Lecture 6

COMPLEX INTEGRATION, Part II

▷ Cauchy integral formulas

▷Application to evaluating contour integrals

▷ Application to boundary value problems

Poisson integral formulas

▷ Corollaries of Cauchy formulas

Liouville theorem

Fundamental theorem of algebra

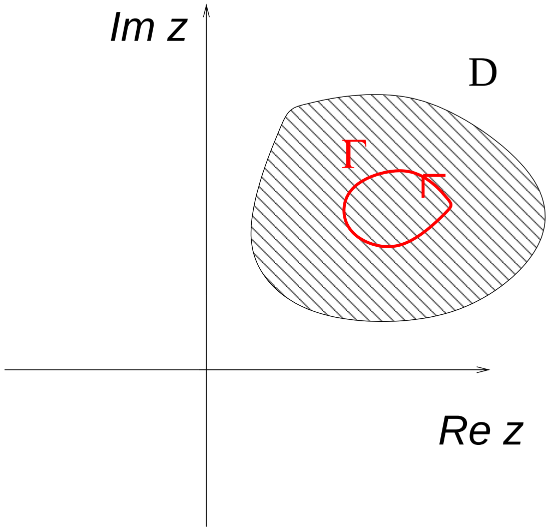
Gauss' mean value theorem

Maximum modulus

A. CAUCHY THEOREM

f holomorphic on
D simply connected domain.
Γ closed path in D. Then

$$\oint_\Gamma f(z)\,dz = 0$$



Implications

- can "deform" integration contour as long as we don't cross singular points
- "primitive function" $F(z) = \int_{z_0}^{z} f(\xi) d\xi$ exists under Cauchy theorem hypotheses and is holomorphic with $F' = f$

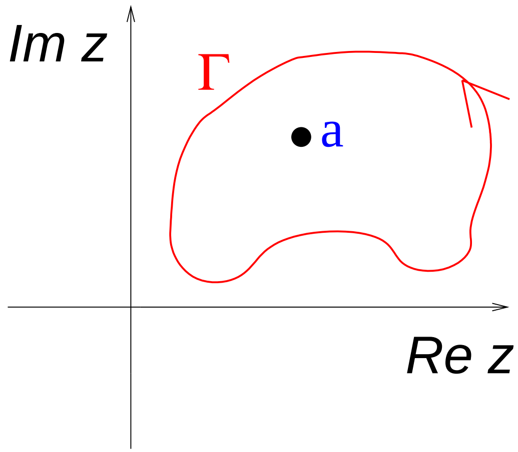
◊ Note: Cauchy-Goursat version of proof does not <u>assume</u> continuity of f'

B. CAUCHY INTEGRAL FORMULAS

B.1 Cauchy integral formula of order 0

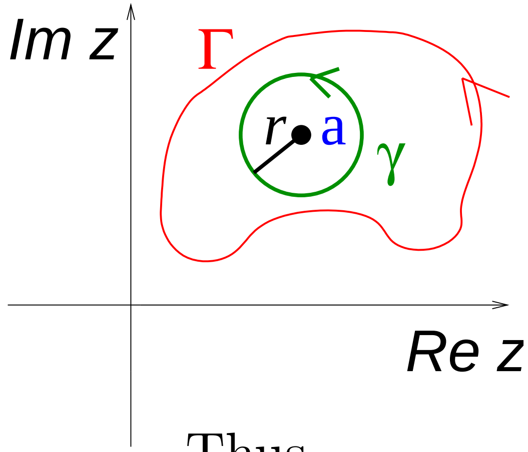
$\diamondsuit$ Let f be holomorphic in simply connected domain D.
Let $a \in D$, and Γ closed path in D encircling a. Then

$$f(a) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z-a}\, dz$$



• value of holomorphic f at any point
fully specified by the values f takes
on any closed path surrounding the point!

Proof



$f/(z-a)$ holomorphic in region between Γ and γ

$$\Rightarrow \oint_\Gamma \frac{f(z)}{z-a}\, dz = \oint_\gamma \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a+re^{i\theta})\, d\theta \ .$$

Thus

$$\left|\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-a}\, dz - f(a)\right| = \left|\frac{1}{2\pi}\int_0^{2\pi}[f(a+re^{i\theta})-f(a)]\, d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi}|f(a+re^{i\theta})-f(a)|\, d\theta$$

f continuous $\Rightarrow$ for any arbitrarily small $\varepsilon > 0$ one can always find r small enough that $|f(a+re^{i\theta})-f(a)| < \varepsilon$, so that

$$\left|\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-a}\, dz - f(a)\right| \le \varepsilon$$

$$\varepsilon \ \text{arbitrary} \quad \Rightarrow \quad \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-a}\, dz = f(a)$$

B.2 Cauchy integral formulas of order n

◊ Let f be holomorphic in simply connected domain D. Let Γ be closed path in D. Then f is infinitely differentiable at any point a encircled by Γ, and

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

• can differentiate a holomorphic f
to arbitrarily high order
by performing a suitable integral of it

for holomorphic functions

• once differentiable $\Rightarrow$ infinitely many times differentiable

(in sharp contrast with real variable case)

Proof

Apply Cauchy integral formula of order 0 to $f(a)$ and $f(b)$ with a and b points surrounded by Γ:

$$f(a) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z-a}\, dz \quad , \quad f(b) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z-b}\, dz$$

$$\Longrightarrow \quad \frac{f(b)-f(a)}{b-a} = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-a)(z-b)}\, dz$$

f holomorphic $\Rightarrow$ f continuous $\Rightarrow$ integral of $f/(z-b)$ continuous function of b.

Let $b \to a$

$$\Longrightarrow \quad \lim_{b\to a} \frac{f(b)-f(a)}{b-a} = \lim_{b\to a} \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-a)(z-b)}\, dz$$

$$= \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-a)^2}\, dz$$

$$\text{i.e.} \quad f'(a) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-a)^2}\, dz$$

Repeat this for derivatives of arbitrary order n.

C. EXAMPLES: EVALUATION OF CONTOUR INTEGRALS BY CAUCHY FORMULAS

$$\oint_{|z-1|=2} \frac{\cos z}{z}\, dz = 2\pi i [\cos z]_{z=0} = 2\pi i$$

$$\oint_{|z|=3} \frac{e^z}{z-2}\, dz = 2\pi i \left[e^z\right]_{z=2} = 2\pi i e^2$$

$$\oint_{|z|=1} \frac{\cosh z}{z^4}\, dz = \frac{2\pi i}{3!} \left[\frac{d^3}{dz^3} \cosh z\right]_{z=0} = 0$$

D. APPLICATION TO BOUNDARY VALUE PROBLEMS

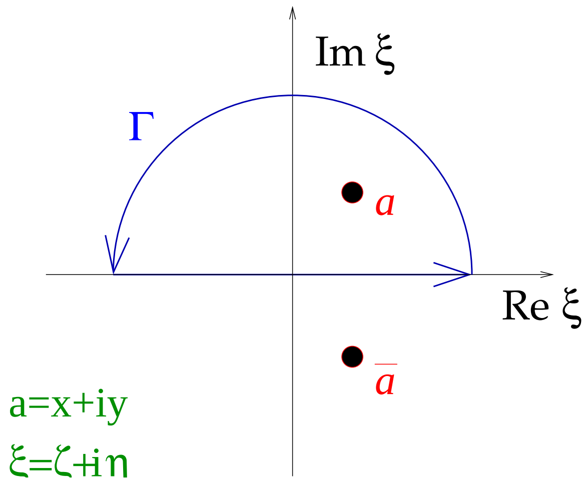
Example : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad , \qquad y > 0$

$$u(x,0) = G(x) \qquad \text{(Dirichlet boundary condition)}$$

- Let $f = u + iv$. Apply Cauchy integral formula and Cauchy theorem as follows.

$$f(a) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{\xi - a}\, d\xi$$

$$0 = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{\xi - \bar{a}}\, d\xi$$



- Subtracting one equation from the other and taking the real part yields

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\zeta \, \frac{y}{(\zeta - x)^2 + y^2} \, u(\zeta, 0) \qquad \text{Poisson integral formula}$$

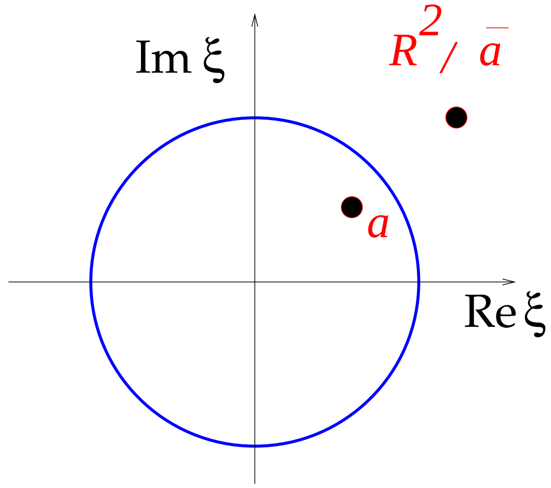
Dirichlet problem for the circle:

$\Delta u = 0$ inside circle C

$u(R, \varphi) = G(\varphi)$ on circle C

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - a} \, d\xi$$

$$0 = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - R^2/\bar{a}} \, d\xi$$



- Subtracting the two eqs. and taking Re yields
  Poisson integral formula for the circle:

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} \, u(R, \varphi)$$

$$\text{Note : for } \; r = 0 \;, \quad u(r = 0) \; = \; \frac{1}{2\pi} \int_0^{2\pi} d\varphi \; u(R, \varphi)$$

- the value of a harmonic function at the center of the circle is the average of its boundary values on the circle

$\mathbf{E}$. COROLLARIES OF CAUCHY FORMULAS

A host of results on holomorphic functions follow from Cauchy integral formulas.

- If $|f| \leq M$ on circle of centre z and radius r, then

$$|f^{(n)}(z)| \leq \frac{n!\ M}{r^n} \quad \text{Cauchy inequality}$$

[Apply Darboux inequality to Cauchy formula of order n.]

- An entire, bounded f must be constant. (Liouville theorem)

[Apply Cauchy inequality at $n = 1$ for arbitrarily high r.]

- Every complex polynomial $P(z)$ of degree $n > 0$ has exactly n roots. (Fundamental theorem of algebra)

[Apply Liouville to $f = 1/P$ to show there is at least 1 root; then proceed iteratively.]

• If f is continuous on simply connected domain D and

$$\oint_\Gamma f(z)dz = 0$$

for any closed path Γ in D, then f is holomorphic in D.
(Morera theorem — a converse of Cauchy theorem)

[Construct primitive function $F(z)$ of f; then use that, from Cauchy integral formulas of order n, the derivative of a holomorphic function is holomorphic.]

• f holomorphic on disk $|z-z_0| \leq r \implies \frac{1}{2\pi}\int_0^{2\pi} d\theta\, f(z_0+re^{i\theta}) = f(z_0)$

(Gauss' mean value theorem)

[Apply Cauchy integral formula of order 0 to the circle of centre z_0 and radius r.]

• If f is holomorphic on a bounded domain R and continuous on the boundary ∂R, then the maximum value of $|f(z)|$ must occur on the boundary, unless f is constant. (Maximum modulus theorem)

[It can be proved starting from Gauss' mean value.]

Summary

COMPLEX INTEGRATION

- Definition of complex integrals in terms of line integrals
- Cauchy theorem
- Cauchy integral formulas: order-0 and order-n
- Boundedness formulas: Darboux inequality, Jordan lemma
- Applications:
  - ▷ evaluation of contour integrals
  - ▷ properties of holomorphic functions
  - ▷ boundary value problems