Waves
Problem Set I

1.1 **Stretched string under gravity.** Consider the transverse oscillations of a string of linear density μ tensioned with force T under the influence of the gravitational acceleration g.

(a) Sketch the forces on an infinitesimal piece of string of length $\mathrm{d}x$ including gravity.

(b) Using the diagram in part (a), show that the equation for the transverse displacements of the spring becomes

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} - g.$$

(c) Solve the equation in part (b) for a stationary string $y_s(x)$ ($\partial y_s/\partial t = 0$) held in place at $x = 0$ and $x = L$, that is, $y_s(0) = 0$ and $y_s(L) = 0$. What is the equation for the oscillations $\tilde{y}(x,t)$ around this solution?

1.2 **Longitudinal and transverse oscillations of a stretched string.** (a) Using the equations for the longitudinal oscillations of a solid bar, find the longitudinal deformation $\xi_s(x)$ of a stationary stretched string ($\partial\xi_s/\partial t = 0$) of Young modulus E, volumetric density ρ, cross section area A and length L. Assume that the string is held in place at $x = 0$, that is, $\xi_s(0) = 0$, and that there are no transverse oscillations. Thus, show that under tension T, the string has stretched a distance $\xi_s(L) = T/K$, and determine the constant K as a function of quantities that you know.

(b) Consider the longitudinal oscillations $\tilde{\xi}(x,t)$ and the transverse oscillations $y(x,t)$ of the string around the steady state, that is, the point of the string originally at $(x+\xi_s(x), 0)$ moves to $(x+\xi_s(x)+\tilde{\xi}(x,t), y(x,t))$. For $\xi/L \ll 1$ and $y/L \ll 1$, show that an infinitesimal piece of string of length $(1 + \mathrm{d}\xi_s/\mathrm{d}x)\,\mathrm{d}x$ elongates to be of length

$$\mathrm{d}l \simeq \left[1 + \frac{\mathrm{d}\xi_s}{\mathrm{d}x} + \frac{\partial\tilde{\xi}}{\partial x} + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]\mathrm{d}x.$$

(c) Using the result in part (b), show that the equation for $\tilde{\xi}(x,t)$ is

$$\frac{\partial^2\tilde{\xi}}{\partial t^2} = \frac{E}{\rho}\frac{\partial}{\partial x}\left[\frac{\partial\tilde{\xi}}{\partial x} + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right].$$

(d) In deriving the equation for the transverse oscillations of the stretched string $y(x,t)$, we neglected the variation of the longitudinal force $F(x,t)$ with x, $\partial F/\partial x = 0$. Keeping this derivative of F in the derivation, show that the longitudinal and transversal oscillations of the string are coupled by the equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left[\left(\frac{T}{\rho A} + \frac{E}{\rho}\left(\frac{\partial\tilde{\xi}}{\partial x} + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right)\right)\frac{\partial y}{\partial x}\right].$$

(e) Using parts (c) and (d), argue that the longitudinal and transversal oscillations decouple when $(y/L)^2 \ll \tilde{\xi}/L \ll T/EA$.

1.3 **Initial conditions for the d'Alembert solution.** (a) Sketch as a function of time the displacement at $x = 0$ and at $x = L$, $y(x = 0, t)$ and $y(x = L, t)$, of an infinite stretched string with wave speed c for the initial conditions

$$y(x, t = 0) = \begin{cases} a(1 + x/L) & \text{for } -L \leq x < 0, \\ a(1 - x/L) & \text{for } 0 \leq x < L, \\ 0 & \text{otherwise,} \end{cases}$$

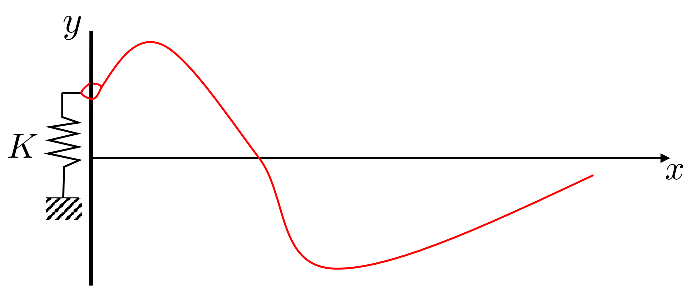
and $\partial y/\partial t(x, t = 0) = 0$.

(b) Sketch $y(x = 0, t)$ and $y(x = L, t)$ for the initial conditions $y(x, t = 0) = 0$ and

$$\frac{\partial y}{\partial t}(x, t = 0) = \begin{cases} V & \text{for } -L \leq x \leq L, \\ 0 & \text{otherwise.} \end{cases}$$

(c) What happens in the case that the string starts with the displacement of part (a) and the velocity of part (b)? For your sketches of $y(x = 0, t)$ and $y(x = L, t)$, assume $VL/c < a$.

1.4 **General boundary condition for the d'Alembert solution.** Consider a semi-infinite stretched string with wave speed c that extends from $x = 0$ to $+\infty$. At $x = 0$, the string is tied with a massless knot to a frictionless column and a spring of constant K, as shown below.



(a) Show that, given the left-traveling wave $g(x + ct)$, the equation for the right-traveling wave $f(x - ct)$ is

$$Tf'(u) - Kf(u) = -Tg'(-u) + Kg(-u),$$

where f' and g' are the derivatives of f and g with respect to their argument.

From here on, consider the initial conditions

$$y(x, t = 0) = \begin{cases} ax/L & \text{for } 0 \leq x < L., \\ a & \text{for } x \geq L, \end{cases}$$

and $\partial y/\partial t(x, t = 0) = 0$

(b) Use the initial conditions to find $f(u)$ for positive u, and $g(v)$.

(c) Employing the differential equation in part (a), solve for $f(u)$ for negative u. Sketch $f(u)$.

(d) What is the solution for $y(x, t)$ for $t \to +\infty$?

1.5 **Two boundary conditions for the d'Alembert solution.** A stretched string with wave speed c is pinned at $x = 0$ and is tied to a frictionless column at $x = L$, that is,

$y(x=0,t)=0$ and $\partial y/\partial x(x=L,t)=0$. The string is plucked at $x=L/4$ at a distance h until it comes to rest. It is then released at $t=0$.

(a) Show that the initial displacement is

$$y(x,t=0)=\begin{cases} 4hx/L & \text{for } 0\le x<L/4, \\ h & \text{for } L/4\le x\le L. \end{cases}$$

(b) Using the initial conditions, determine the right- and left-traveling waves $f(x-ct)$ and $g(x+ct)$ for $0<x-ct<L$ and $0<x+ct<L$.

(c) Show that the boundary condition at $x=0$ imposes $f(u)=-g(-u)$, and that the boundary condition at $x=L$ requires $g(v)=g(L)-f(L)+f(2L-v)$.

(d) With the results in parts (b) and (c), construct $f(x-ct)$ for $x-ct<L$ and $g(x+ct)$ for $x+ct>0$.

(e) Sketch $y(x,t)$ for $t=L/4c, L/c, 5L/4c, 2L/c$. Recall that $y(x,t)$ is only defined for $0<x<L$!

1.6 **Energy equation.** (a) For the longitudinal oscillations of a bar of volumetric density ρ, Young modulus E and cross section area A, show that the energy equation

$$\frac{\partial}{\partial t}\left[\frac{\rho A}{2}\left(\frac{\partial \xi}{\partial t}\right)^2+\frac{EA}{2}\left(\frac{\partial \xi}{\partial x}\right)^2\right]=\frac{\partial}{\partial x}\left(EA\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial t}\right)$$

is satisfied. Explain physically what each of the terms represents.

(b) For the longitudinal oscillations of a gas of pressure p_0, density ρ_0 and adiabatic constant γ in a pipe of cross section A, show that the energy equation

$$\frac{\partial}{\partial t}\left[\frac{\rho_0 A}{2}\left(\frac{\partial \xi}{\partial t}\right)^2+\frac{\gamma p_0 A}{2}\left(\frac{\partial \xi}{\partial x}\right)^2\right]=\frac{\partial}{\partial x}\left(\gamma p_0 A\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial t}\right)$$

is satisfied. Explain physically what each of the terms represents.

1.7 **Energy conservation.** A stretched string with wave speed c has an initial displacement and an initial velocity such that the right- and the left-traveling waves are

$$f(u)=\begin{cases} A\sin(ku) & \text{for } -2\pi/k\le u<-\pi/k, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(v)=\begin{cases} A\sin(kv) & \text{for } \pi/k\le v<2\pi/k, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch $y(x,t=0)$ and $y(x,t=3\pi/2kc)$.

(b) Calculate the energy at $t=0$ and at $t=3\pi/2kc$.

Comment on the results.