FUNCTIONS OF A COMPLEX VARIABLE (S1)

Lecture 7

Power series expansions

▷ Taylor series

▷ Laurent series

▷ Classification of singularities

V. POWER SERIES EXPANSIONS

Basic concepts

- series of functions: $S = \sum_{n=0}^{\infty} s_n(z)$
- power series: $\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad , \quad a_n \in \mathbb{C}$
- radius of convergence: nonnegative real R such that series converges for $|z| < R$, diverges for $|z| > R$

♣ A power series $S = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is holomorphic in the region of convergence.

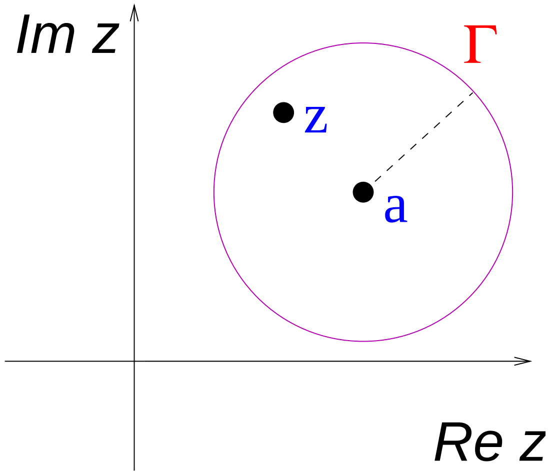
♣ Importantly, converse is also true: a holomorphic function can always be represented by a power series expansion ($\hookrightarrow$ Taylor).

Note: f representable by Taylor series expansion is said to be analytic.
Thus the two above results amount to the equivalence

- holomorphic $=$ analytic

TAYLOR SERIES EXPANSION

◊ Let f be holomorphic in simply connected domain D. Let $a \in D$.



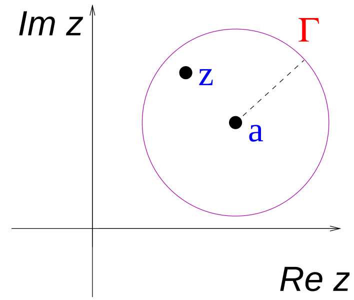
Then a) f has a series expansion in powers of $z - a$:

$$f(z) = f(a) + f'(a)(z-a) + \ldots + (f^{(n)}(a)/n!)(z-a)^n + \ldots$$

b) this expansion is unique;

c) the circle of convergence of the series is the circle of centre a passing through the nearest point to a where f is no longer holomorphic.

(a)



$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{\xi - z} \, d\xi \quad . \qquad |z-a| < |\xi - a| \implies \frac{1}{\xi - z} =$$

$$= \frac{1}{\xi - a} \frac{1}{1 - (z-a)/(\xi - a)} = \frac{1}{\xi - a} + \frac{z-a}{(\xi-a)^2} + \ldots + \frac{(z-a)^n}{(\xi-a)^{n+1}} + \ldots$$

- Integrate term by term:

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{\xi - a} \, d\xi + \frac{z-a}{2\pi i} \oint_\Gamma \frac{f(\xi)}{(\xi-a)^2} \, d\xi + \ldots + \frac{(z-a)^n}{2\pi i} \oint_\Gamma \frac{f(\xi)}{(\xi-a)^{n+1}} \, d\xi + \ldots$$

- Apply Cauchy integral formulas of order n:

$$f(z) = f(a) + f'(a)(z-a) + \ldots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \ldots$$

Taylor series expansion of f about $z = a$

(b)

Show that the expansion is unique.

- Suppose there exists another expansion

$$f(z) = c_0 + c_1(z-a) + \ldots + c_n(z-a)^n + \ldots$$

- Consider $\displaystyle \oint_\Gamma \frac{f(z)}{(z-a)^k}\, dz = c_0 \oint_\Gamma \frac{1}{(z-a)^k}\, dz + \ldots + c_n \oint_\Gamma (z-a)^{n-k}\, dz + \ldots$

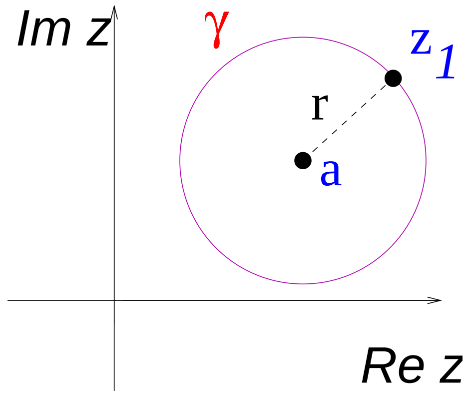
- generic term : $\displaystyle \oint_\Gamma (z-a)^p\, dz = r^{p+1} \underbrace{i \int_0^{2\pi} d\theta\, e^{i(p+1)\theta}}_{= 2\pi i \text{ if } p=-1 \ ,\ 0 \text{ if } p \neq -1}$. Set $r = 1$.

$\Longrightarrow$ only the term with $n = k-1$ is $\neq 0$

Then $\displaystyle \oint_\Gamma \frac{f(z)}{(z-a)^k}\, dz = 2\pi i c_{k-1} \implies c_{k-1} = \frac{f^{(k-1)}(a)}{(k-1)!}$

(c)
radius of convergence R

- Let z_1 be the point nearest to a at which f is no longer holomorphic.
- Let r be the radius of the circle γ centred in a passing through z_1.



◊ f holomorphic in circle of convergence $\Rightarrow$ $R \leq r$

◊ by the same reasoning as in part a) for any z_2 inside γ there exists circle containing z_2 where Taylor expansion holds $\Rightarrow$ $R \geq r$

$$\text{Thus } R = r \ .$$

EXAMPLES OF TAYLOR SERIES

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

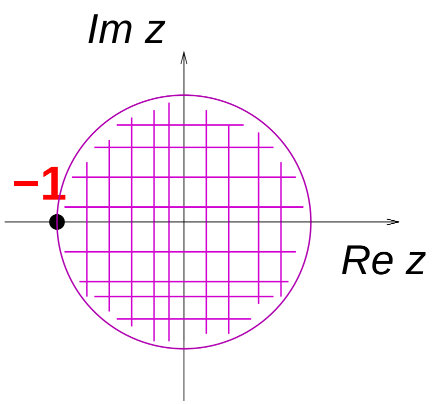
$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

• the above functions are holomorphic over the whole $\mathbb{C}$ ("entire functions")

$\Rightarrow$ radius of convergence of the above series expansions is $R = \infty$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

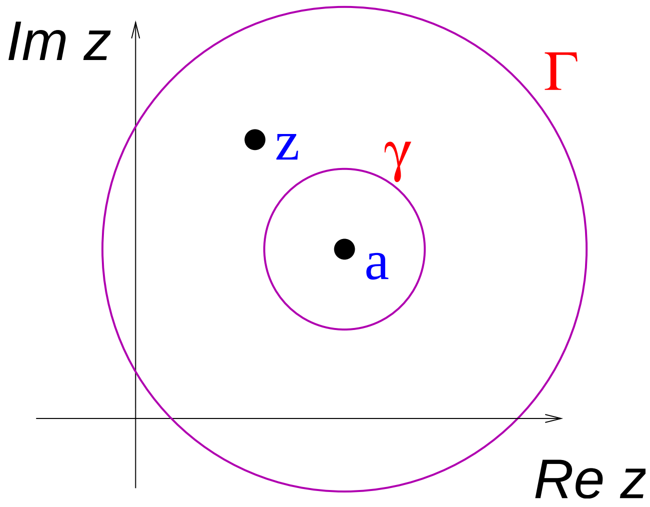


• this series converges for $|z| < 1$, i.e. $R = 1$,

because $z = -1$ is an isolated singularity of the function

LAURENT SERIES EXPANSION

• generalizes Taylor to functions that fail to be holomorphic at isolated points
$\longrightarrow$ extension of Taylor to multiply connected regions D

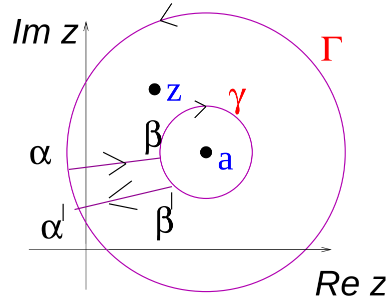


$\Diamond$ f holomorphic in annulus $\mathcal{A}$ between Γ and γ. Then for any $z \in \mathcal{A}$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \ , \quad \text{where} \quad c_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{(\xi - a)^{n+1}} \, d\xi$$

• $n < 0$: "principal part"; $n \geq 0$ "analytic part".

Proof



$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{\xi - z}\, d\xi - \oint_\gamma \frac{f(\xi)}{\xi - z}\, d\xi + \underbrace{\frac{1}{2\pi i} \int_{\alpha\beta} \frac{f(\xi)}{\xi - z}\, d\xi + \int_{\beta'\alpha'} \frac{f(\xi)}{\xi - z}\, d\xi}_{\to\ 0 \text{ for } \alpha' \to \alpha,\ \beta' \to \beta}$$

◇ Rewrite both integrals on Γ and γ by power expanding $1/(\xi - z)$ and integrating term by term:

- $$\frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{\xi - z}\, d\xi = \sum_{n=0}^{\infty} c_n (z-a)^n \quad , \quad \text{where} \quad c_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{(\xi - a)^{n+1}}\, d\xi$$

Note: cannot conclude that $c_n = f^{(n)}(a)/n!$ because Cauchy integral formulas do not apply to Γ contour.

- $$-\frac{1}{2\pi i} \oint_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = \sum_{n=1}^{\infty} \frac{1}{(z-a)^n} \underbrace{\frac{1}{2\pi i} \oint_\gamma f(\xi)(\xi - a)^{n-1}\, d\xi}_{c_{-n} \ \text{(deform contour from } \gamma \text{ to } \Gamma)}$$

◊ Put the two contributions together:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} c_{-n} \frac{1}{(z-a)^n}$$

$$= \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad , \quad \text{where} \quad c_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{(\xi - a)^{n+1}} \, d\xi$$

Laurent series expansion of f about $z = a$

Remarks

♠ Laurent expansion generalizes Taylor expansion:
if D is simply connected all $c_{-|n|}$ are zero.

♠ Laurent expansion is unique.

♠ Region of convergence is annulus $R_\gamma < |z-a| < R_\Gamma$;
as in Taylor, it may be enlarged to nearest singular point.

Examples

$$f(z) = \frac{z+1}{z-1}$$

Taylor series $\quad f(z) = -1 - 2z - 2z^2 + \ldots = -1 - 2\sum_{n=1}^{\infty} z^n \ , \quad |z| < 1$

Laurent series $\quad f(z) = 1 + \frac{2}{z} + \frac{2}{z^2} + \ldots = 1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n} \ , \quad 1 < |z| < \infty$

- By Laurent series one can also expand about a singular point:

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\right) = \underbrace{\frac{1}{z^3} - \frac{1}{6z}}_{\text{principal part}} + \underbrace{\frac{z}{120} + \ldots}_{\text{analytic part}} \ , \quad |z| > 0$$

$$f(z) = \frac{1}{z^2 \sinh z} = \frac{1}{z^2} \frac{1}{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots} = \underbrace{\frac{1}{z^3} - \frac{1}{6z}}_{\text{principal part}} + \underbrace{\frac{7}{360} z + \ldots}_{\text{analytic part}} \ , \quad 0 < |z| < \pi$$

CLASSIFICATION OF SINGULAR POINTS

a is zero of f if $f(a) = 0$.

a is zero of order n if $f(a) = f'(a) = \ldots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$.

♠ a is pole of order n of f if a is zero of order n of $1/f$.

$$\Rightarrow f(z) = \frac{h(z)}{(z-a)^n} \; , \quad h(z) = h_0 + h_1(z-a) + \ldots \; , \quad h_0 \neq 0$$

$$\text{Therefore} \;\; f(z) = \frac{h_0}{(z-a)^n} + \frac{h_1}{(z-a)^{n-1}} + \ldots + h_n + h_{n+1}(z-a) + \ldots$$

The principal part of the Laurent series terminates

at $(z-a)^{-n}$ for a pole of order n

♠ a is essential singularity if the principal part of the Laurent series does not terminate to any finite n, i.e., it has infinitely many terms

$$\text{Example}: e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n}$$

♠ Note: a is said to be a removable singularity of f if f does not exist at $z = a$ but $\lim_{z \longrightarrow a} f$ exists.

$$\text{Example}: f(z) = \frac{\sin z}{z} \text{ at } z = 0$$

Then f can be made holomorphic by redefinition of only one point.

$\Rightarrow$ The principal part of the Laurent series has no terms in the case of a removable singularity