Sinusoidal waves, boundary conditions and transmission problems

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1. Introduction

In these notes, we introduce sinusoidal waves, a very useful tool because it simplifies significantly some of the calculations. In particular, we will show how to use them to obtain solutions with complex boundary conditions and with inhomogeneous wave speed c. In future lectures, we will see that sinusoidal waves can also be used for equations that are not solved by the d'Alembert solution.

2. Sinusoidal waves

We have seen that the wave equation is solved by the d'Alembert solution $y(x,t) = f(x-ct) + g(x+ct)$. A particularly interesting option for $f(u)$ and $g(v)$ are sines and cosines. For example, we can choose

$$f(x-ct) = C\cos(k(x-ct)+\varphi). \tag{2.1}$$

The sinusoidal wave is charaterised by
- **Wavenumber** $= k$,
- **Wavelength** $= 2\pi/k$,
- **Angular frequency** $\omega = kc$,
- **Frequency** $f = \omega/2\pi$,
- **Period** $\tau = 2\pi/\omega$,
- **Amplitude** $= C$, and
- **Phase** $= \varphi$.

It is usually convenient to rewrite the sinusoidal wave as a complex exponential,

$$C\cos(kx-\omega t+\varphi) = \frac{1}{2}Ce^{\mathrm{i}\varphi}e^{\mathrm{i}kx-\mathrm{i}\omega t} + \text{complex conjugate}, \tag{2.2}$$

where the complex number $Ce^{\mathrm{i}\varphi}$ is the complex amplitude. Both the sinusoidal wave and the complex exponential are solutions to the wave equation, and can be used interchangeably.

Why would we focus on sinusoidal solutions to the wave equation? The most important reason is that one can construct any function using them. Indeed, under certain assumptions, one can write any function $f(u)$ as a linear combination of sinusoidal waves,

$$f(u) = \int_{-\infty}^{\infty} F(k)e^{\mathrm{i}ku}\,\mathrm{d}k. \tag{2.3}$$

You will see a proof of this next year, in Mathematical Methods. For this course, it is sufficient to know that this is possible to do, and hence one should be interested in sinusoidal waves.

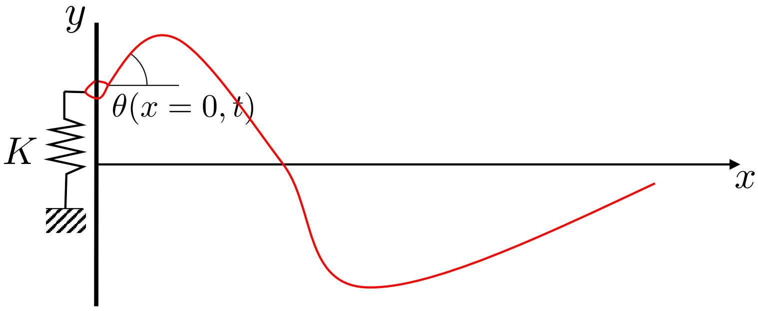

FIGURE 1. Boundary condition at $x = 0$.

3. Boundary conditions for sinusoidal waves

In this section of the notes, we will consider two problems to illustrate the advantages of using sinusoidal waves: a stretched string with a general boundary condition, and transmission lines.

3.1. *Stretched string with general boundary condition*

We consider a semi-infinite string that extends from $x = 0$ to $x = +\infty$. At $x = 0$, the string is knotted around a frictionless column, and it is attached to a vertical spring with spring constant K. The knot is massless. The configuration is sketched in figure 1. Force balance at the knot gives

$$T\frac{\partial y}{\partial x}(x = 0, t) - Ky(x = 0, t) = 0. \tag{3.1}$$

Applying boundary condition (3.1) to the d'Alembert solution is possible but tedious. The problem becomes more tractable if we use sinusoidal waves. We consider a sinusoidal wave traveling from $x = +\infty$ to $x = 0$,

$$g(x + ct) = \frac{1}{2}Ce^{-\mathrm{i}k(x+ct)} + \text{complex conjugate}. \tag{3.2}$$

Here, C can be a complex number. Using boundary condition (3.1), we can obtain the wave traveling to the right (reflected wave), $f(x - ct)$, from the incoming wave $g(x + ct)$. With a sinusoidal incoming wave $g(x + ct)$, obtaining the reflected wave turns out to be trivial. Since all the coefficients in boundary condition (3.1) are constant, the reflected wave is also sinusoidal,

$$f(x - ct) = \frac{1}{2}Re^{\mathrm{i}k(x-ct)} + \text{complex conjugate}. \tag{3.3}$$

Note that only the complex constant R is not known here. Note as well that we have written the complex exponential such that the time dependence is the same for both the right- and the left-traveling waves, i.e. they are both proportional to $e^{-\mathrm{i}\omega t}$. This is a choice that simplifies the equations below. It is always possible to make this choice because the solution is a complex exponential plus its complex conjugate.

We can check that the reflected wave in equation (3.3) is the solution by checking that it satisfies boundary condition (3.1). Since $y(x, t) = f(x - ct) + g(x + ct)$, the different functions that enter in the boundary conditions are

$$y(x = 0, t) = \frac{1}{2}(C + R)e^{-\mathrm{i}kct} + \text{complex conjugate} \tag{3.4}$$

and

$$\frac{\partial y}{\partial x}(x=0,t) = \frac{1}{2}\mathrm{i}k(-C+R)e^{-\mathrm{i}kct} + \text{complex conjugate.} \tag{3.5}$$

Replacing all these results into equation (3.1), we find

$$\mathrm{i}kT(-C+R) - K(C+R) = 0. \tag{3.6}$$

This equation gives R as a function of C,

$$\frac{R}{C} = -\frac{K+\mathrm{i}kT}{K-\mathrm{i}kT}. \tag{3.7}$$

Note that the modulus of R/A is one, and hence

$$\frac{R}{C} = e^{\mathrm{i}\varphi}, \tag{3.8}$$

where

$$\varphi = -\pi + 2\arctan\left(\frac{kT}{K}\right) \tag{3.9}$$

is the phase between the right- and the left-traveling wave. In the limit $kT/K \to 0$, the spring is too stiff and the point at $x=0$ barely moves, giving $y(x=0,t)=0$ and $\varphi = -\pi$, i.e. when the extreme of the string is pinned, the right- and left-traveling wave are in anti-phase (cf with the result that we obtained for the d'Alembert solution). In the limit $kT/K \to \infty$, the spring barely exerts any force, and the spring behaves as if there was no vertical force at $x=0$, leading to $\partial y(x=0,t)/\partial x = 0$ and $\varphi = 0$. In this case, the two waves are in phase, as one would expect from the calculation that we performed in the notes about the d'Alembert solution.

3.2. *Transmission lines and wave impedance*

To transmit electric power at high frequencies, one cannot use simple wires as these would radiate the power away. Instead, we use transmission lines such as co-axial cables. These transmission lines can be conceptualized as two conducting wires next to each other, as sketched in figure 2. One of the conductors absorbs the waves emitted by the other, preventing losses due to radiation. The potential difference $V(x,t)$ between the two conductors and the current $I(x,t)$ flowing through them depend on the position x along the line and the time t. Note that the current on one of the conductors is opposite to the current on the other to maintain zero charge. Focusing on an infinitesimal piece of the line of length $\mathrm{d}x$, we note two important facts:

- Consider the piece of the top conductor of length $\mathrm{d}x$ shown in figure 2 as a dashed box. Charge $\mathrm{d}Q$ accumulates within this piece of conductor due to the different currents at its two extremes,

$$\frac{\partial}{\partial t}\mathrm{d}Q = I(x) - I(x+\mathrm{d}x) = -\frac{\partial I}{\partial x}\,\mathrm{d}x. \tag{3.10}$$

The same charge but with opposite sign accumulates in the other conductor. Thus, the two pieces of conductor of length $\mathrm{d}x$ work as a capacitor. Since the capacitance is proportional to the length of the conductor, we can write the capacitance as $C'\,\mathrm{d}x$, where C' is a capacitance per unit length. Thus, $\mathrm{d}Q = C'V\,\mathrm{d}x$. Using this result and assuming that C' is constant in time, equation (3.10) gives

$$C'\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x}. \tag{3.11}$$

- The currents through the conductor produce time varying magnetic fields that in

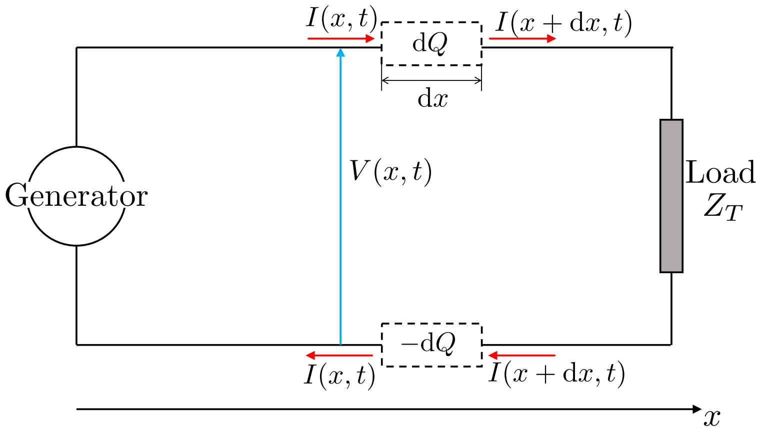

FIGURE 2. Transmission line with currents in red and potential difference in blue.

turn drive electromotive forces in the conductors. The resulting inductance is proportional to the length, and hence, for a piece of the transmission line of length $\mathrm{d}x$, it can be written as $L'\,\mathrm{d}x$, where L' is an inductance per unit length. The potential drop due to the inductance $L'\,\mathrm{d}x$ is

$$V(x) - V(x + \mathrm{d}x) = L' \frac{\partial I}{\partial t}\,\mathrm{d}x. \tag{3.12}$$

This equation then gives

$$-\frac{\partial V}{\partial x} = L' \frac{\partial I}{\partial t}. \tag{3.13}$$

Combining equations (3.11) and (3.13), we find a wave equation. For example, differentiating equation (3.11) with respect to time, we find

$$\frac{\partial^2 V}{\partial t^2} = -\frac{1}{C'}\frac{\partial^2 I}{\partial t \partial x}. \tag{3.14}$$

To calculate $\partial^2 I/\partial t \partial x$, we differentiate equation (3.13) with respect to x (assuming that L' is constant along the transmission line), finally obtaining

$$\frac{\partial^2 V}{\partial t^2} = c^2 \frac{\partial^2 V}{\partial x^2}, \tag{3.15}$$

where the wave speed in this case is given by

$$c = \frac{1}{\sqrt{L'C'}}. \tag{3.16}$$

Thus, we can use the d'Alembert solution for the voltage,

$$V(x,t) = f_V(x - ct) + g_V(x + ct). \tag{3.17}$$

A similar set of manipulations can be used to show that the current $I(x,t)$ also satisfies the wave equation and can hence be written as

$$I(x,t) = f_I(x - ct) + g_I(x + ct). \tag{3.18}$$

Typically, transmission lines end at electric loads that are characterized by their complex impedance Z_T that relates a sinusoidal voltage of frequency ω with the sinusoidal current that this voltage induces in the load. We consider such a load at the end of a semi-infinite transmission line that extends from $x = -\infty$ to $x = 0$. For a voltage

$$V(x = 0, t) = \frac{1}{2} V_0 e^{-\mathrm{i}\omega t} + \text{complex conjugate} \tag{3.19}$$

and a current

$$I(x = 0, t) = \frac{1}{2} I_0 e^{-\mathrm{i}\omega t} + \text{complex conjugate} \tag{3.20}$$

at $x = 0$, the load imposes

$$V_0 = Z_T I_0. \tag{3.21}$$

Let us assume that we have launched a right-traveling wave towards the load with a voltage of the form

$$f_V(x - ct) = \frac{1}{2} A e^{\mathrm{i}k(x-ct)} + \text{complex conjugate.} \tag{3.22}$$

The presence of the boundary condition (3.21) will induce a reflected wave

$$g_V(x + ct) = \frac{1}{2} R e^{-\mathrm{i}k(x+ct)} + \text{complex conjugate.} \tag{3.23}$$

To get the ratio R/A, we can use the useful concept of transmission impedance. This is the ratio of the voltage and the current of the right-traveling wave,

$$Z_L = \frac{f_V(x - ct)}{f_I(x - ct)}. \tag{3.24}$$

Using equation (3.11) and $c = 1/\sqrt{L'C'}$, we find

$$Z_L = \frac{1}{cC'} = \sqrt{\frac{L'}{C'}}. \tag{3.25}$$

The same ratio for the left-traveling wave gives an impedance with the opposite sign,

$$\frac{g_V(x + ct)}{g_I(x + ct)} = -Z_L. \tag{3.26}$$

Using these results in equation (3.21) gives

$$A + R = Z_T \left(\frac{A}{Z_L} - \frac{R}{Z_L} \right), \tag{3.27}$$

which can be solved to obtain

$$\frac{R}{A} = \frac{Z_T - Z_L}{Z_T + Z_L}. \tag{3.28}$$

Thus, we have found that in general loads reflect waves. These reflected waves are undesirable because they mean that not all the energy transmitted has been used by the load, and because they can damage the electric generator that produced the incoming wave. Thus, one needs to design the load such that $Z_T = Z_L$. This is known as impedance matching, and it is achieved by adding capacitors, inductors and resistors to the final load.

4. Transmission problems

Up to this point, we have considered wave propagation in homogeneous media. We will now start considering simple inhomogeneous systems. As an example, we proceed to study a stretched string composed of two strings a and b of different linear density, μ_a and μ_b. These two strings are tied together at $x = 0$.

We know that the vertical motion of both strings, $y_a(x,t)$ and $y_b(x,t)$, is described by the wave equation, but the wave speed is different in each of them. Due to horizontal force balance at the knot, the tension has to be the same in both strings, giving the speeds $c_a = \sqrt{T/\mu_a}$ and $c_b = \sqrt{T/\mu_b}$. But in addition to knowing how each string moves, we need to state how they interact. Since they are tied together at $x = 0$, we must have continuity of vertical displacement,

$$y_a(x = 0, t) = y_b(x = 0, t). \tag{4.1}$$

Assuming that the knot is massless, vertical force balance at the knot gives

$$T\frac{\partial y_a}{\partial x}(x = 0, t) = T\frac{\partial y_b}{\partial x}(x = 0, t). \tag{4.2}$$

As an example of the motion in this two-string system, we consider a right-traveling wave

$$f_a(x - c_a t) = \frac{1}{2}Ae^{\mathrm{i}k(x - c_a t)} + \text{complex conjugate} \tag{4.3}$$

moving towards the knot at $x = 0$. The knot will reflect part of this wave, giving

$$y_a(x,t) = f_a(x - c_a t) + g_a(x + c_a t), \tag{4.4}$$

with

$$g_a(x + c_a t) = \frac{1}{2}Re^{-\mathrm{i}k(x + c_a t)} + \text{complex conjugate}. \tag{4.5}$$

The complex constant R still needs to be determined. The displacement of the knot at $x = 0$ will also launch a right-traveling wave in the second string,

$$y_b(x,t) = f_b(x - c_b t), \tag{4.6}$$

where

$$f_b(x - c_b t) = \frac{1}{2}Be^{\mathrm{i}k_b(x - c_b t)} + \text{complex conjugate}. \tag{4.7}$$

The wavenumber k_b and the complex constant B still need to be determined.

Imposing conditions (4.1) and (4.2), we will be able to find k_b, R and B. Indeed, condition (4.1) gives

$$Ae^{-\mathrm{i}kc_a t} + Re^{-\mathrm{i}kc_a t} = Be^{-\mathrm{i}k_b c_b t}. \tag{4.8}$$

Thus, the frequency of all the waves must coincide,

$$\omega = kc_a = k_b c_b, \tag{4.9}$$

giving

$$k_b = k\frac{c_a}{c_b} = k\sqrt{\frac{\mu_b}{\mu_a}}. \tag{4.10}$$

Then, condition (4.1) simplifies to

$$A + R = B \tag{4.11}$$

Condition (4.2) gives

$$kA - kR = k_b B \tag{4.12}$$

Using all these equations, we finally obtain

$$\frac{B}{A} = \frac{2c_b}{c_a + c_b} = \frac{2\sqrt{\mu_a}}{\sqrt{\mu_a} + \sqrt{\mu_b}} \tag{4.13}$$

and

$$\frac{R}{A} = \frac{c_b - c_a}{c_a + c_b} = \frac{\sqrt{\mu_a} - \sqrt{\mu_b}}{\sqrt{\mu_a} + \sqrt{\mu_b}}. \tag{4.14}$$

Thus, there is always a reflected wave and not all the energy travels through the knot.

It is instructive to consider energy balance in this problem. We have already seen that the work done by a stretched string is

$$-T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = \epsilon_f c - \epsilon_g c. \tag{4.15}$$

where $\epsilon_f := T(f')^2$ and $\epsilon_g := T(g')^2$. For sinusoidal waves

$$f(x - ct) = \frac{1}{2}A_f e^{\mathrm{i}k(x-ct)} + \text{complex conjugate} \tag{4.16}$$

and

$$g(x + ct) = \frac{1}{2}A_g e^{-\mathrm{i}k(x+ct)} + \text{complex conjugate}, \tag{4.17}$$

both ϵ_f and ϵ_g are periodic in time with period $\tau = 2\pi/\omega = 2\pi/kc$. Thus, it is convenient to use the time average

$$\langle F \rangle_\tau = \frac{1}{\tau}\int_t^{t+\tau} F(t')\,\mathrm{d}t'. \tag{4.18}$$

With this time average, the work done by the string becomes

$$-T\left\langle \frac{\partial y}{\partial x}\frac{\partial y}{\partial t} \right\rangle_\tau = \langle \epsilon_f \rangle_\tau c - \langle \epsilon_g \rangle_\tau c, \tag{4.19}$$

where

$$\langle \epsilon_f \rangle_\tau = \frac{1}{2}k^2 T|A_f|^2, \quad \langle \epsilon_g \rangle_\tau = \frac{1}{2}k^2 T|A_g|^2. \tag{4.20}$$

Applying this result to $x = 0$, where the work done by the string a should be equal to the work received by the string b, we find that

$$\frac{1}{2}k^2|A|^2 c_a - \frac{1}{2}k^2|R|^2 c_a = \frac{1}{2}k_b^2|B|^2 c_b, \tag{4.21}$$

that is, the energy of the incoming wave in string a must be equal to the energy of the reflected wave in string a and of the outgoing wave in string b. This equation can be checked to be satisfied using equations (4.10), (4.13) and (4.14).