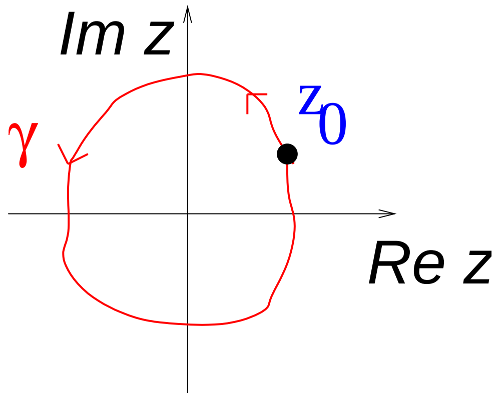
III. MULTI-VALUED FUNCTIONS

$$\text{Ex.}: \quad \ln z = \ln |z| + i(\theta + 2\pi n) \quad , \qquad n = 0, \pm 1, \pm 2, \ldots$$

- γ closed path encircling $z = 0$. Start at z_0 and let z vary along γ.



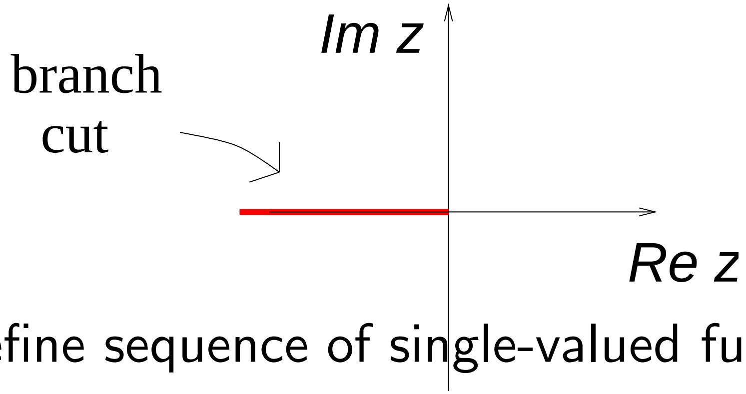
$$\text{after full cycle}: \quad (\ln z_0)_{final} = (\ln z_0)_{initial} + 2\pi i \neq (\ln z_0)_{initial}$$

$$\Uparrow$$

$z = 0$ branch point

Note: $z = \infty$ is also branch point for $\ln z$.

• Imagine "cutting" the complex plane along a line joining the branch points:



• Define sequence of single-valued functions in "cut" plane:

$$f_n(z) = f_n(r,\theta) = \ln r + i(\theta + 2\pi n) \;\;,\;\; -\pi \leq \theta \leq \pi \;\; (n = 0, \pm 1, \pm 2, \ldots)$$

• Each is discontinuous across the cut:

$$\lim_{\varepsilon \longrightarrow 0} [f_n(r, \pi - \varepsilon) - f_n(r, -\pi + \varepsilon)] = 2\pi i$$

• On the other hand, f_n above the cut $=$ f_{n+1} below the cut:

$$\lim_{\varepsilon \longrightarrow 0} f_n(r, \pi - \varepsilon) = \lim_{\varepsilon \longrightarrow 0} f_{n+1}(r, -\pi + \varepsilon)$$

• Construct Riemann surface $=$ stack of cut complex planes, joined along the cut.

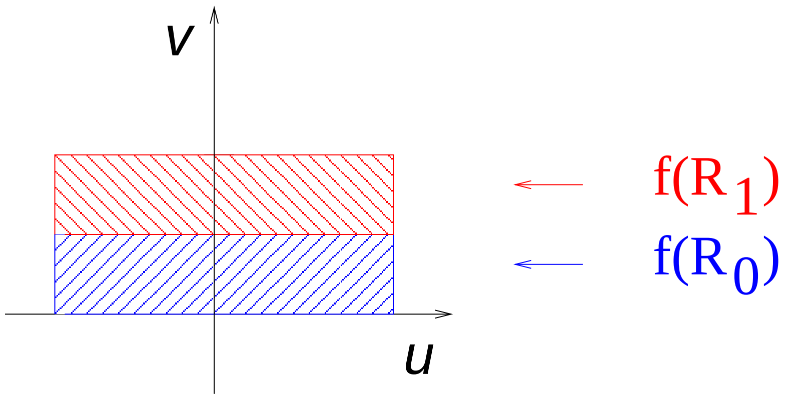
$\Rightarrow$ single-valued f defined on Riemann surface

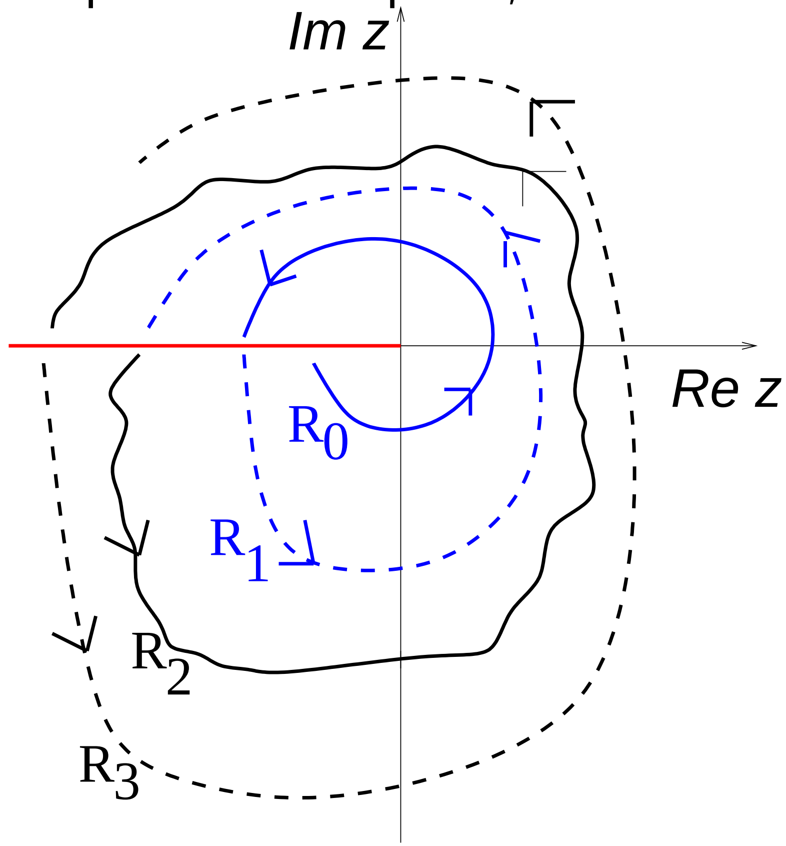
RIEMANN SURFACE

• Helix-like superposition of "cut" planes, with upper edge of cut in n-th plane joined with lower edge of cut in $(n+1)$-th plane. Each plane is "Riemann sheet".

▷ $\ln z$ single-valued and holomorphic, except at branch point, on Riemann surface.

w=ln z maps Riemann sheets R_0, R_1, ... onto horizontal strips:





A cycle around a branch point makes us move to another Riemann sheet.

Classification of branch points

$\Diamond$ branch point of order n: if the function is restored to the starting value by taking $n+1$ cycles around it

Ex.: $\sqrt{z}$ branch point of order 1; $\sqrt[3]{z}$ branch point of order 2

$\Diamond$ branch point of order ∞: if successive cycles around it bring us further and further away from initial Riemann sheet

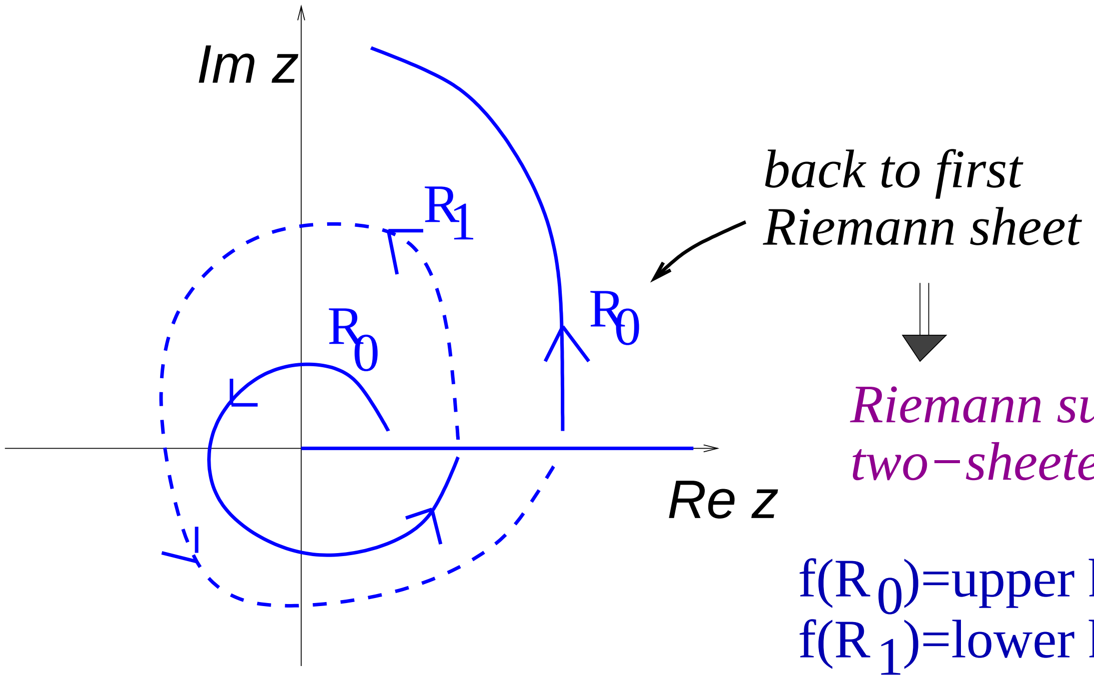
Ex.: $\ln z$ branch point of order ∞

Note 3 distinct representations:

- multi-valued function on complex plane
- sequence of single-valued, discontinuous functions on "cut" plane
- one single-valued, continuous function on Riemann surface

ROOT FUNCTIONS AND THEIR RIEMANN SURFACES

$f(z) = z^{1/2}$: z=0, z=infinity are branch points.



Riemann surface is two−sheeted and closed.

$f(R_0)$=upper half plane
$f(R_1)$=lower half plane

Note : Riemann surface for $f(z)=z^{1/n}$ is a closed, n−sheeted surface.
The n−th sheet is reconnected to the first sheet.

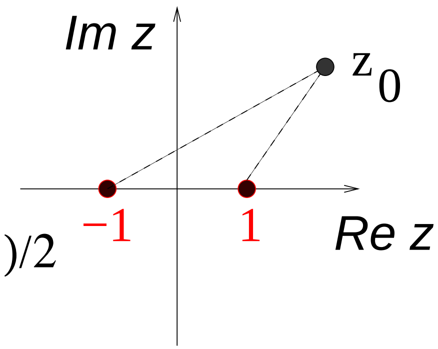
RIEMANN SURFACE OF $f(z) = \sqrt{z^2 - 1}$

$z-1=r_1 e^{i\theta_1}$

$z+1=r_2 e^{i\theta_2}$

$\Downarrow$

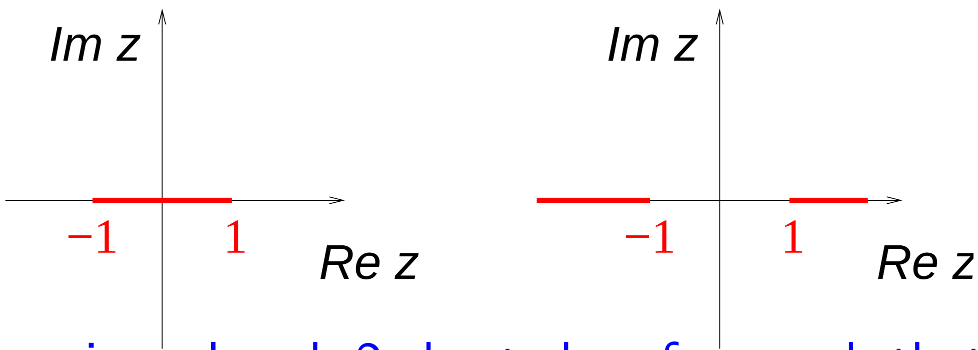
$f(z)=(r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2}$



z=1 and z=–1 are branch points.

z=infinity is not branch point.

Valid branch cuts:

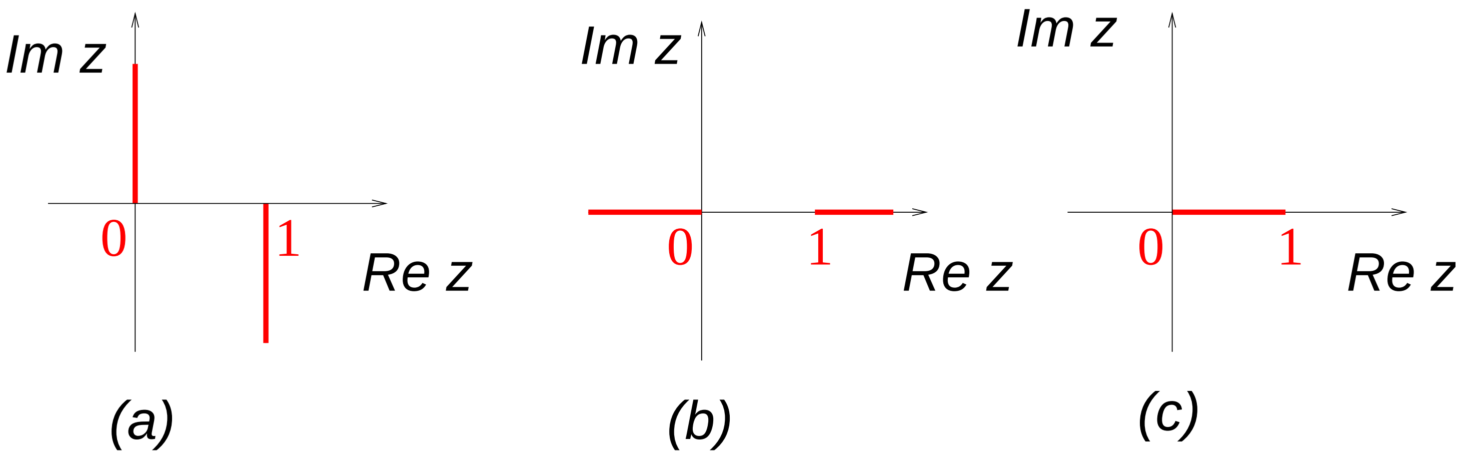


- Riemann surface is a closed, 2-sheeted surface such that any cycle surrounding one of the branch points brings us to a new sheet, whereas any cycle surrounding both branch points restores f to initial value.

$$f(z) = z^{1/2}(z-1)^{1/2}$$

$z = 0$, $z = 1$ are 1st-order branch points

Valid branch cuts:



$$\text{Note}: f(z) = z^{1/3}(z-1)^{1/3}$$

has 2nd-order branch points at $z = 0$, $z = 1$, $z = \infty$;

cuts *(a)* and *(b)* above are valid cuts for this function as well, while *(c)* is not.

INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

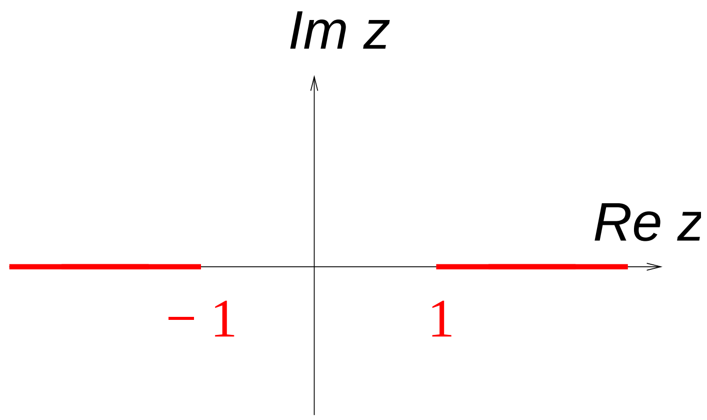
$$\arccos z = \frac{1}{i} \ln(z + \sqrt{z^2 - 1})$$

has branch points of order 1 at $z = 1$, $z = -1$ and

branch point of order ∞ at $z = \infty$.

Valid cuts are semi-infinite lines stretching from ± 1 out to ∞.

For example:



• arccos z has two infinite sets of values for each value of $z \neq \pm 1$:
two possible values of square root and, for each, infinitely many values of log.

◊ Similarly for the other trigonometric and hyperbolic inverse functions.

Summary

◊ Multi-valued functions can be characterized through their

branch points

by specifying a valid set of

branch cuts.

order n — f restored to starting value by taking $n+1$ cycles around it

order ∞ — f never restored to starting value by successive cycles

◊ *Riemann surfaces* provide a setting for defining complex functions — more natural than complex plane itself