_k$$

$$\text{Residue theorem} \implies \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz = \sum_{j_z=1}^{N_z} n_{j_z} - \sum_{j_p=1}^{N_p} n_{j_p} = N - P$$

Part (b)

Let C be parameterized as $z = z(t)$ on $a \leq t \leq b$, with $z(a) = z(b)$. Then

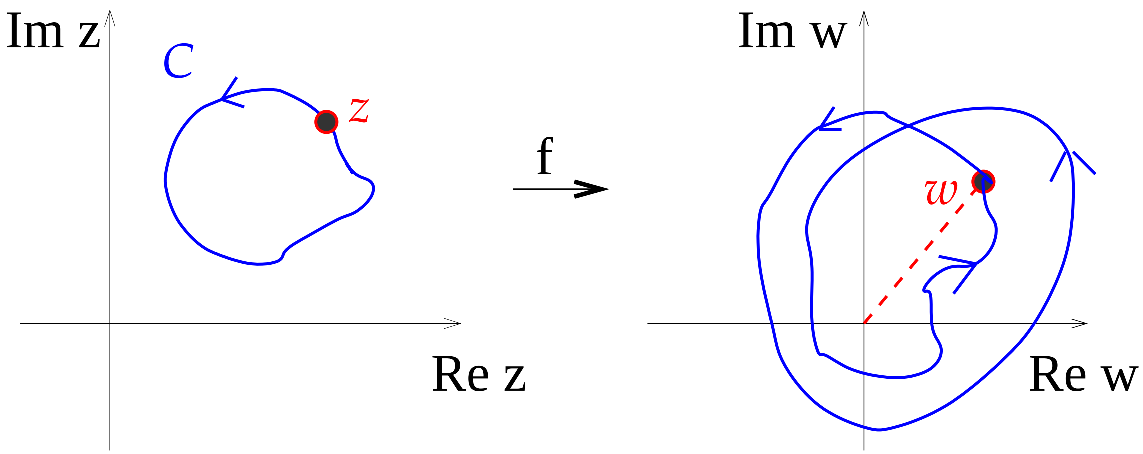
$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_a^b \frac{f'(z(t))}{f(z(t))}\, z'(t)\, dt$$

$$= \frac{1}{2\pi i}\, \ln\left[f(z(t))\right]\Big|_a^b$$

$$= \frac{1}{2\pi i}\, \underbrace{\ln\left[|f(z(t))|\right]\Big|_a^b}_{=\,0} + \frac{1}{2\pi i}\, i\left[\arg f(z(t))\right]\Big|_a^b$$

$$= \frac{1}{2\pi}\, \Delta_C \arg f(z)$$

Winding number



$w = f(z) = |f(z)|e^{i\theta}$

$\Delta_C \arg w/(2\pi)$ gives the number of times the point w winds around the origin in the image curve C' when z moves around C

$\Rightarrow$ "winding number" of C' about the origin

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{C'} \frac{dw}{w} = \frac{1}{2\pi}\,\Delta_C \arg w \equiv J$$

♣ The argument principle establishes a striking relationship between number of zeros $-$ number of poles of f in a domain D and how f maps the boundary ∂D of the domain, namely, the number of times the image of ∂D through f winds around the origin.

$$N - P = J$$

EXAMPLES

$$(a)\ f(z) = z^2 - 1 \quad ; \qquad C : |z-1| = 1$$

$$w = f(z) = |z|^2 e^{2i\theta} - 1$$

$$\Longrightarrow \ \Delta_C \ \arg w = 2\pi \ , \ i.e. \ , \ \ J = 1 \ \Longrightarrow \ N - P = 1$$

Indeed f has 1 zero (at $z = 1$) and no poles inside C.

$$(b)\ f(z) = \frac{z}{(z+1)^2} \quad ; \qquad C : |z| = 10$$

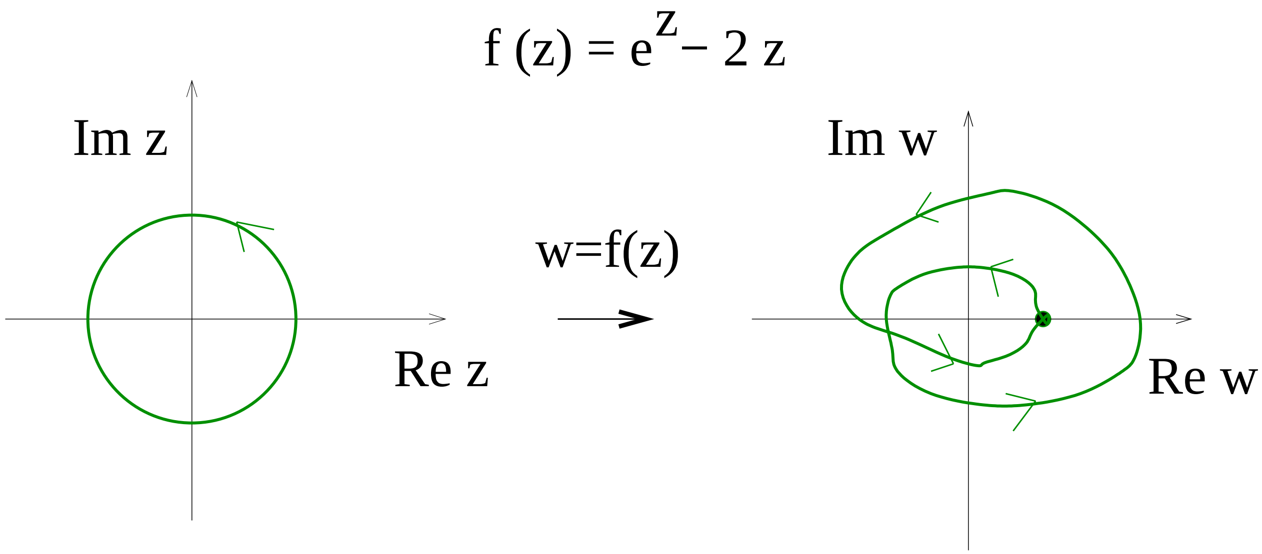
$$w = f(z) = \frac{|z|e^{i\theta}}{(|z|e^{i\theta}+1)^2} \approx \frac{1}{|z|} e^{-i\theta}$$

$$\Longrightarrow \ \Delta_C \ \arg w = -2\pi \ , \ i.e. \ , \ \ J = -1 \ \Longrightarrow \ N - P = -1$$

Indeed f has 1 zero (at $z = 0$) and 1 double pole (at $z = -1$) inside C.

EXAMPLE

- How many solutions does the equation $e^z - 2z = 0$ have inside the circle $|z| = 3$?



▷ image of the circle winds around the origin *twice*

▷ there are *no* poles

$\Rightarrow$ *two solutions*

A COROLLARY OF THE ARGUMENT PRINCIPLE: ROUCHÉ THEOREM

♠ Let $F(z)$ and $G(z)$ be holomorphic on and inside a closed contour C.
If $|F(z)| > |G(z) - F(z)|$ on C,
then $F(z)$ and $G(z)$ have the same number of zeros inside C.

$$\text{Let } \ w = \frac{G}{F} \ ; \ \text{ consider } \ \frac{1}{2\pi i} \oint_C \frac{w'(z)}{w(z)} \, dz \ \ .$$

$$|w(z) - 1| = \frac{|G - F|}{|F|} < 1 \ \text{ on } \ C \ \ .$$

Therefore the image of C lies inside $|w - 1| < 1$

$$\Longrightarrow \ \Delta_C \ \arg w = 0 \ \Longrightarrow \ N = P \ \text{ for } \ w(z) \ .$$

Thus the number of zeros of F (P) equals the number of zeros of G (N).

Rouché theorem may be used to

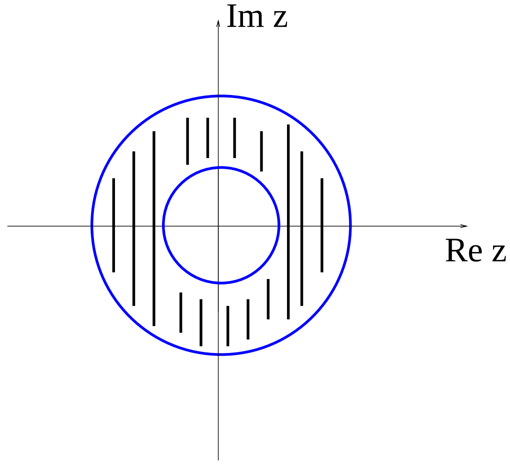
- locate solutions of equations in the complex plane
- arrive at results such as the fundamental theorem of algebra

(alternative proof to that based on Liouville theorem)

and maximum modulus principle.

EXAMPLE

◇ Show that the polynomial $P(z) = z^5 + 14z + 2$ has 4 roots in the annulus $3/2 < |z| < 2$.



• Consider C_2 circle $|z| = 2$. Take $G = P(z)$, $F(z) = z^5$.

$|G - F| < |F|$ on C_2 $\Longrightarrow$ $P(z)$ has as many zeros inside C_2 as $F(z)$, which is 5.

• Next consider C_1 circle $|z| = 3/2$. Take $G = P(z)$, $F(z) = 14z$.

$|G - F| < |F|$ on C_1 $\Longrightarrow$ $P(z)$ has as many zeros inside C_1 as $F(z)$, which is 1.

Thus $5 - 1 = 4$ zeros of $P(z)$ lie between C_2 and C_1.

AN EXTENDED VERSION OF THE ARGUMENT PRINCIPLE

$\Diamond$ If f and C satisfy the same hypotheses of the argument principle and $h(z)$ is holomorphic inside and on C, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\, h(z)\, dz = \sum_{j_z=1}^{N_z} n_{j_z} h(z_{j_z}) - \sum_{j_p=1}^{N_p} n_{j_p} h(z_{j_p})$$

EXPANSIONS BASED ON POLES OF A FUNCTION

- Taylor and Laurent series provide power series expansions of a function f.

- Other kinds of expansions can be useful based on poles of f:

$$z_n \; , \quad n = 1, \ldots, \infty \quad \text{poles of function } f(z)$$

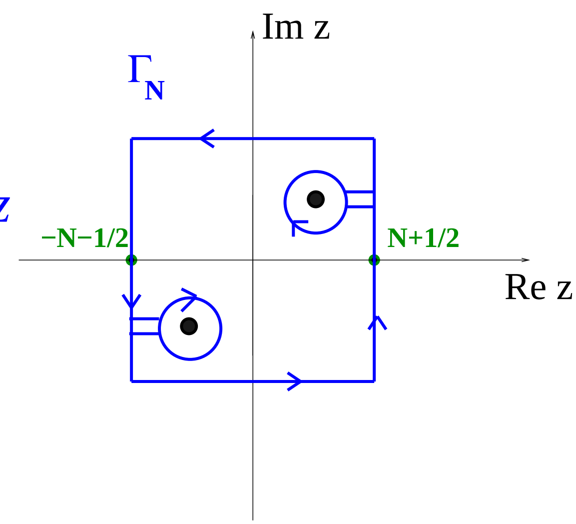
$$i) \quad f(z) = \sum_{n \in poles} g_n(z, n)$$

$$ii) \quad f(z) = \prod_{n \in poles} g_n(z, n)$$

▷ The (extended) argument principle may be used to obtain such expansions.

APPLICATION TO EXPANSION IN SERIES OF FRACTIONS

$$I(\alpha) = \frac{1}{2\pi i} \oint_{\Gamma_N} \frac{\pi \cot \pi z}{\alpha^2 - z^2} \, dz$$



- Use extended argument principle with $f(z) = \sin \pi z$, $h(z) = 1/(\alpha^2 - z^2)$, and $z_k = k \; (k \in \mathbb{Z})$ with $n_k = 1$.

$$\text{So} \;\; I(\alpha) = \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} \; .$$

- Now note that for $N \to \infty$ the integral on the square $\to 0$, and compute the integrals on the circles from the residues at $\pm\alpha$.

$$\Longrightarrow \quad \frac{\pi \cot \pi\alpha}{\alpha} = \sum_{n=-\infty}^{\infty} \frac{1}{\alpha^2 - n^2}$$

$$\Longrightarrow \quad \pi \cot \pi z = z \sum_{n=-\infty}^{\infty} \frac{1}{z^2 - n^2} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} = \frac{1}{z} + \frac{2z}{z^2 - 1} + \frac{2z}{z^2 - 4} + \ldots$$

- expansion of π cot πz in a series of fractions

  based on the poles of the function

◇ Expansion of function f, having poles z_j, $0 < |z_1| \leq \ldots \leq |z_j| \leq \ldots$, with residues r_j:

$$f(z) = f(0) + \sum_j r_j \left(\frac{1}{z - z_j} + \frac{1}{z_j} \right)$$

- For $f(z) =$ cot $z - 1/z$ this gives

$$\cot z = \frac{1}{z} + \sum_{n=\pm 1, \pm 2, \ldots} \left(\frac{1}{z - n\pi} + \frac{1}{n\pi} \right)$$

i.e., the expansion at the top:

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z - n\pi} + \frac{1}{z + n\pi} \right) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2 \pi^2}$$

AN EXAMPLE OF INFINITE PRODUCT EXPANSION

$$\frac{d}{dz} \ln \sin \pi z = \pi \; \cot \; \pi z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}$$

$$\Longrightarrow \ln \sin \pi z = \ln z + c_0 + \sum_{n=1}^{\infty} \; [\ln(z^2 - n^2) + c_n]$$

$$\text{with} \;\; c_0 = \ln \pi \;\; , \quad c_n = - \ln(-n^2)$$

$$\Longrightarrow \quad \sin \pi z = \pi z \; \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right)$$

▷ infinite-product expansion of the function $\sin \pi z$