General solution to the wave equation

Felix I. Parra
Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Oxford OX1 3PU, UK

(This version is of 23 February 2021)

1. Introduction

In these notes, we give the general solution to the wave equation. The wave equation is one of the rare PDEs that we can solve analytically with complete generality. We will use this analytical solution to explain that PDEs need two types of conditions: initial conditions (similar to those needed for Ordinary Differential Equations, ODEs) and boundary conditions (a new concept).

2. The d'Alembert solution

We start from the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{2.1}$$

where c is a constant with units of velocity. To solve this equation, we perform first a change of variables from x and t to

$$u = x - ct \tag{2.2}$$

and

$$v = x + ct. \tag{2.3}$$

Note that, since the relation between (u, v) and (x, t) is univocal, a solution in the new coordinates, $y(u, v)$, is necessarily the solution in x and t.

We need to rewrite equation (2.1) in the new coordinates u and v. We will start rewriting the time derivatives. Using the chain rule, we find

$$\left.\frac{\partial y}{\partial t}\right|_x = \left.\frac{\partial u}{\partial t}\right|_x \left.\frac{\partial y}{\partial u}\right|_v + \left.\frac{\partial v}{\partial t}\right|_x \left.\frac{\partial y}{\partial v}\right|_u. \tag{2.4}$$

For clarity, we have made the coordinate being held constant explicit. Noting that $\partial u/\partial t|_x = -c$ and $\partial v/\partial t|_x = c$, we find

$$\left.\frac{\partial y}{\partial t}\right|_x = -c \left(\left.\frac{\partial y}{\partial u}\right|_v - \left.\frac{\partial y}{\partial v}\right|_u \right). \tag{2.5}$$

Using the chain rule again, we can write

$$\left.\frac{\partial^2 y}{\partial t^2}\right|_x = -c \left[\left.\frac{\partial}{\partial u}\right|_v \left(\left.\frac{\partial y}{\partial t}\right|_x \right) - \left.\frac{\partial}{\partial v}\right|_u \left(\left.\frac{\partial y}{\partial t}\right|_x \right) \right]. \tag{2.6}$$

Substituting equation (2.5) into equation (2.6), we finally obtain

$$\frac{\partial^2 y}{\partial t^2} = c^2 \left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right), \tag{2.7}$$

where we are not indicating the coordinate being held constant to avoid cluttering notation. Similarly,

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \tag{2.8}$$

and

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}. \tag{2.9}$$

Substituting equations (2.7) and (2.9) into equation (2.1), we obtain the extremely simple expression

$$\frac{\partial^2 y}{\partial u \partial v} = 0. \tag{2.10}$$

Equation (2.10) implies that $\partial y/\partial v$ is independent of u, that is,

$$\frac{\partial y}{\partial v} = h(v), \tag{2.11}$$

where $h(v)$ can be any function as long as it only depends on v. Integrating this equation, we find

$$y(u,v) = f(u) + \int h(v)\,\mathrm{d}v. \tag{2.12}$$

Here the function $f(u)$ is the 'constant' of integration. Renaming the indefinite integral $\int h(v)\,\mathrm{d}v$ as $g(v)$, we obtain the complete d'Alembert solution,

$$y(u,v) = f(u) + g(v). \tag{2.13}$$

In the coordinates x and t,

$$y(x,t) = f(x - ct) + g(x + ct). \tag{2.14}$$

Solution (2.14) is the reason why equation (2.1) is known as the wave equation. Any solution to the wave equation can always be split into the two functions $f(u)$ and $g(v)$ in equation (2.14), and these two functions move rigidly along x: the function f towards positive x and the function g towards negative x. Indeed, as time advances, the function that initially was $f(x)$ becomes $f(x - ct)$. Thus, the value f_0 of f located at $x = x_0$ moves to

$$x = x_0 + ct, \tag{2.15}$$

i.e. it moves towards positive x. All values of f move at the same speed c, giving the rigid displacement that we announced above. A similar argument can be applied to show that $g(v)$ moves rigidly towards negative x. These two rigidly moving functions are the waves of the wave equation. From here on, we refer to $f(x - ct)$ as the right-traveling wave, and to $g(x + ct)$ as the left-traveling wave.

We proceed to discuss what determines the functions $f(u)$ and $g(v)$: initial conditions and boundary conditions.

3. Initial conditions

The physical systems that we have studied (the stretched string, the solid bar and the gas in a pipe) are a good justification for the need of initial conditions: to determine the time evolution of a body in mechanics, we need the initial position and velocity. In the case of the stretched string, that means knowing the initial vertical displacement,

$$y(x, t = 0) = y_0(x), \tag{3.1}$$

and the initial vertical velocity,

$$\frac{\partial y}{\partial t}(x, t = 0) = \dot{y}_0(x). \tag{3.2}$$

This is a general property of any PDE: one needs as many initial conditions as the order of the highest order partial derivatives with respect to time in the PDE. In this case, there is a second order derivative with respect to time in the wave equation, and hence we need two initial conditions.

It is possible to find the solution given the initial conditions above (and ignoring boundary conditions, which we will discuss in the next section). Using the d'Alembert solution (2.14) and equation (2.5) for $\partial y/\partial t$, the two initial conditions become

$$f(x) + g(x) = y_0(x) \tag{3.3}$$

and

$$-c\,[f'(x) - g'(x)] = \dot{y}_0(x), \tag{3.4}$$

where f' and g' are the derivatives of f and g with respect to their arguments. We can integrate equation (3.4) in x to obtain

$$f(x) - g(x) = f(0) - g(0) - \frac{1}{c}\int_0^x \dot{y}_0(s)\,\mathrm{d}s. \tag{3.5}$$

We will see shortly that we are free to choose the value of $f(x) - g(x)$ at $x = 0$. We can solve for $f(x)$ and $g(x)$ using equations (3.3) and (3.5), finding

$$f(x) = \frac{y_0(x)}{2} - \frac{1}{2c}\int_0^x \dot{y}_0(s)\,\mathrm{d}s + \frac{f(0) - g(0)}{2} \tag{3.6}$$

and

$$g(x) = \frac{y_0(x)}{2} + \frac{1}{2c}\int_0^x \dot{y}_0(s)\,\mathrm{d}s - \frac{f(0) - g(0)}{2}. \tag{3.7}$$

With these results, equation (2.14) gives

$$y(x, t) = \frac{y_0(x - ct) + y_0(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \dot{y}_0(s)\,\mathrm{d}s. \tag{3.8}$$

Note that the difference $f(0) - g(0)$ does not contribute to the final answer, that is, the constant of integration in equation (3.5) is, in the end, irrelevant. The idea is that the functions $f(u)$ and $g(v)$ are only defined up to a constant: we can add a constant to $f(u)$ if we subtract the same constant from $g(v)$.

We proceed to solve a few examples of initial conditions.

3.1. *Stationary initial condition*

We start with a case in which $\dot{y}_0(x)$ is zero. Consider the infinite stretched string in figure 1(a). It is held in place such that the points at $x = -L$ and L are not displaced, $y(x = \pm L, t) = 0$, and such that the point at $x = 0$ is displaced a distance a, $y(x = 0, t) = a$. Since the string is being held in place, $\partial^2 y/\partial t^2 = 0$, and the wave equation (2.1) simply becomes $\partial^2 y/\partial x^2 = 0$. This equation is valid only for $x \neq -L, 0, L$ because at these points we are applying forces to hold the string in place. From $\partial^2 y/\partial x^2 = 0$, we find that the slope of y is constant, and hence we obtain the piecewise linear function in

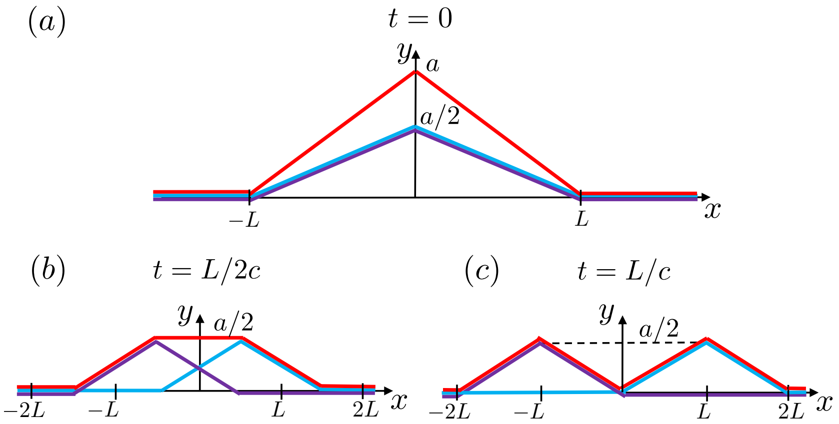

FIGURE 1. (a) Initial position of a stationary string (red) and the resulting right- and left-traveling waves (blue and purple, respectively). For times (b) $t = L/2c$ and (c) $t = L/c$, we sketch, in blue and purple, the right- and left-traveling waves, and in red the position of the string.

red in figure 1(a),

$$y(x, t = 0) = y_0(x) = \begin{cases} a(1 + x/L) & \text{for } -L \leqslant x < 0, \\ a(1 - x/L) & \text{for } 0 \leqslant x < L, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

In addition, we know that $\dot{y}_0(x) = 0$.

Using equations (3.6) and (3.7) and choosing $f(0) - g(0) = 0$, we find

$$f(u) = \frac{y_0(u)}{2} \tag{3.10}$$

and

$$g(v) = \frac{y_0(v)}{2}, \tag{3.11}$$

that is, the solution is composed of two equal waves moving in opposite directions. We represent these waves in blue and purple in figure 1(a). It is unsurprising that there are two waves moving in opposite directions because the initial conditions do not have a preferred direction.

By rigidly moving the two waves $f(u)$ and $g(v)$ and adding them, we can obtain the solution for any time t. We give examples of the solution at $t = L/2c$ and L/c in figures 1(b) and 1(c).

Note that the solution that we have obtained does not have continuous first order derivatives, and one might wonder how it could be a solution to a PDE that contains second order derivatives. We were able to bypass the problem with the second derivatives by using the d'Alembert solution (2.14). Thanks to the d'Alembert solution, we now know that the wave equation works even for discontinuous functions, and it only moves these discontinuities around, without smoothing them out. In a real physical systems, there is usually some dissipation present, and these discontinuities tend to disappear with time.

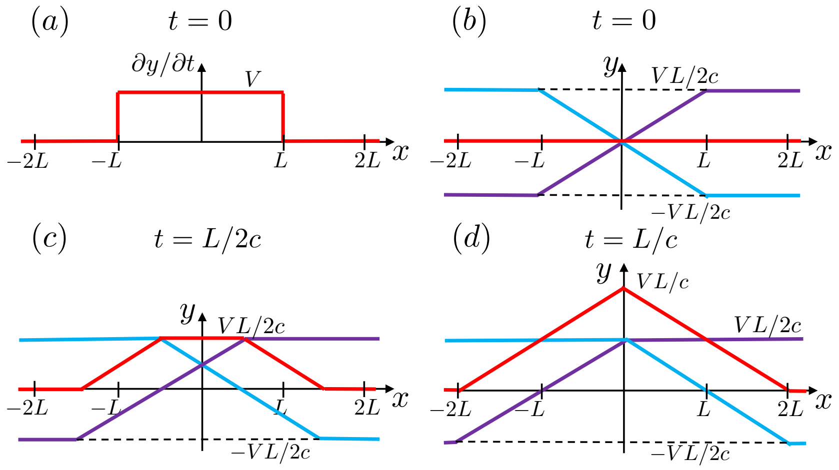

FIGURE 2. (a) Initial velocity of the wave (red). (b) Initial position of the string (red) and right- and left-traveling waves (in blue and purple, respectively). For times (c) $t = L/2c$ and (d) $t = L/c$, we sketch, in blue and purple, the right- and left-traveling waves, and in red the position of the string.

3.2. *Impulsive initial condition*

We now consider a stretched string that starts at $y_0(x) = 0$, but it is given an impulse such that its initial velocity is given by

$$\frac{\partial y}{\partial t}(x, t = 0) = \dot{y}_0 = \begin{cases} V & \text{for } -L \leqslant x \leqslant L, \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

The function $\dot{y}_0$ is sketched in figure 2(a).

Using equations (3.6) and (3.7) with $f(0) - g(0) = 0$, we find

$$f(u) = -\frac{1}{2c}\int_0^u \dot{y}_0(s)\,\mathrm{d}s \tag{3.13}$$

and

$$g(v) = \frac{1}{2c}\int_0^v \dot{y}_0(s)\,\mathrm{d}s, \tag{3.14}$$

where

$$\frac{1}{2c}\int_0^x \dot{y}_0(s)\,\mathrm{d}s = \begin{cases} -VL/2c & \text{for } x < -L, \\ Vx/2c & \text{for } -L \leqslant x < L, \\ VL/2c & \text{for } x \geqslant L. \end{cases} \tag{3.15}$$

These waves are represented as blue and purple curves in figure 2(b). Using them, we can obtain the vertical position of the string at all times t (see figure 2 for a few time slices).

3.3. *Single wave initial condition*

In both cases considered so far, we have obtained two waves traveling in opposite directions. If we wanted a wave of the shape $y_0(x)$ traveling towards positive x, what would we need to do initially? We would simply need to ensure that it has the right initial velocity. Since we want to have

$$y(x, t) = f(x - ct) = y_0(x - ct), \tag{3.16}$$

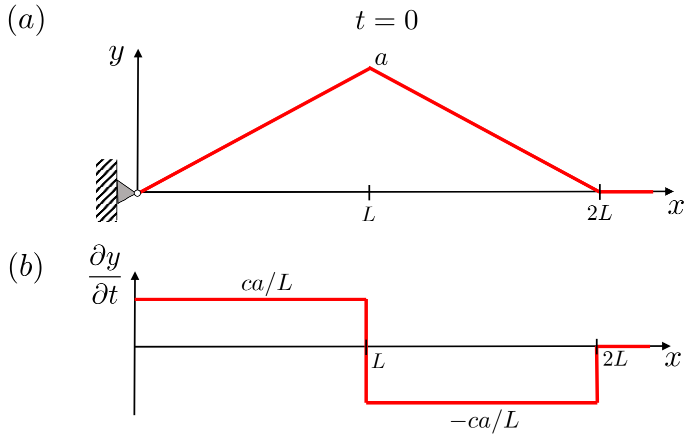

FIGURE 3. Initial (a) position and (b) velocity of a semi-infinite string with a pin at $x = 0$.

we need to impose the initial velocities

$$\dot{y}_0(x) = \frac{\partial y}{\partial t}(x, t = 0) = -c y_0'(x). \tag{3.17}$$

4. Boundary conditions

In all the cases in section 3, we have ignored the ends of the stretched string by assuming that it is infinite, but stretched strings do have ends where they are being pulled to ensure that they are tensioned. For example, we can consider a semi-infinite string that extends from $x = 0$ to $+\infty$, as sketched in figure 3. This string is kept in tension by being pinned to the wall at $x = 0$ and by the pull of something or someone a very long distance away from $x = 0$. The motion of such a string should be solvable given initial conditions.

To show the problems that arise for a semi-infinite string, we will try to solve the problem with initial conditions

$$y(x, t = 0) = y_0(x) = \begin{cases} ax/L & \text{for } 0 \leqslant x < L, \\ a(2 - x/L) & \text{for } L \leqslant x < 2L, \\ 0 & \text{for } x \geqslant 2L, \end{cases} \tag{4.1}$$

and

$$\frac{\partial y}{\partial t}(x, t = 0) = \dot{y}_0(x) = c y_0'(x) = \begin{cases} ca/L & \text{for } 0 \leqslant x < L, \\ -ca/L & \text{for } L \leqslant x < 2L, \\ 0 & \text{for } x \geqslant 2L, \end{cases} \tag{4.2}$$

sketched in figure 3. Using formulas (3.6) and (3.7) with $f(0) - g(0) = 0$, we find

$$f(u) = \begin{cases} ?? & \text{for } u < 0, \\ 0 & \text{for } u \geqslant 0, \end{cases} \tag{4.3}$$

and

$$g(v) = \begin{cases} ?? & \text{for } v < 0, \\ y_0(v) & \text{for } v \geqslant 0. \end{cases} \tag{4.4}$$

We cannot determine $f(u)$ or $g(v)$ for negative values of their arguments! This is not a problem for $g(v)$ since $v = x + ct$ is positive at all positive times for $x > 0$, but we need to do something for $f(u)$ since sufficiently small values of x will give negative values of

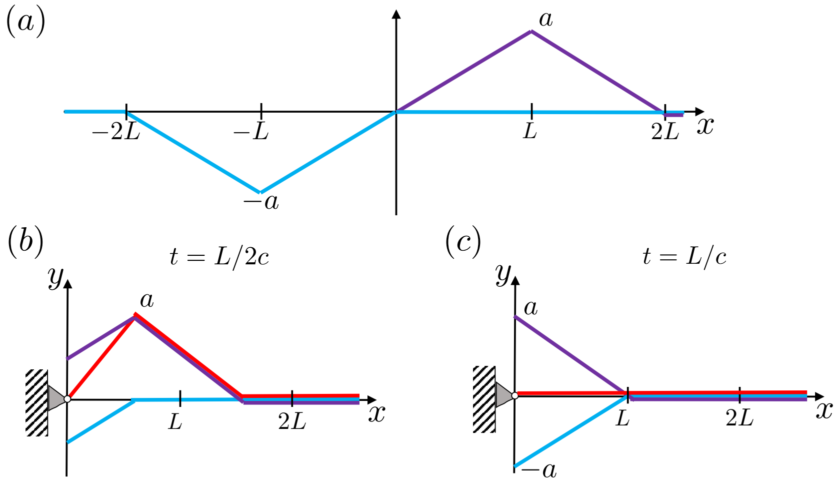

FIGURE 4. (a) Right- and left-traveling waves (in blue and purple, respectively) for the initial conditions and the boundary condition in figure 3. For times (b) $t = L/2c$ and (c) $t = L/c$, we sketch, in blue and purple, the right- and left-traveling waves, and in red the position of the string. Note that at $t = L/c$, the whole string satisfies $y(x, t = L/c) = 0$. Note as well that at all times, $y(x = 0, t) = 0$, as we wanted.

$u = x - ct$ at finite times t. The value of $f(u)$ for $u < 0$ is determined by the boundary condition at $x = 0$: we need to specify what the string is doing at $x = 0$.

In general, a PDE requires as many boundary conditions as the order of its highest order derivative with respect the spatial variable. The wave equation has a second order spatial derivatives and hence it requires two boundary conditions, one on each end of the string. One then might wonder how we solved the problems in section 3 without imposing boundary conditions on the 'ends' of the infinite string. We did in fact impose hidden boundary conditions: we did not let any waves travel into our domain from infinity, that is, we assumed that no other person was twiddling with the string at a large distance from us (here large means large compared to the extent of the initial condition). These boundary conditions are known as outgoing wave boundary conditions because we allow waves to leave our domain but we do not allow any waves to come into it.

We proceed to discuss three examples of boundary conditions for the semi-infinite string and one example with two simultaneous boundary conditions.

4.1. *Dirichlet boundary condition*

The Dirichlet boundary conditions are the ones in which the value of the function $y(x, t)$ is set at one point. One example of a Dirichlet boundary condition is the case in figure 3 that shows a string pinned at $x = 0$,

$$y(x = 0, t) = 0. \tag{4.5}$$

Imposing boundary condition (4.5) on the d'Alembert solution (2.14), we find

$$f(-ct) + g(ct) = 0. \tag{4.6}$$

Using the change of variables $u = -ct$, we obtain

$$f(u) = -g(-u), \tag{4.7}$$

that is, $f(u)$ for negative values of u is a reflection of $g(v)$ with respect to the point

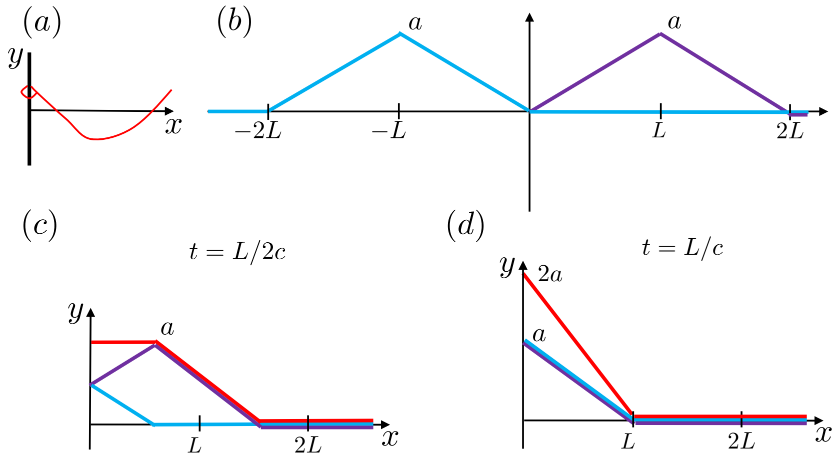

FIGURE 5. (a) Neumann boundary condition for a stretched string. (b) Right- and left-traveling waves (in blue and purple, respectively) for the initial conditions in figure 3. For times (c) $t = L/2c$ and (d) $t = L/c$, we sketch, in blue and purple, the right- and left-traveling waves, and in red the position of the string. Note that $\partial y/\partial x(x = 0, t)$ vanishes for almost every time. Time $t = L/c$ is one of the times at which the boundary condition is not satisfied, but note that it is satisfied for times infinitesimally close to $t = L/c$. This problem is due to the fact that the initial condition does not have continuous first derivatives.

$x = 0, y = 0$. Thus, the function $f(u)$ is

$$f(u) = \begin{cases} -y_0(-u) & \text{for } u < 0, \\ 0 & \text{for } u \geqslant 0. \end{cases} \tag{4.8}$$

The functions $f(u)$ and $g(v)$ are sketched in figure 4(a). By rigidly moving these functions and summing them, we find $y(x, t)$, as shown in figure 4.

4.2. *Neumann boundary condition*

The Neumann boundary conditions are the ones in which the value of the spatial derivative $\partial y(x, t)/\partial x$ is set at one point. One example of Neumann boundary condition is the set-up shown in figure 5(a). The string is kept stretched in this case by tying it in a knot around a frictionless column (represented as a thick vertical line in figure 5(a)). Since the column is frictionless, the only vertical force on the knot is the one exerted by the string,

$$F_y(x = 0, t) \simeq T\theta(x = 0, t) \simeq T\frac{\partial y}{\partial x}(x = 0, t). \tag{4.9}$$

We assume that the knot has negligibly small mass, giving $F_y(x = 0, t) = 0$, that is,

$$\frac{\partial y}{\partial x}(x = 0, t) = 0. \tag{4.10}$$

To show the effect of the Neumann boundary condition on the string, we solve the problem with the initial conditions (4.1) and (4.2) that lead to the partial solutions for $f(u)$ and $g(v)$ in equations (4.3) and (4.4). Imposing boundary condition (4.10) on the d'Alembert solution (2.14), we find

$$f'(-ct) + g'(ct) = 0. \tag{4.11}$$

Using the change of variable $u = -ct$, this expression becomes

$$f'(u) = -g'(-u). \tag{4.12}$$

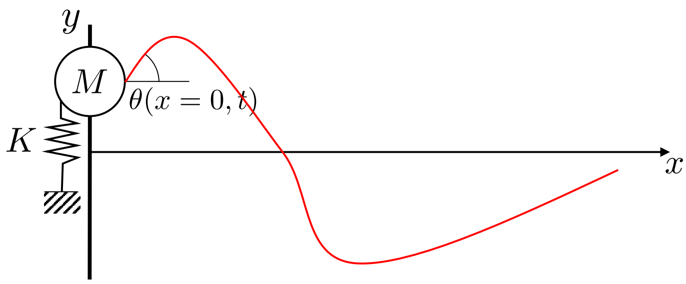

FIGURE 6. Example of general boundary condition for a stretched string.

Integrating this equation once and recalling that we had chosen $f(0) = g(0)$ to obtain equations (4.3) and (4.4), we find

$$f(u) = g(-u), \tag{4.13}$$

that is, $f(u)$ for negative u is a reflection of $g(v)$ with respect to the axis $x = 0$. With this result, we obtain the value of $f(u)$ for negative u,

$$f(u) = \begin{cases} y_0(-u) & \text{for } u < 0, \\ 0 & \text{for } u \geqslant 0. \end{cases} \tag{4.14}$$

The functions $f(u)$ and $g(v)$ are sketched in figure 5(b), and we can construct the solution for $t > 0$ by displacing them rigidly and summing them, as shown in figure 5 for a few times.

4.3. *General boundary condition*

In general, boundary conditions give $\partial y/\partial x$ at a given point x as a function of y and its time derivatives at that point. An example of a general boundary condition is given in figure 6. In this case, the string is knotted around a frictionless column, but the knot has a mass M and it is tied to a vertical spring with spring constant K. Thus, in addition to the vertical force due to the string, $T(\partial y/\partial x|_{x=0})$, the knot is pulled down by the force of the spring, $-Ky(x = 0, t)$, giving the equation of motion

$$M\frac{\partial^2 y}{\partial t^2}(x = 0, t) = T\frac{\partial y}{\partial x}(x = 0, t) - Ky(x = 0, t). \tag{4.15}$$

The application of this boundary condition to the d'Alembert solution is left as an exercise for the reader. We will see in a few lectures that these complex boundary conditions are usually easier to treat by using a technique different from the one that we have used so far.

4.4. *Problems with two boundary conditions*

We will now consider a finite stretched string with two boundary conditions. The string is of length $2L$ and it is pinned at both ends, located at $x = 0$ and $2L$. Thus, it has the Dirichlet boundary conditions

$$y(x = 0, t) = 0 \tag{4.16}$$

and

$$y(x = 2L, t) = 0. \tag{4.17}$$

We consider the motion of this string for the initial conditions

$$y(x, t = 0) = y_0(x) = \begin{cases} ax/L & \text{for } 0 \leqslant x < L, \\ a(2 - x/L) & \text{for } L \leqslant x < 2L, \end{cases} \tag{4.18}$$

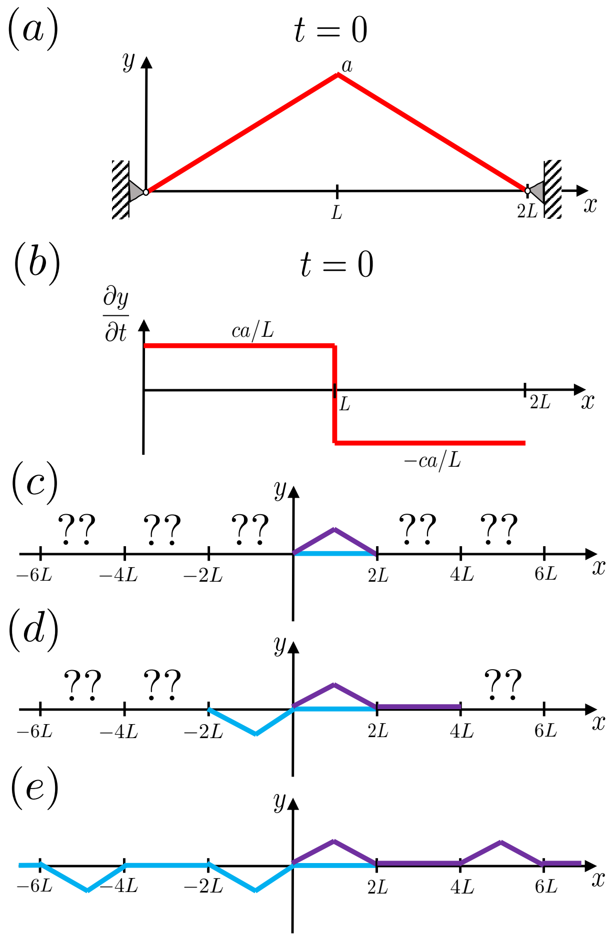

FIGURE 7. (a) Initial position of a string pinned at $x = 0$ and $x = 2L$. (b) Initial velocity of the same string. (c) Right- and left-traveling waves (blue and purple, respectively) given by the initial condition. (d) Right- and left-traveling waves after applying the boundary conditions to the functions in (c). (e) Final form of the right- and left-traveling waves after repeated application of the boundary conditions.

and

$$\frac{\partial y}{\partial t}(x, t = 0) = \dot{y}_0(x) = c y_0'(x) = \begin{cases} ca/L & \text{for } 0 \leqslant x < L, \\ -ca/L & \text{for } L \leqslant x < 2L. \end{cases} \tag{4.19}$$

These initial conditions are sketched in figures 7(a) and 7(b). Using formulas (3.6) and (3.7) with $f(0) - g(0) = 0$, we find

$$f(u) = \begin{cases} ?? & \text{for } u < 0, \\ 0 & \text{for } 0 \leqslant u \leqslant 2L, \end{cases} \tag{4.20}$$

and

$$g(v) = \begin{cases} y_0(v) & \text{for } 0 \leqslant v < 2L, \\ ?? & \text{for } v \geqslant 2L, \end{cases} \tag{4.21}$$

These pieces of the right- and left-traveling waves are sketched in figure 7(c).

Applying the same technique that we used to solve the wave equation with one Dirichlet boundary condition, we find that $f(u) = -g(-u)$ due to the boundary condition at $x = 0$, and $g(v) = -f(4L - v)$ due to the boundary condition at $x = 2L$. Given the solutions in equations (4.20) and (4.21), we can use $f(u) = -g(-u)$ and $g(v) = -f(4L - v)$ to determine $f(u)$ for $-2L \leqslant u < 0$ and $g(v)$ for $2L \leqslant v < 4L$,

$$f(u) = \begin{cases} ?? & \text{for } u < -2L, \\ -y_0(-u) & \text{for } -2L \leqslant u < 0, \\ 0 & \text{for } 0 \leqslant u \leqslant 2L, \end{cases} \tag{4.22}$$

and

$$g(v) = \begin{cases} y_0(v) & \text{for } 0 \leqslant v < 2L, \\ 0 & \text{for } 2L \leqslant v < 4L, \\ ?? & \text{for } v \geqslant 4L, \end{cases} \tag{4.23}$$

These extended right- and left-traveling waves are sketched in figure 7(d). With the left-traveling wave $g(v)$ known for $2L \leqslant v < 4L$ and $f(u) = -g(-u)$, we can obtain the right-traveling wave $f(u)$ from $-4L \leqslant u < -2L$, and with $f(u)$ known for $-2L \leqslant u < 0$ and $g(v) = -f(4L - v)$, we can calculate $g(v)$ for $4L \leqslant v < 6L$. Thus, we can continue constructing $f(u)$ and $g(v)$ to obtain the solution to the motion of the string for all positive times, as shown in figure 7(e).

The most important lesson to learn from this calculation is that the wave equation needs two boundary conditions, and each of these two boundary conditions must be applied at the two opposite extremes of the string. Over-constraining the solution by imposing two boundary conditions at $x = 0$ and no boundary conditions at $x = 2L$ would not work. At both $x = 0$ and $x = 2L$, we need sufficient information to determine the right- and left-traveling waves. At $x = 2L$, the initial condition determines the right-traveling wave, but without a boundary condition we would not be able to find the left-traveling wave. Conversely, at $x = 0$, the extra boundary condition would be impossible to impose because the left-traveling wave is determined by the initial condition, leaving only the right-traveling wave as an unknown.

Understanding how many boundary conditions are needed and where they should be imposed is one of the fundamental problems in Partial Differential Equations (PDEs). The wave equation demonstrates that to solve this problem, we need to understand how the information is transported through the system.

It is clear how to use the d'Alembert solution to construct solutions in finite strings with two boundary conditions, but it is also clear that such construction can be tedious, and it can hide fundamental properties of the solutions, such as the fact that it is periodic, as we will prove shortly. We will present a different technique to solve these problems with two boundary conditions in a few lectures. Before doing so, we discuss the energy balance for the stretched string to demonstrate how energy is transported via waves in such a system.