Energy conservation in the stretched string

Felix I. Parra

Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Oxford OX1 3PU, UK

(This version is of 7 January 2021)

1. Introduction

In these notes, we present the energy conservation equation for the stretched string. The equation for energy conservation will reveal interesting physics, such as the fact that waves can transport energy from one place to another even though matter is not being transported in the process.

2. Energy density

We start by calculating the energy in an infinitesimally small piece $\mathrm{d}x$ of a stretched string with linear density μ tensioned by a force T. The string and the infinitesimal piece of interest are represented in figure 1. The infinitesimal piece of string contains infinitesimal amounts of kinetic and potential energy.

The kinetic energy of the infinitesimal piece of string is

$$\mathrm{d}E_k = \frac{1}{2}\mu \left(\frac{\partial y}{\partial t} \right)^2 \mathrm{d}x, \tag{2.1}$$

where we have used the fact that the mass of the piece of string is $\mu\,\mathrm{d}x$. Note that we have ignored the velocity of the infinitesimal piece of string in the x direction. We can do this because the motion in the x and y direction are decoupled from each other.

The potential energy is due to the work done by the tension of the string. To move the string in the y direction, we have increased the original length of the string from $\mathrm{d}x$ to

$$\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\partial y}{\partial x} \right)^2}\, \mathrm{d}x \simeq \left[1 + \frac{1}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right] \mathrm{d}x, \tag{2.2}$$

where we have used $\partial y/\partial x \sim y/L \ll 1$. Due this increase in length, the rest of the string is doing work on the infinitesimal length of string of interest, and this work is

$$\mathrm{d}E_p = T(\mathrm{d}l - \mathrm{d}x) \simeq \frac{1}{2} T \left(\frac{\partial y}{\partial x} \right)^2 \mathrm{d}x. \tag{2.3}$$

The total energy of the string can be calculated by summing the kinetic and potential energies of all the infinitesimal pieces of string,

$$E = \int \mathrm{d}E_k + \int \mathrm{d}E_p = \int \left[\frac{1}{2}\mu \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} T \left(\frac{\partial y}{\partial x} \right)^2 \right] \mathrm{d}x. \tag{2.4}$$

The quantity

$$\epsilon = \frac{1}{2}\mu \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} T \left(\frac{\partial y}{\partial x} \right)^2 \tag{2.5}$$

is the energy density and gives the energy per unit length.

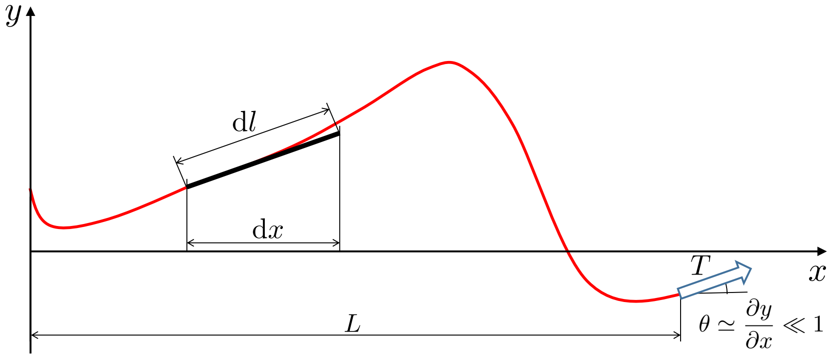

FIGURE 1. Stretched string in red and an infinitesimal piece of it in black.

3. Energy conservation equation

We proceed to study the time evolution of the energy density. By simple differentiation, we find

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2\right] = \mu\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2} + T\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial t \partial x}. \tag{3.1}$$

Employing the wave equation to write $\partial^2 y/\partial t^2 = c^2(\partial^2 y/\partial x^2)$, we rewrite this equation as

$$\frac{\partial \epsilon}{\partial t} = \mu c^2 \frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial x^2} + T\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial t \partial x}. \tag{3.2}$$

Finally, recalling that $c^2 = T/\mu$, we find the nice expression

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right). \tag{3.3}$$

To understand what the term in the right side of equation (3.3) means, we consider the energy of a finite piece of string that extends from $x = 0$ to $x = L$. Integrating equation (3.3) from $x = 0$ to $x = L$, we obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} = T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\bigg|_{x=L} - T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\bigg|_{x=0}. \tag{3.4}$$

The fact that the term in the right side of equation (3.3) is a spatial derivative implies that the time evolution of the energy of a finite length of string only depends on what happens at its boundaries $x = 0$ and $x = L$. Physically, this is not surprising because the boundary terms are the work done by the rest of the string on the piece of string that we are considering. Indeed, in figure 1 we show that the vertical force F_y exerted by the rest of the string on $x = L$ is

$$F_y = T\frac{\partial y}{\partial x}(x = L, 0). \tag{3.5}$$

Thus, the first term in the right side of equation (3.4) is the power due to this vertical force, $F_y v_y$, where $v_y = \partial y/\partial t$ is the vertical velocity of the string. A similar result can be found for the term at $x = 0$.

The boundary terms in equation (3.4) can also be interpreted as energy fluxes in and out of the length of string of interest. Using the d'Alembert solution $y(x,t) = f(x - ct) +$

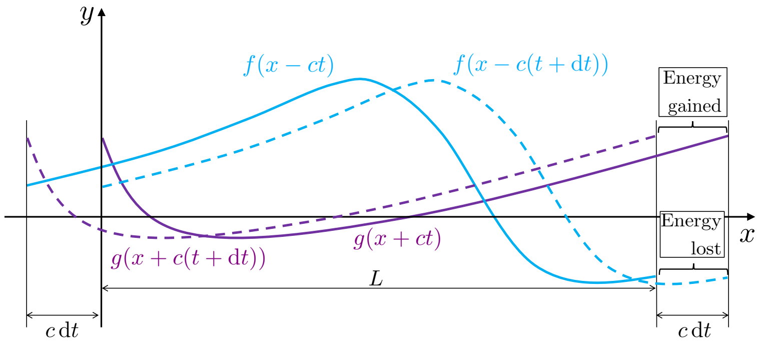

FIGURE 2. Right- and left-traveling waves (in blue and purple, respectively) at times t (solid lines) and $t + \mathrm{d}t$ (dashed lines).

$g(x + ct)$, we can rewrite the energy density and the boundary terms as

$$\epsilon = \frac{1}{2}\mu(-cf' + cg')^2 + \frac{1}{2}T(f' + g')^2 \tag{3.6}$$

and

$$T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = T(f' + g')(-cf' + cg'), \tag{3.7}$$

where f' and g' are the derivatives of $f(u)$ and $g(v)$ with respect to their arguments. Using $c^2 = T/\mu$, these expressions can be converted into

$$\epsilon = T(f')^2 + T(g')^2 \tag{3.8}$$

and

$$T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = -T(f')^2 c + T(g')^2 c. \tag{3.9}$$

Both the energy density and the boundary terms can be split into two additive terms (even though they are nonlinear quantities!). In particular, the energy density can be split into the energy density of the right-traveling wave, $\epsilon_f = T(f')^2$, and the energy density of the left-traveling wave, $\epsilon_g = T(g')^2$. As a result, the boundary term can be written as

$$T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = -\epsilon_f c + \epsilon_g c, \tag{3.10}$$

and equation (3.4) becomes

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -c\,\epsilon_f\big|_{x=L} + c\,\epsilon_g\big|_{x=L} + c\,\epsilon_f\big|_{x=0} - c\,\epsilon_g\big|_{x=0}\,. \tag{3.11}$$

To understand these terms, we consider what happens at $x = L$ with the right-traveling wave $f(x - ct)$. In a time $\mathrm{d}t$, the wave moves a distance $c\,\mathrm{d}t$ to the right – see figure 2. The energy of the piece of $f(x - ct)$ that leaves the region between $x = 0$ and $x = L$, $\epsilon_f c\,\mathrm{d}t$, is lost, giving a rate of energy loss of $-\epsilon_f c$, as seen in equation (3.11). Similarly, the left-traveling wave $g(x + ct)$ at $x = L$ moves into the region between $x = 0$ and $x = L$, feeding energy into the piece of string of interest at a rate $\epsilon_g c$. For this reason, the boundary terms in equation (3.4) can be interpreted as energy fluxes.

We finish by noting that the energy conservation equation is satisfied for the left- and right-traveling waves independently. By simply noting that f depends on $x - ct$ and g depends on $x + ct$, it can be shown that

$$\frac{\partial \epsilon_f}{\partial t} + c\frac{\partial \epsilon_f}{\partial x} = 0 \tag{3.12}$$

and

$$\frac{\partial \epsilon_g}{\partial t} - c\frac{\partial \epsilon_g}{\partial x} = 0. \tag{3.13}$$