The wave equation

Felix I. Parra

Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Oxford OX1 3PU, UK

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1. Introduction

In these notes, we will derive the wave equation by considering the transverse motion of a stretched string, the compression and expansion of a solid bar, and the compression and expansion of gas in a pipe. All these systems can be described by the wave equation, a Partial Differential Equation (PDE) that we will study thoroughly for the rest of the term. We will learn why it is referred to as the wave equation in the next set of notes.

2. Transverse oscillations of a stretched string

Consider a string with linear density μ (kg/m) that is stretched from both sides with a force T. Here, the letter T stands for 'tension'. The string is sketched in figure 1. We want to understand how this string responds to small transverse displacements. We assume that the tension is sufficiently large that the string is horizontal to lowest order, that is, in the $x-y$ plane in figure 1, the string is very close to being the line $y=0$. This is, of course, only an approximation, and the string has in fact a vertical displacement $y(x,t)$ that depends on the position along the x-axis and the time t. For a string of length L, we assume that

$$\frac{y(x,t)}{L} \ll 1. \tag{2.1}$$

To describe the motion of the string, we use simple mechanics. We consider an infinitesimally small piece of string such as the one singled out in figure 1. Its length is given by

$$\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}\,\mathrm{d}x \simeq \mathrm{d}x, \tag{2.2}$$

where we have used $\partial y/\partial x \sim y/L \ll 1$. Thus, the mass of this infinitesimal section is $\mu\,\mathrm{d}x$. This string piece is subjected to horizontal and vertical forces. Newton's second law gives

$$\mu a_x(x,t)\,\mathrm{d}x = F(x+\mathrm{d}x,t)\cos\theta(x+\mathrm{d}x,t) - F(x,t)\cos\theta(x,t) \tag{2.3}$$

and

$$\mu a_y(x,t)\,\mathrm{d}x = F(x+\mathrm{d}x,t)\sin\theta(x+\mathrm{d}x,t) - F(x,t)\sin\theta(x,t), \tag{2.4}$$

where $a_x(x,t)$ and $a_y(x,t)$ are the acceleration of the string in the x and y directions, and $F(x,t)$ and $F(x+\mathrm{d}x,t)$ are the forces that the rest of the string exerts on the infinitesimal piece that we are considering. Note that these forces must be of the order of T and that the string can only exert a force $F(x,t)$ parallel to itself. The angle $\theta(x,t)$ is determined by slope of $y(x,t)$,

$$\tan\theta(x,t) = \frac{\partial y}{\partial x} \sim \frac{y}{L} \ll 1. \tag{2.5}$$

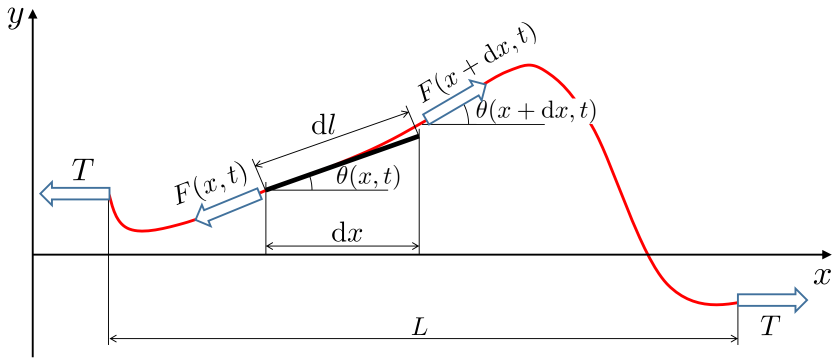

FIGURE 1. String (red line) stretched with tension T. Forces are represented as empty blue arrows. The thick black line represents an infinitesimally small piece of string.

Since $\tan\theta$ is small, we can use the small angle approximations to find

$$\theta(x,t) \simeq \frac{\partial y}{\partial x} \sim \frac{y}{L} \ll 1, \tag{2.6}$$

and hence the equations of motion simplify to

$$\mu a_x(x,t)\,\mathrm{d}x \simeq F(x+\mathrm{d}x,t) - F(x,t) \tag{2.7}$$

and

$$\mu a_y(x,t)\,\mathrm{d}x \simeq F(x+\mathrm{d}x,t)\theta(x+\mathrm{d}x) - F(x,t)\theta(x). \tag{2.8}$$

Using partial derivatives, we finally obtain

$$\mu a_x \simeq \frac{\partial F}{\partial x} \tag{2.9}$$

and

$$\mu a_y \simeq \frac{\partial}{\partial x}(F\theta). \tag{2.10}$$

We do not expect the displacement of the string in the x-direction to be much larger than the displacement in the y direction, giving $a_x \sim a_y$. The size of a_y can be obtained from equation (2.10): $a_y \sim T\theta/\mu L$. Using this result in equation (2.9), we find

$$\underbrace{\mu a_x}_{\sim T\theta/L \ll T/L} \simeq \underbrace{\frac{\partial F}{\partial x}}_{\sim T/L}. \tag{2.11}$$

Thus, the acceleration in the x direction can be neglected, giving that F is constant and equal to the tension exerted at the ends of the string,

$$F(x,t) = T. \tag{2.12}$$

Finally, using equation (2.12) and the fact that the acceleration in the y direction is

$$a_y(x,t) = \frac{\partial^2 y}{\partial t^2}, \tag{2.13}$$

equation (2.10) becomes

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{2.14}$$

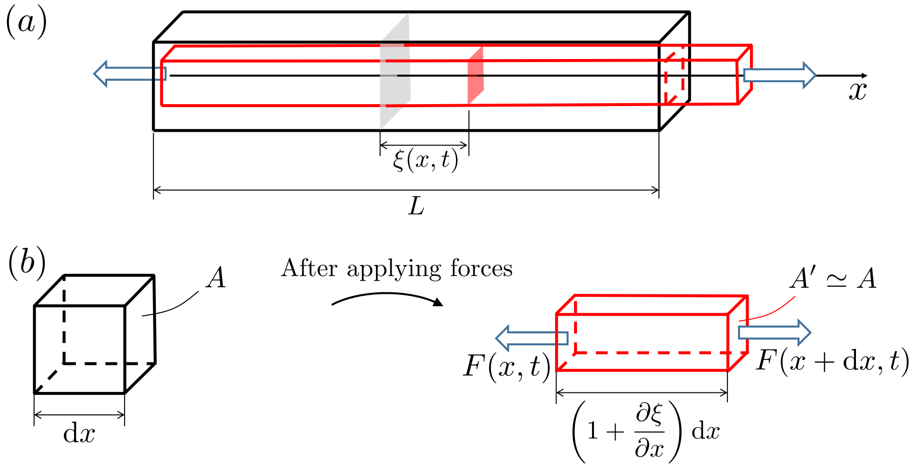

FIGURE 2. (a) Sketch of the deformation of a solid bar of constant cross section area A. The black lines represent the shape of the bar before deformation, and the red lines, the shape after deformation. Matter that was physically at the transparent grey square has moved to the position represented by the transparent red square. (b) Evolution of an infinitesimally small piece of the bar.

where

$$c = \sqrt{\frac{T}{\mu}} \tag{2.15}$$

is a constant with units of velocity. Equation (2.14) is the wave equation, and it will be our workhorse for most of this course.

3. Longitudinal oscillations of a solid bar

The wave equation also describes other systems of interest. For example, a solid bar can be stretched by pulling on it as shown in figure 2(a). We consider the displacements in the direction of the exerted force: a point initially located in the position x will move to the position $x + \xi(x, t)$ due to the forces on both sides of the bar (the Greek letter ξ is 'xi'). Note that, in addition to stretching (or compressing) in the direction of the exerted force, the bar will contract (or expand) in the direction perpendicular to the exerted force. These perpendicular deformations change the cross section area of the bar, but the changes in the cross section area turn out to be negligible in the limit that we consider in this course.

We consider the motion of an infinitesimally small piece of the bar of length $\mathrm{d}x$, shown in figure 2(b). If the volumetric density of the bar is ρ (kg/m^3), we find

$$\rho A a_x(x, t)\,\mathrm{d}x = F(x + \mathrm{d}x, t) - F(x, t), \tag{3.1}$$

where $a_x(x, t)$ is the acceleration of the infinitesimal piece of bar in the x-direction, A is the cross section area, and $F(x, t)$ and $F(x + \mathrm{d}x, t)$ are the forces exerted by the rest of the bar on the infinitesimal piece of interest. Using partial derivatives and the fact that the acceleration in the x direction is

$$a_x(x, t) = \frac{\partial^2 \xi}{\partial t^2}, \tag{3.2}$$

we find

$$\rho A \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial F}{\partial x}. \tag{3.3}$$

The relationship between the displacement ξ and the force F depends on the geometry of the bar and constitutive properties of the material:

• A bar with a larger cross section area is more difficult to deform. The exerted force is spread over a wider area, and hence it will have a smaller effect. In other words, the relevant quantity is the stress F/A.

• The length of the bar is also crucial. It is not the same to extend 1 mm a bar of 1 m than a bar of 2 mm. It is the relative deformation that matters. An infinitesimally small piece of the bar that initially was of length $\mathrm{d}x$ ends up with a length

$$[x + \mathrm{d}x + \xi(x + \mathrm{d}x, t)] - [x + \xi(x, t)] = \mathrm{d}x + \frac{\partial \xi}{\partial x}\mathrm{d}x. \tag{3.4}$$

Thus, it has increased its length by $(\partial\xi/\partial x)\,\mathrm{d}x$ compared to its initial length of $\mathrm{d}x$: its relative deformation is $\partial\xi/\partial x$. This relative deformation depends on F/A. If the deformation is small, $\partial\xi/\partial x \sim \xi/L \ll 1$, the relationship is linear

$$\frac{\partial \xi}{\partial x} \propto \frac{F}{A}. \tag{3.5}$$

In this linear limit, the cross section area of the bar after the deformation, A', is approximately equal to the area before the deformation, A.

• The constant of proportionality that we are missing in equation (3.5) is a property of the material. It is known as Young modulus and we will denote it by E,

$$\frac{\partial \xi}{\partial x} = \frac{F}{EA}. \tag{3.6}$$

Typical values of the Young modulus range from 10^{11} N/m^2 for metals to 10^5 N/m^2 for rubber.

Solving for F in equation (3.6) and substituting the result into equation (3.3), we obtain the wave equation (2.14) again,

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}, \tag{3.7}$$

where the velocity c is now

$$c = \sqrt{\frac{E}{\rho}}. \tag{3.8}$$

4. Acoustic waves

By now, we can see a pattern in these calculations: the forces on both sides of an infinitesimal piece of material almost balance, leaving a residual force proportional to the spatial derivative of the force to accelerate the infinitesimal volume of material. The same technique can be applied to gas in a pipe (see figure 3). The only difference between the bar and the gas is the constitutive equation that relates the force and the deformation. Unlike in the bar, the forces on the infinitesimal volume of gas in figure 3 are always compression forces $F_C(x,t)$, i.e. they try to reduce the volume of the gas. Thus, Newton's equation gives

$$\rho A \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial F_C}{\partial x}, \tag{4.1}$$

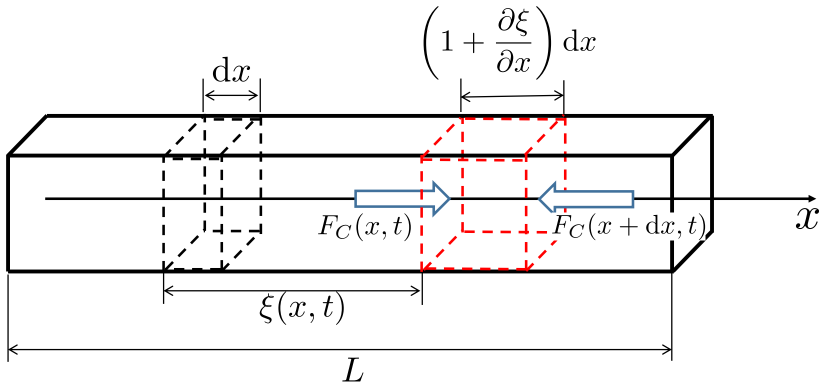

FIGURE 3. Pipe full of gas. The gas that at rest was within the infinitesimally small black dashed volume moves to the red dashed volume due to the compression forces $F_C(x,t)$ exerted by the rest of the gas.

where ρ is the density of the gas and A is the cross section area of the pipe.

The compression force F_C is proportional to the cross section of the pipe and the gas pressure $p(x,t)$,

$$F_C(x,t) = Ap(x,t). \tag{4.2}$$

The pressure depends on the properties of the gas, and in particular on its density and temperature. For rapid changes, the relationship between the pressure p and the density ρ is

$$p \propto \rho^{\gamma}, \tag{4.3}$$

that is, when the density decreases, the pressure decreases, and when it increases, the pressure increases. The constant γ is known as adiabatic constant and it depends on the gas and the temperature.

For simplicity, from here on, we consider a gas that is at a pressure p_0 and density ρ_0, and we assume that the changes in the pressure and the density are small, that is,

$$p(x,t) = p_0 + p_1(x,t), \quad \rho(x,t) = \rho_0 + \rho_1(x,t), \tag{4.4}$$

with $p_1/p_0 \ll 1$ and $\rho_1/\rho_0 \ll 1$. Then, relation (4.3) gives

$$p_0 + p_1 = \frac{p_0}{\rho_0^{\gamma}}(\rho_0 + \rho_1)^{\gamma}. \tag{4.5}$$

Expanding in the smallness of $\rho_1/\rho_0 \ll 1$, we find

$$p_1 \simeq \frac{\gamma p_0}{\rho_0}\rho_1. \tag{4.6}$$

Now, we only need to relate the perturbation to the density, ρ_1, with the horizontal displacement ξ. Since the mass between $x + \xi(x,t)$ and $x + \mathrm{d}x + \xi(x + \mathrm{d}x, t)$ is the same that the mass between x and $x + \mathrm{d}x$, the relation between the density ρ_0 and the perturbed density $\rho_0 + \rho_1$ is given by

$$(\rho_0 + \rho_1)A\left[\mathrm{d}x + \xi(x + \mathrm{d}x, t) - \xi(x,t)\right] = \rho_0 A\,\mathrm{d}x. \tag{4.7}$$

Hence,

$$1 + \frac{\rho_1}{\rho_0} = \frac{1}{1 + \partial\xi/\partial x}. \tag{4.8}$$

For ρ_1/ρ_0 to be small, we need $\partial\xi/\partial x \sim \xi/L \ll 1$ to be small. Then,

$$\frac{\rho_1}{\rho_0} \simeq -\frac{\partial \xi}{\partial x} \tag{4.9}$$

and equation (4.6) finally becomes

$$p_1 \simeq -\gamma p_0 \frac{\partial \xi}{\partial x}. \tag{4.10}$$

Thus, the compressional force in equation (4.2) is

$$F_C(x,t) \simeq A p_0 \left(1 - \gamma \frac{\partial \xi}{\partial x}\right). \tag{4.11}$$

Substituting equation (4.11) into equation (4.1) and using $\rho \simeq \rho_0$ give, yet again, the wave equation. In this case, the velocity c is

$$c = \sqrt{\frac{\gamma p_0}{\rho_0}}. \tag{4.12}$$

The pressure and density perturbations that we have described are what sound is: once excited on air, these perturbation travel and reach our ears, where they move the eardrum and we interpret them as sound.