Waves
Problem Set II

2.1 **General boundary condition for sinusoidal waves.** A semi-infinite string of linear density μ extends from $x = 0$ to $+\infty$ and is under tension T. At $x = 0$ the string is tied to a rod that lies perpendicular to the string. The friction force between the string and the rod is γv_y, where γ is a constant and v_y is the velocity of the knot in the direction perpendicular to the string. Determine the relation between the reflected wave and an incident sinusoidal wave of amplitude A. Calculate the average power lost due to friction and discuss the limits $T \ll \gamma c$ and $T \gg \gamma c$, where $c = \sqrt{T/\mu}$.

2.2 **More general boundary conditions for sinusoidal waves.** A bar of uniform cross section A, density ρ and Young's modulus E transmits longitudinal elastic waves. Waves of frequency ω traveling in the bar are reflected at an end which has a mass M rigidly attached to it. Find the phase change on reflection and discuss the cases $M = 0$ and $M \to \infty$.

2.3 **Transmission lines.** A semi-infinite transmission line, of capacitance C' and inductance L' per unit length, extends from $-\infty$ to $x = 0$ and is terminated by an impedance Z_T at $x = 0$. Find the ratio of the amplitude and the phase difference for the reflected and incident waves if

(a) $Z_T = \sqrt{L'/C'}$,

(b) $Z_T = 2\sqrt{L'/C'}$,

(c) Z_T is a capacitor of capacitance C_0.

In (a) and (b), what type of impedance is required?

2.4 **Transmission problem in a stretched string.** Two long strings lie along the x-axis under tension T. They are joined at $x = 0$ so that for $x < 0$ the line density $\mu = \mu_a$, and for $x > 0$, $\mu = \mu_b$. A mass M is attached to the join and it is connected to a fixed support by a light spring of stiffness K. This spring exerts a transverse force on the mass when the latter is displaced from $y = 0$. Show that at the join

$$T\left(\left.\frac{\partial y}{\partial x}\right|_{x=0^+} - \left.\frac{\partial y}{\partial x}\right|_{x=0^-}\right) = M\left.\frac{\partial^2 y}{\partial t^2}\right|_{x=0} + Ky(x=0).$$

Small transverse sinusoidal oscillations propagate along these strings from $x = -\infty$. Show that the phase of the transmitted wave lags behind that of the incident wave by an angle

$$\arctan\left(\frac{c_a c_b(M\omega^2 - K)}{\omega T(c_a + c_b)}\right),$$

where c_a and c_b are the speeds of the waves for $x < 0$ and $x > 0$, respectively.

Check that the reflected and transmitted waves satisfy energy conservation.

2.5 **Transmission problem in pipe full of gas.** A pipe of constant cross section A contains two gases with densities ρ_a (at $x < 0$) and ρ_b (at $x > 0$) separated by an elastic membrane at $x = 0$. The elastic membrane moves to the left or the right following the law

$$A[p(x=0^-) - p(x=0^+)] = K\xi(x=0),$$

where K is a constant. We consider perturbations to a background pressure p_0 that is the same for both gases.

(a) Find the relationships between $\xi(x = 0^-)$, $\xi(x = 0^+)$, $\partial\xi/\partial x|_{x=0^-}$ and $\partial\xi/\partial x|_{x=0^-}$.

(b) Use these relationships to determine the reflected and transmitted waves that result from an incident harmonic wave with frequency ω traveling from $-\infty$ to $x = 0$.

(c) Discuss the limits of small and large ω.

2.6 **Separation of variables.** (a) For a stretched string with wave velocity c that satisfies the boundary conditions $\partial y/\partial x(x = 0) = 0 = \partial y/\partial x(x = L)$, show that the general solution is of the form

$$y(x, t) = A_0 + \widetilde{A}_0 t + \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{k\pi ct}{L}\right) + B_k \sin\left(\frac{k\pi ct}{L}\right) \right] \cos\left(\frac{k\pi x}{L}\right).$$

(b) For a stretched string with wave velocity c that satisfies the boundary conditions $y(x = 0) = 0 = \partial y/\partial x(x = L)$, show that the general solution is of the form

$$y(x, t) = \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{(k - 1/2)\pi ct}{L}\right) + B_k \sin\left(\frac{(k - 1/2)\pi ct}{L}\right) \right] \sin\left(\frac{(k - 1/2)\pi x}{L}\right).$$

2.7 **Initial conditions for separation of variables.** (a) Determine the motion of a stretched string with wave velocity c held in place at $x = 0$ and $x = L$ with the initial conditions $y(x, t = 0) = h \sin(\pi x/L) + 2h \sin(2\pi x/L)$ and $\partial y/\partial t(x, t = 0) = 0$.

(b) Determine the motion of a stretched string with wave velocity c that can slide along frictionless columns at $x = 0$ and $x = L$ with the initial conditions $y(x, t = 0) = 0$ and $\partial y/\partial t(x, t = 0) = V \sin^2(\pi x/L)$.

(c) Determine the motion of a stretched string with wave velocity c that is held in place at $x = 0$ and can slide along a frictionless column at $x = L$ with the initial conditions $y(x, t = 0) = h \sin(\pi x/2L)$ and $\partial y/\partial t(x, t = 0) = V \sin(3\pi x/2L) \cos(\pi x/L)$.

2.8 **Dispersion.** (a) Show that an alternative expression for group velocity v_g is

$$v_g = v_p + k\frac{\mathrm{d}v_p}{\mathrm{d}k},$$

where v_p is the phase velocity.

(b) Evaluate v_p and v_g as a functions of k for the following cases:

i. Long wavelength surface waves on water $\omega = \sqrt{gk}$ (where g is the acceleration due to gravity).

ii. Short wavelength ripples on water $\omega = \sqrt{\sigma k^3/\rho}$ (where σ is the surface tension and ρ the density).

iii. In the crossover region where both effects are important $\omega^2 = gk + \sigma k^3/\rho$.

iv. Guided electromagnetic waves in a waveguide (with a non-zero longitudinal component of either $\mathbf{E}$ or $\mathbf{B}$) $\omega^2 = \omega_0^2 + k^2c^2$ (where c is the speed of light).

(c) In the first two cases but not the other two you should have found $v_g = \alpha v_p$, where the constant α is different in the two cases. What type of dispersion relation leads to this result?

(d) In the fourth case you should have found $v_p v_g = c^2$, so that either v_p or v_g is greater than c. Which is it, and why does this *not* allow signalling faster than the speed of light?

2.9 **Stationary phase or group velocity.** In the long wavelength limit of question 2.8(b)i., v_p and v_g are decreasing functions of k, while in the short-wavelength limit of 2.8(b)ii., they increase with k. Thus in the cross-over region of question 2.8(b)iii., both pass through minima.

(a) At the minimum of v_p, we have, using the result of question 2.8(a), $v_p = v_g$. Find the values of k and ω at which this occurs. Verify this using the dispersive wavepacket plotter on the course web page, and describe the propagation of a wavepacket centred around this wavenumber (for example, $k_{\text{min}} = k - 25\,\text{m}^{-1}$, $k_{\text{max}} = k + 25\,\text{m}^{-1}$). Don't forget that the length unit used in this section of the DWP is 10 cm.

(b) Calculate the wavenumber k and frequency ω at which v_g has its minimum. Verify this using the DWP. (Note that the displayed value of v_g is evaluated for the centre frequency of the wavepacket, so that it can be read out at steps of $5\,\text{m}^{-1}$ in the cross-over region by using appropriate combinations of k_{min} and k_{max}.) Around this frequency, v_g is essentially constant, making the envelope approximation particularly accurate.