Separation of variables and stationary waves

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1. Introduction

In these notes, we show how to obtain solutions for the wave equation with two boundary conditions without resorting to D'Alembert's solution. The new technique is known as separation of variables, and can simplify the problem significantly in some cases.

2. Separation of variables

The idea behind separation of variables is to reproduce the calculation that we have already performed for systems of ODEs. The motion of a system of oscillators is described by a system of ODEs of the form

$$\frac{\mathrm{d}^2\mathbf{q}}{\mathrm{d}t^2} = \mathbf{D}\cdot\mathbf{q}, \tag{2.1}$$

where $\mathbf{q}(t)$ is the vector that contains the position of the different oscillators, and $\mathbf{D}$ is a constant square matrix. To solve system (2.1), we propose solutions of the form

$$\mathbf{q}(t) = \Theta(t)\mathbf{Q}, \tag{2.2}$$

where the vector $\mathbf{Q}$ does not depend on time. With this assumption, $\Theta(t)$ and $\mathbf{Q}$ must satisfy

$$\mathbf{D}\cdot\mathbf{Q} = \frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}t^2}\mathbf{Q}. \tag{2.3}$$

Thus, $\mathbf{Q}$ must be an eigenvector of matrix $\mathbf{D}$, and $\Theta^{-1}(\mathrm{d}^2\Theta/\mathrm{d}t^2)$ the corresponding eigenvalue. For matrices $\mathbf{D}$ that have a complete basis of eigenvectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ with their corresponding eigenvalues $\{\Lambda_1, \Lambda_2, \dots, \Lambda_n\}$, the most general solution is

$$\mathbf{q}(t) = \sum_{i=1}^{n}\left[A_i\cos(\sqrt{-\Lambda_i}t) + B_i\sin(\sqrt{-\Lambda_i}t)\right]\mathbf{e}_i, \tag{2.4}$$

where the constants A_i and B_i are determined by the initial conditions. Note that we have written the solution in a convenient form for negative eigenvalues Λ_i, but the expression is still valid for complex eigenvalues.

The wave equation can be written analogously to equation (2.1),

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}. \tag{2.5}$$

Here, the operator $c^2(\partial^2/\partial x^2)$ plays the role of matrix $\mathbf{D}$. We solved system (2.1) by assuming that all oscillators moved in unison, but with different amplitudes, as shown in equation (2.2). For the wave equation, we can use the continuum limit of this idea, that is, each position x has its corresponding oscillator that moves with amplitude $Q(x)$, giving

$$y(x,t) = \Theta(t)Q(x). \tag{2.6}$$

With this assumption, we can write equation (2.5) as

$$\frac{1}{c^2\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}t^2} = \frac{1}{Q}\frac{\mathrm{d}^2 Q}{\mathrm{d}x^2}. \tag{2.7}$$

Note that here we are setting a function of t equal to a function of x. Since these are functions of different variables, they can only be equal if they are constant, that is,

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}t^2} = \Lambda c^2 \Theta \tag{2.8}$$

and

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}x^2} = \Lambda Q, \tag{2.9}$$

where the constant Λ can only take a limited set of values – in complete analogy to the eigenvalues of matrix $\mathbf{D}$. To determine these special values of Λ that are eigenvalues of the operator $\partial^2/\partial x^2$, we need to consider the boundary conditions. We proceed to find the eigenvalues for a series of simple examples. You can also find an example of separation of variables with a more general boundary condition in Appendix A.

2.1. *Two Dirichlet boundary conditions*

We consider a stretched string that is held in place at $x = 0$ and $x = L$, that is, we use the boundary conditions

$$y(x = 0, t) = 0, \quad y(x = L, t) = 0. \tag{2.10}$$

For a solution of the form (2.6) to satisfy these boundary conditions, we need to impose

$$Q(0) = 0, \quad Q(L) = 0. \tag{2.11}$$

Solving for $Q(x)$ using equation (2.9), we find

$$Q(x) = C_+ e^{\sqrt{\Lambda}x} + C_- e^{-\sqrt{\Lambda}x}. \tag{2.12}$$

Imposing conditions (2.11), we obtain a linear system of equations for the unknown constants C_+ and C_-,

$$\begin{pmatrix} 1 & 1 \\ e^{\sqrt{\Lambda}L} & e^{-\sqrt{\Lambda}L} \end{pmatrix} \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.13}$$

This linear system of equations for C_+ and C_- gives a nontrivial solution only if the matrix is singular. By setting the determinant of the matrix equal to zero, we find the equation

$$e^{2\sqrt{\Lambda}L} = 1. \tag{2.14}$$

Using $1 = e^{2\pi k \mathrm{i}}$, with k an integer number, this equation leads to

$$\Lambda = -\frac{k^2\pi^2}{L^2}. \tag{2.15}$$

Since k appears squared, we only need to consider $k \geqslant 0$ from here on.

In principle, k could be equal to zero, giving $\Lambda = 0$. For this value of Λ, the solution for $Q(x)$ is not equation (2.12), but $Q(x) = C + \widetilde{C}x$, where C and $\widetilde{C}$ are constants that we need to determine using boundary conditions (2.11). It is not possible to find $C \neq 0$ and $\widetilde{C} \neq 0$ that satisfy these boundary conditions, so $\Lambda = 0$ is not of interest. Thus, the final set of eigenvalues is

$$\Lambda_k = -\frac{k^2\pi^2}{L^2}, \tag{2.16}$$

with $k = 1, 2, 3, \ldots$ The corresponding eigenfunctions $Q_k(x)$ satisfy equation (2.9) and boundary conditions (2.11),

$$Q_k(x) = \sin\left(\frac{k\pi x}{L}\right). \tag{2.17}$$

The functions $Q_k(x)$ are defined up to a constant that can be absorbed into $\Theta(t)$. The functions $Q_k(x)$ are the **modes** of the system.

For eigenvalue $\Lambda_k = -k^2\pi^2/L^2$, we can solve equation (2.8) for $\Theta(t)$, finding

$$\Theta_k(t) = A_k \cos\left(\frac{k\pi ct}{L}\right) + B_k \sin\left(\frac{k\pi ct}{L}\right). \tag{2.18}$$

The constants A_k and B_k must be determined by the initial conditions, as we will see shortly.

Combining all these results, we find that the most general solution to this problem is

$$y(x,t) = \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{k\pi ct}{L}\right) + B_k \sin\left(\frac{k\pi ct}{L}\right)\right] \sin\left(\frac{k\pi x}{L}\right). \tag{2.19}$$

Note that this solution is periodic in time with period $\tau = 2L/c$.

We finish by showing how one imposes initial conditions. The initial conditions for the string are its position and velocity,

$$y(x, t=0) = y_0(x), \quad \frac{\partial y}{\partial t}(x, t=0) = \dot{y}_0(x). \tag{2.20}$$

It turns out that any function of x between 0 and L can be written as an infinite series of sines,

$$f(x) = \sum_{k=1}^{\infty} F_k \sin\left(\frac{k\pi x}{L}\right), \tag{2.21}$$

and the coefficients $\{F_1, F_2, F_3, \ldots\}$ of this decomposition are unique, that is, there is no other set of coefficients $\{F_1', F_2', F_3', \ldots\}$ that gives the same function $f(x)$. Next year, in Mathematical Methods, you will learn how to calculate these coefficients from the function $f(x)$. For now, it is sufficient to know that they exist and that they are unique. Thus, we can write $y_0(x)$ and $\dot{y}_0(x)$ as

$$y_0(x) = \sum_{k=1}^{\infty} Y_k \sin\left(\frac{k\pi x}{L}\right), \quad \dot{y}_0(x) = \sum_{k=1}^{\infty} \dot{Y}_k \sin\left(\frac{k\pi x}{L}\right). \tag{2.22}$$

With these decompositions of $y_0(x)$ and $\dot{y}_0(x)$, we can impose initial conditions (2.20) on the solution (2.19) to find

$$A_k = Y_k, \quad \frac{k\pi c}{L} B_k = \dot{Y}_k. \tag{2.23}$$

For this course, you will usually be given initial conditions that are finite sums of sines so that it is easy for you to identify the coefficients Y_k and $\dot{Y}_k$. For example, for

$$y(x, t=0) = h \sin\left(\frac{\pi x}{L}\right), \quad \frac{\partial y}{\partial t}(x, t=0) = 0, \tag{2.24}$$

the coefficients $\dot{Y}_k$ are zero for all values of k, and the coefficients Y_k are zero for $k \neq 1$. For $k = 1$, $Y_1 = h$, giving the final solution

$$y(x,t) = h \cos\left(\frac{\pi ct}{L}\right) \sin\left(\frac{\pi x}{L}\right). \tag{2.25}$$

2.2. *Two Neumann boundary conditions*

In this section, we consider a stretched string that slides along frictionless columns at $x = 0$ and $x = L$,

$$\frac{\partial y}{\partial x}(x = 0, t) = 0, \quad \frac{\partial y}{\partial x}(x = L, t) = 0. \tag{2.26}$$

Following the procedure above for this case gives the general solution

$$y(x, t) = A_0 + \widetilde{A}_0 t + \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{k\pi ct}{L}\right) + B_k \sin\left(\frac{k\pi ct}{L}\right) \right] \cos\left(\frac{k\pi x}{L}\right). \tag{2.27}$$

In this case, the string can slide freely along the columns, as demonstrated by the term linear in time. Apart from this linear term, the solution is again periodic in time with period $\tau = 2L/c$.

2.3. *One Dirichlet boundary condition and one Neumann boundary condition*

As an example of a wave equation with two different boundary conditions, we consider a stretched string that is held in place at $x = 0$ and slides along a frictionless column at $x = L$,

$$y(x = 0, t) = 0, \quad \frac{\partial y}{\partial x}(x = L, t) = 0. \tag{2.28}$$

In this case, separation of variables gives the general solution

$$y(x, t) = \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{(k - 1/2)\pi ct}{L}\right) + B_k \sin\left(\frac{(k - 1/2)\pi ct}{L}\right) \right] \sin\left(\frac{(k - 1/2)\pi x}{L}\right). \tag{2.29}$$

This solution is periodic in time with period $\tau = 4L/c$.

3. Stationary waves

The solutions that we have found using separation of variables are stationary waves: they are oscillating patterns that do not move along x.

It is instructive to see how to construct stationary waves from the D'Alembert solution. The stationary wave solutions that we have found are of the form $\sin(kx+\theta)\sin(kct+\varphi)$, and they can be constructed from left- and right-traveling sinusoidal waves of the same amplitude. Indeed, for $f(x - ct) = A\cos(k(x - ct) + \theta - \varphi)$ and $g(x + ct) = -A\cos(k(x + ct) + \theta + \varphi)$, we find

$$y(x, t) = f(x - ct) + g(x + ct) = 2A\sin(kx + \theta)\sin(kct + \varphi). \tag{3.1}$$

The time-averaged energy fluxes due to the left- and right-traveling waves cancel each other because the two waves have the same amplitude. Indeed, using the time average over one period,

$$\langle \ldots \rangle_\tau = \frac{kc}{2\pi} \int_t^{t+2\pi/kc} (\ldots)\, \mathrm{d}t', \tag{3.2}$$

the time averaged energy fluxes are $\langle \epsilon_f \rangle_\tau c = Tk^2A^2c/2$ and $-\langle \epsilon_g \rangle_\tau c = -Tk^2A^2c/2$. Thus, these fluxes cancel at every point x, ensuring that the energy does not grow or decay at any given point on average, although there are instantaneous changes in the energy as it sloshes from one x location to another.

4. Energy and modes

The modes that we have calculated in the examples in sections 2.1, 2.2 and 2.3 have one surprising property: the total energy contained in any wave can be split into the sum of the energies in each mode despite the fact that the energy is a quadratic quantity. This is a consequence of the simple boundary conditions that we have imposed.

The solutions that we have found in sections 2.1, 2.2 and 2.3 are of the form

$$y(x,t) = A_0 + \widetilde{A}_0 t + \sum_{k=1}^{\infty} C_k \cos\left(\sqrt{-\Lambda_k} ct - \varphi_k\right) Q_k(x), \tag{4.1}$$

where the functions $Q_k(x)$ satisfy equation

$$\frac{\mathrm{d}^2 Q_k}{\mathrm{d}x^2} = \Lambda_k Q_k \tag{4.2}$$

and boundary conditions of the type $Q_k = 0$ or $\mathrm{d}Q_k/\mathrm{d}x = 0$ at $x = 0$ and $x = L$. For example, solution (2.19) can be recovered with $A_0 = 0$, $\widetilde{A}_0 = 0$, $C_k = \sqrt{A_k^2 + B_k^2}$, $\tan\varphi_k = B_k/A_k$, $\Lambda_k = -k^2\pi^2/L^2$ and $Q_k(x) = \sin(k\pi x/L)$.

If $y(x,t)$ is the transverse displacement of a stretched string of linear density μ and tension T, the energy contained in the string motion is

$$E = \frac{\mu}{2}\int_0^L \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x + \frac{T}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x. \tag{4.3}$$

Integrating the second term by parts, we find

$$E = \int_0^L \left[\frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{Ty}{2}\frac{\partial^2 y}{\partial x^2}\right] \mathrm{d}x + \frac{T}{2}\left[y\frac{\partial y}{\partial x}\right]_{x=0}^{x=L}. \tag{4.4}$$

The combination $y(\partial y/\partial x)$ vanishes due to the boundary conditions imposed on $Q_k(x)$ at $x = 0$ and $x = L$. Using this fact and equation (4.2), we rewrite the energy as

$$\begin{aligned} E = \int_0^L \Bigg[&\frac{\mu}{2}\left(\widetilde{A}_0 - c\sum_{k=1}^{\infty}\sqrt{-\Lambda_k} C_k \sin\left(\sqrt{-\Lambda_k} ct - \varphi_k\right) Q_k(x)\right)^2 \\ &- \frac{T}{2}\left(\sum_{k=1}^{\infty} C_k \cos\left(\sqrt{-\Lambda_k} ct - \varphi_k\right) Q_k(x)\right)\left(\sum_{l=1}^{\infty} \Lambda_l C_l \cos\left(\sqrt{-\Lambda_l} ct - \varphi_l\right) Q_l(x)\right)\Bigg] \mathrm{d}x. \end{aligned} \tag{4.5}$$

If equation (4.5) is expanded, we find integrals of the form $\widetilde{A}_0 C_k \int_0^L Q_k(x)\,\mathrm{d}x$ and $C_k C_l \int_0^L Q_k(x) Q_l(x)\,\mathrm{d}x$, giving a complicated expression for the energy E. This is hardly surprising given the quadratic nature of the energy. What is surprising is that the modes satisfy $\widetilde{A}_0 \int_0^L Q_k(x)\,\mathrm{d}x = 0$ and $\int_0^L Q_k(x) Q_l(x)\,\mathrm{d}x = 0$. This result can be checked by direct integration, or using equation (4.2), as we proceed to show. Multiplying equation (4.2) by $Q_l(x)$ and integrating over x, we find

$$\int_0^L Q_l \frac{\mathrm{d}^2 Q_k}{\mathrm{d}x^2}\,\mathrm{d}x = \Lambda_k \int_0^L Q_l Q_k\,\mathrm{d}x. \tag{4.6}$$

Integrating the term on the left side of the equation by parts, we obtain

$$\left[Q_l \frac{\mathrm{d}Q_k}{\mathrm{d}x}\right]_{x=0}^{x=L} - \int_0^L \frac{\mathrm{d}Q_l}{\mathrm{d}x}\frac{\mathrm{d}Q_k}{\mathrm{d}x}\,\mathrm{d}x = \Lambda_k \int_0^L Q_l Q_k\,\mathrm{d}x. \tag{4.7}$$

The combination $Q_l(\mathrm{d}Q_k/\mathrm{d}x)$ vanishes at $x = 0$ and $x = L$ due to the boundary conditions, leaving

$$-\int_0^L \frac{\mathrm{d}Q_l}{\mathrm{d}x}\frac{\mathrm{d}Q_k}{\mathrm{d}x}\,\mathrm{d}x = \Lambda_k \int_0^L Q_l Q_k \,\mathrm{d}x. \tag{4.8}$$

We obtained this expression by multiplying equation (4.2) for $Q_k(x)$ by $Q_l(x)$ and integrating over x. One can multiply the equation for $Q_l(x)$ by $Q_k(x)$ and integrate over x to find the alternative equation

$$-\int_0^L \frac{\mathrm{d}Q_l}{\mathrm{d}x}\frac{\mathrm{d}Q_k}{\mathrm{d}x}\,\mathrm{d}x = \Lambda_l \int_0^L Q_l Q_k \,\mathrm{d}x. \tag{4.9}$$

Note that the left sides of equations (4.8) and (4.9) are the same, and thus, if we subtract one equation from the other, we find

$$(\Lambda_k - \Lambda_l)\int_0^L Q_l Q_k \,\mathrm{d}x = 0. \tag{4.10}$$

Thus,

$$\int_0^L Q_l Q_k \,\mathrm{d}x = 0 \quad \text{for } k \neq l. \tag{4.11}$$

This equation also proves that $\widetilde{A}_0 \int_0^L Q_k(x)\,\mathrm{d}x$ vanishes. The coefficient $\widetilde{A}_0$ is different from zero only when $\Lambda_0 = 0$ is an eigenvalue and hence $Q_0(x) = 1$ is a mode. As a result, either $\widetilde{A}_0 = 0$ or $\int_0^L Q_0 Q_k \,\mathrm{d}x = \int_0^L Q_k \,\mathrm{d}x = 0$ for $k \geqslant 1$, giving

$$\widetilde{A}_0 \int_0^L Q_k \,\mathrm{d}x = 0 \quad \text{for } k \geqslant 1. \tag{4.12}$$

Using equations (4.11) and (4.12), the cross-terms in equation (4.5) for the energy vanish, giving the much simpler result

$$\begin{aligned} E =& \frac{\mu}{2}\left(\widetilde{A}_0^2 L - c^2 \sum_{k=1}^{\infty} \Lambda_k C_k^2 \sin^2\left(\sqrt{-\Lambda_k} ct - \varphi_k\right)\int_0^L Q_k^2(x)\,\mathrm{d}x\right) \\ & - \frac{T}{2}\sum_{k=1}^{\infty} \Lambda_k C_k^2 \cos^2\left(\sqrt{-\Lambda_k} ct - \varphi_k\right)\int_0^L Q_k^2(x)\,\mathrm{d}x. \end{aligned} \tag{4.13}$$

Using $c = \sqrt{T/\mu}$ finally gives the explicitly time-independent expression

$$E = \frac{\mu L \widetilde{A}_0^2}{2} - \frac{T}{2}\sum_{k=1}^{\infty} \Lambda_k C_k^2 \int_0^L Q_k^2(x)\,\mathrm{d}x. \tag{4.14}$$

Thus, the total energy is simply the sum of the energy of each mode. Expression (4.14) can be further simplified by realizing that the functions $Q_k(x)$ in sections 2.1, 2.2 and 2.3 satisfy $\int_0^L Q_k^2(x)\,\mathrm{d}x = L/2$, finally leading to

$$E = \frac{\mu L \widetilde{A}_0^2}{2} - \frac{TL}{4}\sum_{k=1}^{\infty} \Lambda_k C_k^2. \tag{4.15}$$

For the example in equation (2.19), with $\widetilde{A}_0 = 0$, $\Lambda_k = -k^2\pi^2/L^2$ and $C_k = \sqrt{A_k^2 + B_k^2}$, the energy is then

$$E = \frac{\pi^2 T}{4L}\sum_{k=1}^{\infty} k^2 (A_k^2 + B_k^2). \tag{4.16}$$

As we pointed out at the start of this section, the split of the energy into a sum of the energies of the individual modes was possible due to the simple boundary conditions considered in sections 2.1, 2.2 and 2.3. For more complex boundary conditions, one needs to carefully choose what to include or exclude in the definition of the energy of the system. See the end of Appendix A for an example of these more general boundary conditions.

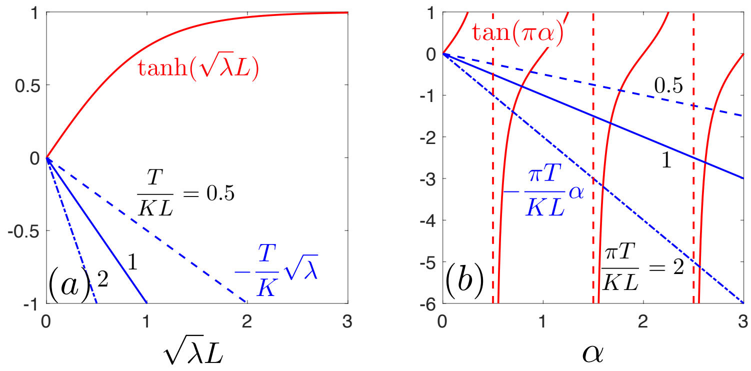

FIGURE 1. Solutions to equation (A 4). (a) For $\Lambda > 0$, the solutions are intersections between the curves $\tanh(\sqrt{\Lambda}L)$ (in red) and $-(T/K)\sqrt{\Lambda}$ (in blue for different values of T/KL: dashed line for $T/KL = 0.5$, solid for $T/KL = 1$ and dash-dot for $T/KL = 2$). (b) For $\Lambda < 0$, we use $\alpha = \sqrt{-\Lambda}L/\pi$. The solutions are intersections between the curves $\tan(\pi\alpha)$ (in red) and $-(\pi T/KL)\alpha$ (in blue for different values of $\pi T/KL$: dashed line for $\pi T/KL = 0.5$, solid for $\pi T/KL = 1$ and dash-dot for $\pi T/KL = 2$).

Appendix A. Example of separation of variables for a general boundary condition

We finish these notes by considering a situation in which the eigenvalues and modes are not as simple as the ones described so far: a stretched string that is held in place at $x = 0$ and that, at $x = L$, is knotted to a frictionless column and to a spring of constant K,

$$y(x = 0, t) = 0, \quad \frac{\partial y}{\partial x}(x = L, t) + \frac{K}{T} y(x = L, t) = 0. \tag{A 1}$$

For assumption (2.6) to work, we need to impose

$$Q(0) = 0, \quad \frac{\mathrm{d}Q}{\mathrm{d}x}(L) + \frac{K}{T} Q(L) = 0. \tag{A 2}$$

Using solution (2.12) for $Q(x)$, we find that these boundary conditions lead to the linear system of equations

$$\begin{pmatrix} 1 & 1 \\ (\sqrt{\Lambda} + K/T)e^{\sqrt{\Lambda}L} & (-\sqrt{\Lambda} + K/T)e^{-\sqrt{\Lambda}L} \end{pmatrix} \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{A 3}$$

for the constants C_+ and C_-. This system of equations gives nontrivial solutions only when the matrix is singular. Requiring that the determinant of the matrix vanishes leads to

$$\tanh(\sqrt{\Lambda}L) = -\frac{T}{K}\sqrt{\Lambda}. \tag{A 4}$$

One can show that Λ is real. Thus, we need to distinguish two cases to solve this equation:

- **Positive** Λ**.** For positive values of Λ, the solutions to equation (A 4) are the intersections between a hyperbolic tangent and a straight line, as shown in figure 1(a). Only $\Lambda = 0$ is a solution, and as we have seen, the solution for this value of Λ is $Q(x) = C + \widetilde{C}x$. Since it is not possible to find values $C \neq 0$ and $\widetilde{C} \neq 0$ that satisfy the boundary conditions, Λ is in fact not an eigenvalue.
- **Negative** Λ**.** For negative values of Λ, we can rewrite equation (A 4) using the new

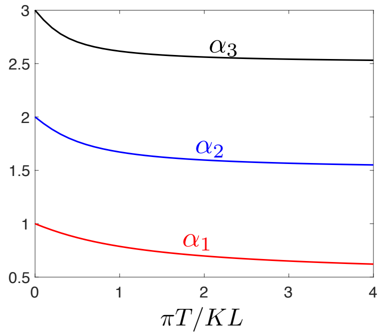

FIGURE 2. Coefficients α_1, α_2 and α_3 for different values of $\pi T/KL$.

variable $\alpha = \sqrt{-\Lambda}L/\pi$,

$$\tan(\pi\alpha) = -\frac{\pi T}{KL}\alpha. \tag{A 5}$$

The solutions to this equation are then the intersections between $\tan(\pi\alpha)$ and a straight line, as shown in figure 1(b). There are infinitely many solutions $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$. For example, for $\pi T/KL = 1$, we find

$$\alpha_1 = 0.788, \quad \alpha_2 = 1.672, \quad \alpha_3 = 2.616, \quad \alpha_4 = 3.587, \ldots \tag{A 6}$$

In figure 2 we plot how α_1, α_2 and α_3 depend on $\pi T/KL$. For $\pi T/KL = 0$, we find $\alpha_k = k$, whereas for $\pi T/KL \to \infty$, $\alpha_k \to k - 1/2$.

As a result of the discussion above, we find that the eigenvalues are

$$\Lambda_k = -\frac{\alpha_k^2\pi^2}{L^2}, \tag{A 7}$$

with $k = 1, 2, 3, \ldots$ The corresponding general solution is then

$$y(x,t) = \sum_{k=1}^{\infty}\left[A_k\cos\left(\frac{\alpha_k\pi ct}{L}\right) + B_k\sin\left(\frac{\alpha_k\pi ct}{L}\right)\right]\sin\left(\frac{\alpha_k\pi x}{L}\right). \tag{A 8}$$

In this system, the energy that splits into a nice sum of the energies of the modes must include the potential energy of the spring at $x = L$,

$$E = \frac{\mu}{2}\int_0^L\left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x + \frac{T}{2}\int_0^L\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x + \frac{1}{2}Ky^2(x=L,t) \tag{A 9}$$

$$= \frac{\pi^2 T}{2L^2}\sum_{k=1}^{\infty}\alpha_k^2(A_k^2 + B_k^2)\int_0^L \sin^2\left(\frac{\alpha_k\pi x}{L}\right)\mathrm{d}x. \tag{A 10}$$

The energy of the stretched string alone, $(\mu/2)\int_0^L(\partial y/\partial t)^2\,\mathrm{d}x + (T/2)\int_0^L(\partial y/\partial x)^2\,\mathrm{d}x$, does not satisfy this property. It is easy to see that this is the case: the energy of the spring

$$\frac{1}{2}Ky^2(x=L,t) = \frac{K}{2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left[A_k\cos\left(\frac{\alpha_k\pi ct}{L}\right) + B_k\sin\left(\frac{\alpha_k\pi ct}{L}\right)\right]$$
$$\times\left[A_l\cos\left(\frac{\alpha_l\pi ct}{L}\right) + B_l\sin\left(\frac{\alpha_l\pi ct}{L}\right)\right]\sin(\alpha_k\pi)\sin(\alpha_l\pi) \tag{A 11}$$

cannot be split into a sum of the energies of the modes and hence the energy of the stretched string, given by $E - Ky^2(x=L,t)/2$, cannot either.