Dispersive waves, phase velocity and group velocity

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1. Introduction

In these notes, we use sinusoidal waves to solve linear PDEs. The resulting waves do not behave as the solution to the wave equation, and wavepackets are one interesting instance in which the difference between general sinusoidal solutions and the solutions to the wave equation is very clear. By studying the behavior of wavepackets, we will learn about phase and group velocities.

2. Dispersive waves and dispersion relations

We have used sinusoidal solutions of the form $e^{\mathrm{i}kx-\mathrm{i}\omega t}$ to solve the wave equation. We found that, to be solutions of the wave equation, the frequency of these sinusoidal waves had to satisfy $\omega(k) = \pm kc$. Importantly, solutions of the form $e^{\mathrm{i}kx-\mathrm{i}\omega t}$ are not only solutions to the wave equation, but they can be solutions to a variety of systems of linear PDEs as long as the frequency has the right value, $\omega(k)$, determined by the PDEs. The relation between the frequency ω and the wavenumber k is known as **dispersion relation**. To show that we can use sinusoidal waves for a variety of linear PDEs, we consider two examples:

- to demonstrate that sinusoidal waves are solutions to linear PDEs with constant coefficients, we solve for the corrected motion of a stretched string, and
- to show that dispersion relations arise even in systems of linear PDEs without constant coefficients, we discuss the motion of the interface surface between a liquid and the atmosphere.

2.1. *Bending correction to the motion of a stretched string*

For the motion of a stretched string of linear density μ and tension T, one of the approximations that led to the wave equation was that the force exerted by the rest of the string on an infinitesimal piece of string can only point along the string. In reality, there is always a small component of the force perpendicular to the string, $S(x,t)$, as shown in figure 1. This force resists bending, that is, it opposes changes of the angle $\theta(x,t) \simeq \partial y/\partial x$ with x. By introducing the force $S(x,t)$, we prevent the appearance of sharp corners in $y(x,t)$.

The force $S(x,t)$ is related to the derivatives of $y(x,t)$ by

$$S(x,t) = -TL_b^2 \frac{\partial^3 y}{\partial x^3}, \tag{2.1}$$

where L_b is a length that depends on the tension T, properties of the material and the shape of the cross section of the string. See Appendix A for a very brief discussion of how to obtain equation (2.1).

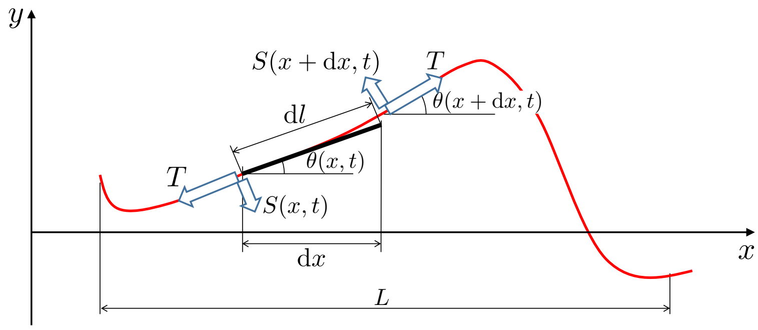

FIGURE 1. Transverse oscillations of a stretched string of linear density μ tensioned by a force T including the small forces $S(x,t)$ perpendicular to the string.

When we keep $S(x,t)$ in the equations for the motion of the string, vertical force balance of the infinitesimal piece of string in figure 1 gives

$$\mu\,\mathrm{d}x\frac{\partial^2 y}{\partial t^2} = T\sin\theta(x+\mathrm{d}x,t) + S(x+\mathrm{d}x,t)\cos\theta(x+\mathrm{d}x,t) - T\sin\theta(x,t) - S(x,t)\cos\theta(x,t). \tag{2.2}$$

Using $\theta(x,t) \ll 1$, we find

$$\mu\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T\theta + S\right). \tag{2.3}$$

Employing $\theta \simeq \partial y/\partial x$ and equation (2.1) for the force $S(x,t)$, we find the modified wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial x^2} - L_b^2\frac{\partial^4 y}{\partial x^4}\right), \tag{2.4}$$

where $c = \sqrt{T/\mu}$.

Equation (2.4) does not have a d'Alembert solution, but it can be solved by using the assumption $y(x,t) \propto e^{\mathrm{i}kx-\mathrm{i}\omega t}$. With this assumption for $y(x,t)$, differentiation becomes an algebraic operation,

$$\frac{\partial y}{\partial t} = -\mathrm{i}\omega y, \quad \frac{\partial y}{\partial x} = \mathrm{i}ky, \tag{2.5}$$

and since the coefficients c and L_b are constants, the PDE in equation (2.4) gives

$$\left[k^2c^2(1+k^2L_b^2) - \omega^2\right]y(x,t) = 0. \tag{2.6}$$

Thus, the sinusoidal wave is a solution if ω satisfies

$$\omega(k) = \pm kc\sqrt{1+k^2L_b^2}. \tag{2.7}$$

The sinusoidal waves that we have found move rigidly with a velocity known as **phase velocity** v_p, $y(x,t) = f(x - v_pt)$. Indeed, for a general dispersion relation $\omega(k)$, we find

$$y(x,t) = Ce^{\mathrm{i}kx-\mathrm{i}\omega(k)t} = Ce^{\mathrm{i}k(x-v_p(k)t)}, \tag{2.8}$$

where the phase velocity is

$$v_p(k) = \frac{\omega(k)}{k}. \tag{2.9}$$

In the case of the wave equation, we find that $v_p(k) = \pm c$ does not depend on k, and all sinusoidal waves, regardless of their wavenumber, move at the same velocity. For waves such as the ones represented by the dispersion relation (2.7), however, the phase velocity depends on k, and waves with different wavenumber k move at different speeds. Thus, a shape constructed by adding several sinusoidal waves will not keep its shape while moving, but will disperse in a collection of waves that move at different speeds. For this reason, waves that do not have a phase velocity independent of the wavenumber are known as **dispersive waves**.

Note that equation (2.7) contains two possible branches for $\omega(k)$, given by

$$\omega_+(k) = kc\sqrt{1+k^2L_b^2}, \quad \omega_-(k) = -kc\sqrt{1+k^2L_b^2}. \tag{2.10}$$

These two branches represent two waves traveling in opposite direction. We have constructed the two branches of the solution in (2.10) to satisfy the property

$$\omega(-k) = -\omega^*(k), \tag{2.11}$$

where * indicates complex conjugation. This condition is satisfied by any dispersion relation that solves a linear PDE with real coefficients. The solutions to the equation have to be real, and to be able to construct real solutions using the wave $e^{\mathrm{i}kx-\mathrm{i}\omega(k)t}$, its complex conjugate $e^{-\mathrm{i}kx+\mathrm{i}\omega^*(k)t}$ has to be a solution to the PDE. This is only possible if condition (2.11) is satisfied.

The two different waves in the dispersion relation (2.10) are required when imposing the two necessary initial conditions: displacement and velocity of the wave. Indeed, consider the two initial conditions

$$y(x,t=0) = \frac{1}{2}Ye^{\mathrm{i}kx} + \text{complex conjugate} \tag{2.12}$$

and

$$\frac{\partial y}{\partial t}(x,t=0) = \frac{1}{2}\dot{Y}e^{\mathrm{i}kx} + \text{complex conjugate,} \tag{2.13}$$

where Y and $\dot{Y}$ are two complex constants. The solution for $t>0$ will only have the two waves

$$y(x,t) = \frac{1}{2}\left(C_+e^{\mathrm{i}kx-\mathrm{i}\omega_+(k)t} + C_-e^{\mathrm{i}kx-\mathrm{i}\omega_-(k)t}\right) + \text{complex conjugate.} \tag{2.14}$$

The constants C_+ and C_- are determined by the initial conditions,

$$\begin{pmatrix} 1 & 1 \\ -\mathrm{i}\omega_+(k) & -\mathrm{i}\omega_-(k) \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} Y \\ \dot{Y} \end{pmatrix}. \tag{2.15}$$

Solving this system of equations, we find the solution

$$y(x,t) = \frac{1}{2}\left(\frac{\mathrm{i}\omega_-(k)Y+\dot{Y}}{\mathrm{i}\omega_-(k)-\mathrm{i}\omega_+(k)}e^{\mathrm{i}kx-\mathrm{i}\omega_+(k)t} - \frac{\dot{Y}+\mathrm{i}\omega_+(k)Y}{\mathrm{i}\omega_-(k)-\mathrm{i}\omega_+(k)}e^{\mathrm{i}kx-\mathrm{i}\omega_-(k)t}\right)$$
$$+\text{complex conjugate.} \tag{2.16}$$

In complete analogy to this initial condition problem, boundary conditions can be imposed and transmission problems can be solved using sinusoidal waves with the appropriate dispersion relation $\omega(k)$. In the particular case of PDE (2.4), we need four boundary conditions (as many conditions as the order of the highest order derivative with respect to x in the equation). By symmetry, two of these boundary conditions would have to be imposed on one end of the stretched string, and the other two on the other end. The treatment of boundary conditions and transmission problems for dispersive waves is outside of the scope of this course, so we will stop here and consider a different type of problem.

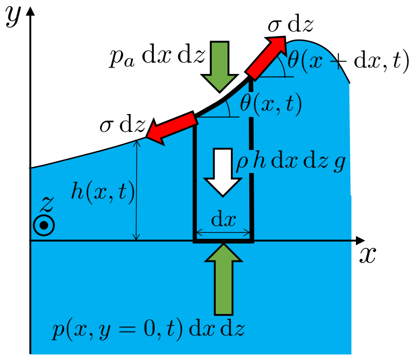

FIGURE 2. Interface $h(x,t)$ between a liquid and the atmosphere. The z axis is pointing towards the reader. The thick black line is a volume of liquid of height $h(x,t)$ and base $\mathrm{d}x\,\mathrm{d}z$ (the extent in the z direction is not represented). The forces with vertical components applied on this volume are gravity (in white), pressure (in green) and surface tension (in red).

2.2. *Gravity waves with surface tension*

Dispersion relations can also be found for more complex problems. As an example, we consider a liquid of density ρ that has a free surface in contact with the atmosphere at or near $y=0$, as shown in figure 2. The atmosphere is at a pressure p_a.

We proceed to find the equations for small displacements $h(x,t)$ of the free surface around $y=0$. At $y=0$, pressure not only opposes the atmospheric pressure p_a, but also sustains the weight of the liquid above it and balances the surface tension. The surface tension σ is a force per unit length that appears at the interface between different fluids and points along the interface – see, for example, the surface tension on the infinitesimal piece of interface highlighted in figure 2. For a water-air interface, surface tension is $\sigma = 7.4\times 10^{-2}$ N/m. To calculate the pressure $p(x,y,t)$ at $y=0$, we apply vertical force balance to the volume of height $h(x,t)$ and base $\mathrm{d}x\,\mathrm{d}z$ shown in figure 2,

$$p(x,y=0,t)\,\mathrm{d}x\,\mathrm{d}z - p_a\,\mathrm{d}x\,\mathrm{d}z - \rho g h(x,t)\,\mathrm{d}x\,\mathrm{d}z + \sigma\sin\theta(x+\mathrm{d}x,t)\,\mathrm{d}z \\ -\sigma\sin\theta(x,t)\,\mathrm{d}z = 0. \tag{2.17}$$

Note that we have neglected inertia because it is quadratic in the small displacement $h(x,t)$ – it scales as $h(\partial h/\partial t)$. For small $h(x,t)$, $\theta(x,t)\simeq \partial h/\partial x \ll 1$, giving the result

$$p(x,y=0,t) = p_a + \rho g h(x,t) - \sigma\frac{\partial^2 h}{\partial x^2}(x,t). \tag{2.18}$$

The non-uniform pressure at $y=0$ pushes the liquid to move horizontally. Indeed, considering an infinitesimal volume of liquid of size $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ at $y=0$, shown in figure 3, we find that the horizontal forces give

$$\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,a_x(x,y=0,t) = p(x,y=0,t)\,\mathrm{d}y\,\mathrm{d}z - p(x+\mathrm{d}x,y=0,t)\,\mathrm{d}y\,\mathrm{d}z, \tag{2.19}$$

where $a_x(x,y=0,t)$ is the acceleration in the x direction of the infinitesimal volume. Replacing equation (2.18) into equation (2.19), we obtain the equation for the horizontal acceleration

$$a_x(x,y=0,t) = -\frac{\partial}{\partial x}\left(gh(x,t) - \frac{\sigma}{\rho}\frac{\partial^2 h}{\partial x^2}(x,t)\right). \tag{2.20}$$

Due to this acceleration, the infinitesimal volume of liquid that we are considering moves in the x direction, displacing other parts of the liquid in its path and ultimately affecting the shape of the interface $h(x,t)$. To find the equations for the evolution of the interface

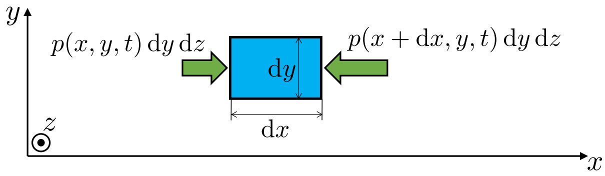

FIGURE 3. Horizontal forces on an infinitesimally small volume of liquid (the extent in the z direction is not represented; the z axis is pointing towards the reader).

shape, we need to relate the acceleration $a_x(x, y = 0, t)$ to the interface shape $h(x, t)$, and to obtain this relationship, we need to solve the 2D fluid motion below the surface, described by a system of PDEs given in Appendix B. The resulting oscillations of the interface are known as **gravity waves**.

The problem might seem daunting, but it can be simplified significantly. The coefficients of the problem depend on y (they are discontinuous at the interface), but they do not depend on x or t. Hence, by using the assumption $h(x, t) \propto e^{\mathrm{i}kx-\mathrm{i}\omega t}$, we eliminate all the derivatives with respect to x and t, and equation (2.20) gives

$$a_x(x, y = 0, t) = -\mathrm{i}k \left(g + \frac{\sigma k^2}{\rho} \right) h(x, t). \tag{2.21}$$

Solving the 2D fluid motion below the surface (see Appendix C) gives the unsurprising result that the horizontal acceleration of the fluid at $y = 0$ is related to the vertical acceleration of the interface $\partial^2 h/\partial t^2 = -\omega^2 h$,

$$a_x(x, y = 0, t) = -\frac{\mathrm{i}k}{|k|}\omega^2 h(x, t). \tag{2.22}$$

With this result, equation (2.21) gives the dispersion relation

$$\omega(k) = \pm\sqrt{g|k| + \frac{\sigma|k|^3}{\rho}}. \tag{2.23}$$

Note that we got a dispersion relation despite the fact that the problem was not uniform in the y direction.

There are many other examples of linear PDEs that lead to dispersion relations (Schrödinger's equation for the motion of a quantum particle, electromagnetic waves in a wave guide, electromagnetic waves in an ionized gas...). We will now focus on wavepackets.

3. Wavepackets: phase velocity and group velocity

We have already established that, unlike the solutions to the wave equation, dispersive waves do not maintain its initial shape while they move. There is a particular case in which analytical progress can be made: wavepackets. Wavepackets are composed of a carrier wave with wavelength k_c and an envelope that can have any shape (see the example in figure 4).

At $t = 0$, a wavepacket has the form

$$y(x, t = 0) = E(x)\cos(k_c x + \varphi), \tag{3.1}$$

where $E(x)$ is the envelope. We stated (without proof) that most functions can be ex-

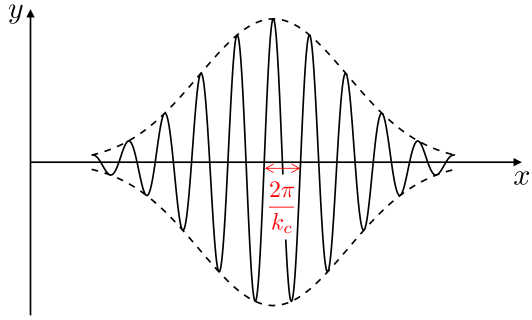

FIGURE 4. Example of wavepacket (solid black line) with a carrier wave of wavelength k_c. The envelope is the dashed black line.

pressed as sums or integrals of sines and cosines. Thus,

$$E(x) = \int C(K) e^{\mathrm{i}Kx} \,\mathrm{d}K, \tag{3.2}$$

where the coefficients $C(K)$ have to satisfy the property $C(-K) = C^*(K)$ to ensure that $E(x)$ is real. With this form for $E(x)$, the wavepacket can be written as

$$y(x, t = 0) = \frac{1}{2} \int C(K) e^{\mathrm{i}(k_c + K)x + \mathrm{i}\varphi} + \text{complex conjugate}, \tag{3.3}$$

and hence the solution for $t > 0$ is

$$y(x, t) = \frac{1}{2} \int C(K) e^{\mathrm{i}(k_c + K)x - \mathrm{i}\omega(k_c + K)t + \mathrm{i}\varphi} + \text{complex conjugate}. \tag{3.4}$$

This solution can be rewritten as

$$y(x, t) = \frac{1}{2} e^{\mathrm{i}k_c x - \mathrm{i}\omega(k_c)t + \mathrm{i}\varphi} \int C(K) e^{\mathrm{i}Kx - \mathrm{i}\Omega(K)t} \,\mathrm{d}K + \text{complex conjugate}, \tag{3.5}$$

where we have used the shorthand $\Omega(K) = \omega(k_c + K) - \omega(k_c)$. In general, the integral $\int C(K) e^{\mathrm{i}Kx - \mathrm{i}\Omega(K)t + \mathrm{i}\varphi} \,\mathrm{d}K$ gives a complex function $F(x, t)e^{\mathrm{i}\psi(x,t)}$, and the solution for $t > 0$ is $y(x, t) = F(x, t) \cos(k_c x - \omega(k_c)t + \varphi + \psi(x, t))$. This solution will rapidly become unrecognizable as a wavepacket if the function $\psi(x, t)$ becomes large. Fortunately, there is a limit in which this does not happen.

For an envelope with a characteristic length much longer than the wavelength of the carrier wave k_c,

$$\frac{1}{k_c} \left| \frac{1}{E} \frac{\mathrm{d}E}{\mathrm{d}x} \right| \sim \frac{K}{k_c} \ll 1, \tag{3.6}$$

the variable K is much smaller than k_c, and we can expand the dispersion relation,

$$\omega(k_c + K) \simeq \omega(k_c) + \frac{\mathrm{d}\omega}{\mathrm{d}k}(k_c)K. \tag{3.7}$$

Then, $\Omega(K)$ in equation (3.5) simplifies to $\Omega(K) \simeq (\mathrm{d}\omega/\mathrm{d}k)K$, giving

$$\int C(K) e^{\mathrm{i}Kx - \mathrm{i}\Omega(K)t} \,\mathrm{d}K \simeq \int C(K) e^{\mathrm{i}K[x - \mathrm{i}(\mathrm{d}\omega/\mathrm{d}k)t]} \,\mathrm{d}K = E\left(x - \frac{\mathrm{d}\omega}{\mathrm{d}k} t\right). \tag{3.8}$$

Substituting this result into equation (3.5), we obtain the nice formula

$$y(x, t) = E\left(x - \frac{\mathrm{d}\omega}{\mathrm{d}k} t\right) \cos(k_c x - \omega(k_c)t + \varphi). \tag{3.9}$$

The envelope and the carrier wave move at different speeds. The carrier wave moves at the phase velocity $v_p(k_c) = \omega(k_c)/k_c$, whereas the envelope moves at the **group velocity**

$$v_g(k_c) = \frac{\mathrm{d}\omega}{\mathrm{d}k}(k_c). \tag{3.10}$$

For non-dispersive waves, $\omega(k) = kc$, both velocities coincide, but for dispersive waves they do not. In the Dispersive Wavepacket Plotter (DWP), created by Christopher Palmer and available in the wavepage of the course, one can try different initial wavepackets for different dispersion relation – all of them based on dispersion relation (2.23) for gravity waves with surface tension.

The coherence of the wavepacket has a finite life time. The next order corrections to $\Omega(K)$ become important if we wait for a sufficiently long time. The next order term in the Taylor expansion of $\Omega(K)$ gives a phase

$$\frac{K^2}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}t, \tag{3.11}$$

and this phase cannot be neglected when the time t is of order

$$t \sim \frac{1}{K^2}\frac{1}{\mathrm{d}^2\omega/\mathrm{d}k^2} \gg \frac{1}{\omega}. \tag{3.12}$$

Larger envelopes (= smaller K) take a longer time to disperse. The carrier wave also matters. If we choose a k_c for which $\mathrm{d}^2\omega/\mathrm{d}k^2$ is small, the envelope will keep its shape for longer.

The group velocity is important for more than wavepackets. It also matters for energy and information transfer, as you will see in future courses.

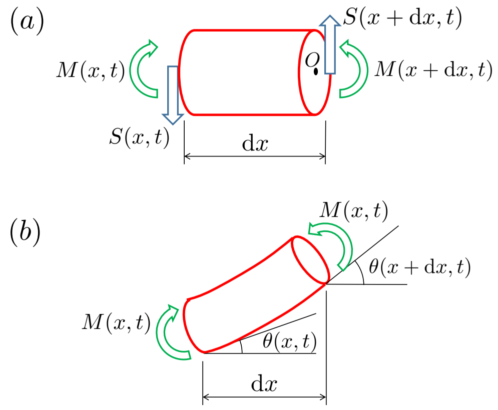

FIGURE 5. (a) Torque balance with respect to the point O for an infinitesimal length of string. (b) Bending of an infinitesimal piece of string due to torques on its ends.

Appendix A. Relation between the force $S(x,t)$ and the derivatives of $y(x,t)$

The main effect of the small force $S(x,t)$ in figure 1 is to introduce torques $M(x,t)$. Indeed, to keep the infinitesimal piece of string shown in figure 5(a) in equilibrium, we must balance torques. We proceed to calculate torques with respect to point O in the middle of the cross section at $x+\mathrm{d}x$. The torque due to the force $S(x,t)$ at x, of size $S(x,t)\,\mathrm{d}x$, must be balanced by other torques. Thus, the rest of the string not only needs to apply the tension T along the direction of the string and the forces $S(x,t)$ and $S(x+\mathrm{d}x,t)$ in the direction perpendicular to it, but also torques $M(x,t)$ and $M(x+\mathrm{d}x,t)$, represented in figure 5(a) as green curved arrows. These torques can be applied due to the finite thickness of string: one can exert a stronger force on the top of the cross section than on the bottom. To balance all the torques with respect to the point O, we need $M(x+\mathrm{d}x,t) = M(x,t) - S(x+\mathrm{d}x,t)\,\mathrm{d}x$, giving

$$S(x,t) = -\frac{\partial M}{\partial x}. \tag{A 1}$$

Note that we have neglected the angular momentum of the infinitesimal piece of string because it scales as $(\mathrm{d}x)^2$.

The torques $M(x,t)$ in turn lead to bending, as shown in figure 5(b). The change in the slope of the string shown in figure 5(b), $\theta(x+\mathrm{d}x) - \theta(x)$, is proportional to the torque $M(x,t)$, and the length of the piece of string being bent, $\mathrm{d}x$, as longer strings are easier to bend. Hence, $\theta(x+\mathrm{d}x,t)-\theta(x,t) \propto M(x,t)\,\mathrm{d}x$. The constant of proportionality depends on the cross section of the string, of area A, and on its Young modulus E. In fact, the constant of proportionality is EAr_b^2,

$$M(x,t)\,\mathrm{d}x = EAr_b^2\left[\theta(x+\mathrm{d}x,t) - \theta(x,t)\right], \tag{A 2}$$

where r_b is a length of the order of the thickness of the string that only depends on the shape of the cross section. For our purposes, it is convenient to write the constant of proportionality as TL_b^2, where $L_b = r_b\sqrt{EA/T} \gg r_b$. Then,

$$M(x,t)\,\mathrm{d}x = TL_b^2\left[\theta(x+\mathrm{d}x,t) - \theta(x,t)\right]. \tag{A 3}$$

Using partial derivatives and $\theta(x,t) \simeq \partial y/\partial x$, we obtain

$$M(x,t) = TL_b^2\frac{\partial^2 y}{\partial x^2}. \tag{A 4}$$

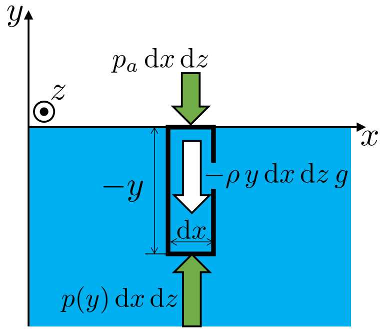

FIGURE 6. Horizontal interface between a liquid and the atmosphere. The z axis is pointing towards the reader. The thick black line is a volume of height $-y$ and base $\mathrm{d}x\,\mathrm{d}z$ (the extent in the z direction is not represented). The forces with vertical components applied on the liquid inside this volume are gravity (in white) and pressure (in green). The surface tension is not included in the vertical force balance because it is pointing in the horizontal direction.

With this result and equation (A 1), we find formula (2.1) for the force $S(x, t)$ perpendicular to the string.

Appendix B. Equations for gravity waves with surface tension

We start by calculating the steady state pressure of the liquid. We consider the case in which the interface is horizontal and is located at $y = 0$, as sketched in figure 6. To calculate the pressure within the liquid, we use infinitesimal volumes of height $-y$ and base $\mathrm{d}x\,\mathrm{d}z$, such as the one represented in figure 6. The vertical forces on these volumes are the atmospheric pressure force $p_a\,\mathrm{d}x\,\mathrm{d}z$ on the top, the pressure force $p(y)\,\mathrm{d}x\,\mathrm{d}z$ on the bottom, and the weight of the volume of liquid $-\rho\, y\,\mathrm{d}x\,\mathrm{d}z\, g$. Balancing all these forces, we find $p(y) = p_a - \rho g y$. Thus, the pressure increases as we go deeper because the pressure must sustain all the weight of the fluid above it.

We now proceed to perturb the shape of the interface. To describe the motion of the liquid below the interface, we use the displacements $\xi(x, y, t)$ and $\eta(x, y, t)$: a point originally at (x, y, z) moves to $(x + \xi(x, y, t), y + \eta(x, y, t), z)$. The vertical displacement η of the line $y = 0$ is the interface shape that we are trying to calculate,

$$h(x, t) = \eta(x, y = 0, t). \tag{B 1}$$

Due to the displacement $h(x, t)$, the pressure is perturbed, that is,

$$p(x, y, t) = p_a - \rho g y + p_1(x, y, t). \tag{B 2}$$

The displacements ξ and η are not independent of each other in a liquid because the density ρ is constant. Thus, infinitesimal volumes of liquid cannot change their volume as they move. Consider the infinitesimal volume of sides $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ sketched in figure 7. The result of its deformation, also sketched in figure 7, has a volume

$$\left[\left(1 + \frac{\partial \xi}{\partial x}\right)\left(1 + \frac{\partial \eta}{\partial y}\right) - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{B 3}$$

For small ξ and η, this volume simplifies to $(1 + \partial\xi/\partial x + \partial\eta/\partial y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. For this volume to be the same as the original volume, $\xi(x, y, t)$ and $\eta(x, y, t)$ must satisfy the condition

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} = 0. \tag{B 4}$$

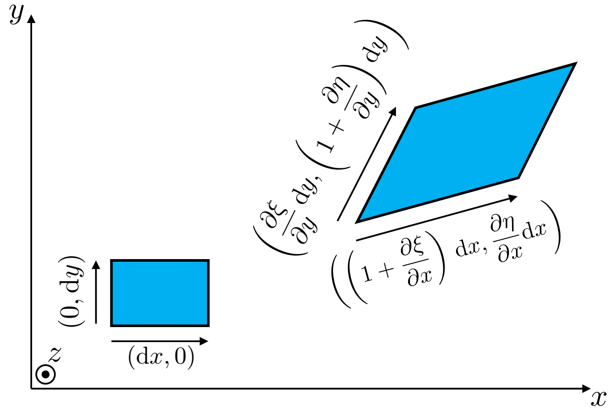

FIGURE 7. Volume of sides $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$, and its deformation due to the motion of the fluid (the extent of the volumes in the z direction are not represented; the z axis is pointing towards the reader).

We proceed to calculate the equations of motion for ξ and η. The horizontal force balance that we used to obtain equation (2.20) is valid for any y, and not only for $y = 0$. Thus, $\rho a_x(x, y, t) = -\partial p/\partial x$. Using expression (B 2) and $a_x = \partial^2\xi/\partial t^2$, we can write the equation of motion for ξ in terms of the perturbed pressure $p_1(x, y, t)$,

$$\rho\frac{\partial^2\xi}{\partial t^2} = -\frac{\partial p_1}{\partial x}. \tag{B 5}$$

To obtain the equation of motion for the vertical displacement, we consider the vertical forces on an infinitesimal volume such as the one in figure 8, where can see that the volume's weight contributes a force $-\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,g$. We find that

$$\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\partial^2\eta}{\partial t^2}(x, y, t) = p(x, y, t)\,\mathrm{d}x\,\mathrm{d}z - p(x, y + \mathrm{d}y, t)\,\mathrm{d}x\,\mathrm{d}z - \rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,g, \tag{B 6}$$

leading to

$$\rho\frac{\partial^2\eta}{\partial t^2} = -\frac{\partial p}{\partial y} - \rho g. \tag{B 7}$$

Using equation (B 2), we finally obtain

$$\rho\frac{\partial^2\eta}{\partial t^2} = -\frac{\partial p_1}{\partial y}. \tag{B 8}$$

Equations (B 4), (B 5) and (B 8) are the PDEs that we need to solve to obtain the motion of the liquid below the interface. These equations have to be solved with the boundary condition given by equation (2.18). Using equation (B 2), equation (2.18) implies that

$$p_1(x, y = 0, t) = gh(x, t) - \frac{\sigma}{\rho}\frac{\partial^2 h}{\partial x^2}(x, t), \tag{B 9}$$

where $h(x, t)$ is related to $\eta(x, y, t)$ by equation (B 1). In addition to this boundary condition, we impose that the displacements $\xi(x, y, t)$ and $\eta(x, y, t)$ and the pressure perturbation $p_1(x, y, t)$ vanish far away from the the liquid interface, that is, they go to zero for $y \to -\infty$.

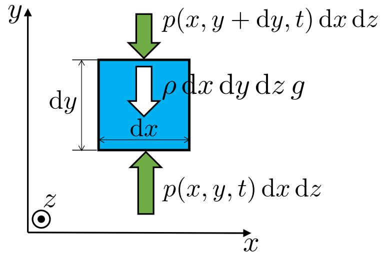

FIGURE 8. Vertical forces on an infinitesimally small volume of liquid (the extent in the z direction is not represented; the z axis is pointing towards the reader).

Appendix C. Solution to equations (B 4), (B 5) and (B 8) for a sinusoidal interface $h(x, t)$

The pressure perturbation $p_1(x, y, t)$ has to ensure that equation (B 4) is satisfied at all times. Differentiating equation (B 5) with respect to x and equation (B 8) with respect to y, and summing these equations, we find

$$\rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y}\right) = -\frac{\partial^2 p_1}{\partial x^2} - \frac{\partial^2 p_1}{\partial y^2}. \tag{C 1}$$

Thus, to satisfy equation (B 4), $p_1(x, y, t)$ must satisfy the equation

$$\frac{\partial^2 p_1}{\partial x^2} + \frac{\partial^2 p_1}{\partial y^2} = 0. \tag{C 2}$$

For $h(x, t) \propto e^{\mathrm{i}kx - \mathrm{i}\omega t}$, equation (B 9) gives $p_1(x, y = 0, t) \propto e^{\mathrm{i}kx - \mathrm{i}\omega t}$. This suggests looking for solutions of the form $p_1(x, y, t) = P_1(y) e^{\mathrm{i}kx - \mathrm{i}\omega t}$. With this assumption, equation (C 2) becomes

$$\frac{\mathrm{d}^2 P_1}{\mathrm{d}y^2} - k^2 P_1 = 0. \tag{C 3}$$

Recalling that we expect $P_1 \to 0$ for $y \to -\infty$, we can solve this ODE to find

$$p_1(x, y, t) = p_1(x, y = 0, t) e^{|k| y}. \tag{C 4}$$

This solution implies that

$$\frac{\partial p_1}{\partial x} = \mathrm{i}k p_1, \quad \frac{\partial p_1}{\partial y} = |k| p_1, \tag{C 5}$$

and using these results in equations (B 5) and (B 8) for the accelerations in the x and y directions, we obtain relation (2.22).