Collisionless Plasma Physics
Problem Set II
Due: Wednesday 24 February 2021

2.1 (10 points) The derivation of the slab ITG mode for large η_i in the notes seems to indicate that the mode is stable for $-4/27 \leq k_z^2 T_i \eta_i / m_i \omega_{*e}^2 \leq 0$. Show that this is not the case by following the steps below.

(a) For $k_z v_{ti}/\omega \sim 1$, show that the perturbed ion density, Fourier transformed in space and time, $\tilde{n}_i = 2\pi \int B \tilde{g}_i \, \mathrm{d}v_\parallel \, \mathrm{d}\mu$, is

$$\tilde{n}_i = \left\{ \frac{\omega_{*i}}{|k_z| v_{ti}} \left[\mathcal{Z}(\zeta_i) + \eta_i \left[\zeta_i + \left(\zeta_i^2 - \frac{1}{2} \right) \mathcal{Z}(\zeta_i) \right] \right] - 1 - \zeta_i \mathcal{Z}(\zeta_i) \right\} \frac{Ze\tilde{\phi}}{T_i} n_i,$$

where $\zeta_i = \omega / |k_z| v_{ti}$, $v_{ti} = \sqrt{2T_i/m_i}$ and $\mathcal{Z}(\zeta)$ is the plasma dispersion function.

(b) Using the Maxwell-Boltzmann response for the electrons, and assuming that $\omega \sim \omega_{*i} \sim \omega_{*e}$, $|\zeta_i| \sim \sqrt{\eta_i} \gg 1$, show that the dispersion relation is

$$\frac{\omega_{*e}}{\omega} \left(1 + \frac{k_z^2 T_i \eta_i}{m_i \omega^2} \right) + \frac{1}{2\zeta_i^2} \left[\frac{ZT_e}{T_i} + \frac{\omega_{*e}}{\omega} \left(1 + \frac{6 k_z^2 T_i \eta_i}{m_i \omega^2} \right) \right] - \mathrm{i}\sqrt{\pi} \frac{\omega_{*e} \eta_i}{|k_z| v_{ti}} \zeta_i^2 \exp(-\zeta_i^2) - 1 = 0.$$

(c) Using the dispersion relation in (b), show that to lowest order in $\eta_i^{-1} \ll 1$, ω is equal to the solution given in the notes. From here on, we refer to this solution as the lowest order solution $\omega^{(0)}$.

(d) For the parameter range $-4/27 \leq k_z^2 T_i \eta_i / m_i \omega_{*e}^2 \leq 0$, $\omega = \omega^{(0)} + \omega_r^{(1)} + \mathrm{i}\gamma$, where $\omega^{(0)}$ is purely real, $\omega_r^{(1)} \sim \omega^{(0)}/|\zeta_i|^2$ is the correction to the real frequency $\omega^{(0)}$, and $\gamma \sim \omega^{(0)} \exp(-|\zeta_i|^2)$ is the growth rate. Find $\omega_r^{(1)}$ and γ in the parameter range $-4/27 \leq k_z^2 T_i \eta_i / m_i \omega_{*e}^2 \leq 0$ (to avoid lengthy expressions, do not substitute the value of $\omega^{(0)}$). Show that the mode is unstable (although its growth rate is exponentially small).

2.2 (25 points) Consider a uniform plasma with one ion species with charge Ze and mass m_i, and electrons with charge $-e$ and mass m_e in a strong magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. Assume that the ion and electron distribution functions are bi-Maxwellians,

$$\langle f_s \rangle_\varphi (v_\parallel, \mu) = f_{Bs}(v_\parallel, \mu) \equiv n_s \left(\frac{m_s}{2\pi T_{s\parallel}} \right)^{1/2} \frac{m_s}{2\pi T_{s\perp}} \exp\left(-\frac{m_s v_\parallel^2}{2T_{s\parallel}} - \frac{m_s \mu B}{T_{s\perp}} \right).$$

(a) Following the procedure shown in the notes, linearize the kinetic MHD equations and show that the plasma displacement $\boldsymbol{\xi}_\perp$ and the parallel electric field $\delta E_\parallel$ satisfy the equations

$$\begin{pmatrix} W_{11} & 0 & 0 \\ 0 & W_{22} & W_{23} \\ 0 & -W_{23} & W_{33} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{\xi}}_\perp \cdot (\mathbf{k}_\perp \times \hat{\mathbf{b}}) \\ \mathbf{k}_\perp \cdot \tilde{\boldsymbol{\xi}}_\perp \\ e\tilde{E}_\parallel / k_\parallel T_{e\perp} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$W_{11} = \frac{B^2 + \mu_0(P_\perp - P_\parallel)}{\mu_0 n_e T_{e\perp}} - \frac{m_i \omega^2}{Z T_{e\perp} k_\parallel^2},$$

$$W_{22} = \frac{k^2 B^2 + \mu_0 k_\parallel^2 (P_\perp - P_\parallel)}{k_\perp^2 \mu_0 n_e T_{e\perp}} - \frac{m_i \omega^2}{Z T_{e\perp} k_\perp^2} + \frac{2T_{i\perp}}{Z T_{e\perp}} \left\{ 1 - \frac{T_{i\perp}}{T_{i\parallel}} [1 + \zeta_i \mathcal{Z}(\zeta_i)] \right\}$$
$$+ 2 \left\{ 1 - \frac{T_{e\perp}}{T_{e\parallel}} [1 + \zeta_e \mathcal{Z}(\zeta_e)] \right\},$$

$$W_{23} = \frac{T_{i\perp}}{T_{i\parallel}} [1 + \zeta_i \mathcal{Z}(\zeta_i)] - \frac{T_{e\perp}}{T_{e\parallel}} [1 + \zeta_e \mathcal{Z}(\zeta_e)],$$

$$W_{33} = \frac{Z T_{e\perp}}{T_{i\parallel}} [1 + \zeta_i \mathcal{Z}(\zeta_i)] + \frac{T_{e\perp}}{T_{e\parallel}} [1 + \zeta_e \mathcal{Z}(\zeta_e)].$$

Here $P_\perp = n_i T_{i\perp} + n_e T_{e\perp}$, $P_\parallel = n_i T_{i\parallel} + n_e T_{e\parallel}$, $\mathcal{Z}(\zeta)$ is the plasma dispersion function and $\zeta_s = \omega / |k_\parallel| v_{ts\parallel}$ with $v_{ts\parallel} = \sqrt{2T_{s\parallel}/m_s}$.

(b) Using the dispersion relation in (a), prove that the shear Alfven wave is unstable for $P_\parallel - P_\perp > B^2/\mu_0$. When this condition is satisfied, the Alfven wave is said to be *firehose unstable*. By studying the kinetic MHD forces on a thin magnetic flux tube around a magnetic field line (see the notes), can you explain why this instability resembles that of a firehose?

(c) To show that the plasma is unstable in the limit $\beta = 2\mu_0 n_e T_{e\perp}/B^2 \gg 1$ if $T_{s\perp}/T_{s\parallel} > 1$ are sufficiently large, assume $T_{i\parallel} \simeq T_{e\parallel} \simeq T_{i\perp} \simeq T_{e\perp} = T$, $\Delta_s = 2\mu_0 n_s (T_{s\perp} - T_{s\parallel})/B^2 \sim 1$ and $|\zeta_e| \ll |\zeta_i| \sim \beta^{-1} \ll 1$ to find

$$\zeta_i = \mathrm{i} \frac{Z}{\sqrt{\pi}} \frac{B^2}{2\mu_0 n_e T} \frac{k^2}{k_\perp^2} \left[(\Delta_i + \Delta_e) \left(1 - \frac{3}{2} \frac{k_\parallel^2}{k^2} \right) - 1 \right].$$

Determine the range of values of $k_\parallel^2/k^2$ for which there is instability for both $\Delta_i + \Delta_e > 0$ and $\Delta_i + \Delta_e < 0$. If $\Delta_i + \Delta_e$ is positive, what is the condition for instability? When this conditions is satisfied, the plasma is *mirror unstable*. By considering the polarization of the wave, can you justify the name of this instability?

2.3 (5 points) Consider the propagation of cold plasma waves in a plasma with one ion species with charge Ze and mass m_i, and electrons with charge $-e$ and mass m_e in a constant magnetic field. Use the basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ defined such that $\mathbf{B} = B\hat{\mathbf{z}}$ and $\mathbf{k} = k_\perp \hat{\mathbf{x}} + k_\parallel \hat{\mathbf{z}}$.

(a) Show that the cold plasma dielectric tensor in the limit $\omega \ll \Omega_i \ll \omega_{pi} \ll \Omega_e \sim \omega_{pe}$ is

$$\boldsymbol{\epsilon} \simeq \begin{pmatrix} c^2/v_A^2 & 0 & 0 \\ 0 & c^2/v_A^2 & 0 \\ 0 & 0 & -\omega_{pe}^2/\omega^2 \end{pmatrix},$$

where $v_A = B/\sqrt{\mu_0 n_i m_i}$ is the Alfven speed. Estimate the size of the neglected terms.

(b) Prove that the cold plasma waves in the regime $\omega \ll \Omega_i \ll \omega_{pi} \ll \Omega_e \sim \omega_{pe}$ are the compressional Alfven wave, $\omega = k v_A$, and the wave

$$\omega = \frac{k_\parallel v_A}{\sqrt{1 + k_\perp^2 d_e^2}}.$$

Here $d_e = c/\omega_{pe}$ is the electron skin depth. What are the polarizations of these two waves?

2.4 (10 points) Consider a cylindrical magnetic field $\mathbf{B} = B(r)\hat{\boldsymbol{\theta}}$. Assume that the ion density $n_i(r)$ and the magnetic field magnitude $B(r)$ are tailored such that the Alfven speed $v_A = \sqrt{B^2/\mu_0 n_i m_i}$ is uniform in the region of interest.

(a) Using ray tracing techniques, prove that the wavevector of a shear Alfven wave ($\omega = k_\| v_A$) that is launched at $\theta = 0$ with $\mathbf{k}(\theta = 0) = k_0\hat{\boldsymbol{\theta}}$ evolves as $\mathbf{k}(\theta) = k_0\theta\hat{\mathbf{r}} + k_0\hat{\boldsymbol{\theta}}$. Describe the wave front motion.

(b) Describe the wave front motion of a compressional Alfven wave ($\omega = kv_A$) that is launched at $\theta = 0$ with $\mathbf{k}(\theta = 0) = k_0\hat{\boldsymbol{\theta}}$.

2.5 (15 points) (a) Write the cold plasma dielectric tensor for a plasma with one ion species with charge Ze and mass m_i, and electrons with charge $-e$ and mass m_e in a time-independent magnetic field in the limit $\Omega_i \ll \omega_{pi} \ll \Omega_e \sim \omega_{pe} \ll \omega$. Use the basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ defined such that $\mathbf{k} = k\hat{\mathbf{z}}$ and $\mathbf{B} = B_x\hat{\mathbf{x}} + B_z\hat{\mathbf{z}}$ (note the change with respect to the basis that we have used in class!). Keep terms of up to order $\omega_{pe}^2\Omega_e/\omega^3$, and neglect any ion contribution.

(b) In the limit described in (a), the waves are $\omega \simeq kc$. Prove that the correction to this lowest order dispersion relation is

$$k \simeq \frac{\omega}{c}\left(1 - \frac{\omega_{pe}^2}{2\omega^2} \mp \frac{\omega_{pe}^2\Omega_e}{2\omega^3}\frac{B_z}{B}\right). \tag{1}$$

Find the polarizations that correspond to these two possible solutions.

(c) A wave that satisfies the assumptions in (a) is launched with linear polarization. By projecting this initial condition onto the two possible polarizations and following the path of the wave through the plasma, show that the polarization of the electric field rotates with respect to the initial linear polarization by an angle

$$\alpha = -\int \frac{\omega_{pe}^2\Omega_e}{2\omega^3}\hat{\mathbf{b}}\cdot\mathbf{k}\,\mathrm{d}l, \tag{2}$$

where l is the arc length of the path perpendicular to the wave fronts. This rotation of the wave electric field is known as *Faraday rotation*, and it can be used to measure the magnetic field.

2.6 (35 points) In fusion plasmas, the second and higher harmonics of the ion gyrofrequency, $\omega = q\Omega_i$ with $q \geq 2$, are used to heat the ions to the necessary temperatures for fusion energy. To study the second or higher harmonic resonance, consider a plasma with one ion species with charge Ze and mass m_i, and electrons with charge $-e$ and mass m_e in a constant magnetic field. In the Cartesian basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$, $\mathbf{B} = B\hat{\mathbf{z}}$ and $\mathbf{k} = k_\perp\hat{\mathbf{x}} + k_\|\hat{\mathbf{z}}$. Assume that both the ion and the electron distribution functions are Maxwellians with densities $n_s(x)$ and temperature $T_s(x)$ that only depend on x. Here $s = i, e$ is the index indicating the species.

Using first the cold plasma dispersion relation, we proceed to determine the characteristics that a wave with $\omega = q\Omega_i$ must satisfy to reach the center of a fusion device.

(a) Using ray tracing, show that the parallel wavevector $k_\parallel$ is constant.

(b) Assuming $\omega \sim k v_A \sim \Omega_i \ll \omega_{pi} \ll \Omega_e \sim \omega_{pe}$, where $v_A = B/\sqrt{\mu_0 n_i m_i}$ is the Alfven speed, show that $k_\perp$ is given by the equation

$$\frac{k_\perp^2 v_A^2}{\omega^2} = \frac{\omega^2/\Omega_i^2}{(\omega^2/\Omega_i^2 - 1)[1 + (k_\parallel^2 v_A^2/\omega^2)(\omega^2/\Omega_i^2 - 1)]} - \frac{1}{\omega^2/\Omega_i^2 - 1} - \frac{k_\parallel^2 v_A^2}{\omega^2}.$$

(c) Using the formula in (b) and taking $k_\parallel \neq 0$, argue that the wave is evanescent in vacuum ($n_e = 0$). As we will see in part (f) of this problem, $k_\parallel$ must be different from zero to achieve heating, but locating the antenna inside the plasma to launch the wave erodes it. As a compromise, $k_\parallel$ is typically much smaller than $k_\perp$.

(d) Taking $k_\parallel \ll k_\perp \sim \Omega_i/v_A$ and $\omega = q\Omega_i$, show that

$$k_\perp = \frac{q\Omega_i}{v_A}.$$

From here on, we use the hot plasma dispersion relation to study the absorption due to the resonance with $q\Omega_i$.

(e) For $|k_\parallel| v_{ti} \sim |\omega - q\Omega_i| \ll |k_\parallel| v_{te} \ll \omega \simeq q\Omega_i \ll \omega_{pi} \ll \Omega_e \sim \omega_{pe}$ with $q \geq 2$, $b_e = k_\perp^2 T_e/m_e\Omega_e^2 \ll b_i = k_\perp^2 T_i/m_i\Omega_i^2 \ll 1$ and $\Omega_i b_i^{q-1}/|\omega - q\Omega_i| \sim 1$, show that the dispersion relation is $D = 0$, where

$$D \simeq \det \begin{pmatrix} \epsilon_\perp - k_\parallel^2 c^2/q^2\Omega_i^2 & \mathrm{i}g & k_\parallel k_\perp c^2/q^2\Omega_i^2 \\ -\mathrm{i}g & \epsilon_\perp - k^2 c^2/q^2\Omega_i^2 & 0 \\ k_\parallel k_\perp c^2/q^2\Omega_i^2 & 0 & -\omega_{pe}^2/q^2\Omega_i^2 - k_\perp^2 c^2/q^2\Omega_i^2 \end{pmatrix}$$

and

$$\epsilon_\perp = \frac{c^2}{v_A^2}\left(-\frac{1}{q^2-1} + \frac{1}{2(q-1)!}\frac{\Omega_i b_i^{q-1}}{|k_\parallel| v_{ti}} \mathcal{Z}(\zeta_{i,q})\right),$$
$$g = \frac{c^2}{v_A^2}\left(-\frac{q}{q^2-1} + \frac{1}{2(q-1)!}\frac{\Omega_i b_i^{q-1}}{|k_\parallel| v_{ti}} \mathcal{Z}(\zeta_{i,q})\right).$$

Here $\zeta_{i,q} = (\omega - q\Omega_i)/|k_\parallel| v_{ti}$ and $\mathcal{Z}(\zeta)$ is the plasma dispersion function. [Hint: note that the dispersion relation is the cold plasma dispersion relation except for the terms that represent the resonance with $q\Omega_i$.]

(f) Perform a subsidiary expansion in $\zeta_{i,q} \gg 1$, $\Omega_i b_i^{q-1}/|\omega - q\Omega_i| \ll 1$ and $|k_\parallel| \ll k_\perp \sim \Omega_i/v_A$ in the plasma dispersion relation deduced in (e). Show that $k_\perp$ is given to lowest order by

$$k_\perp \simeq \frac{q\Omega_i}{v_A}\left[1 + \mathrm{i}\sqrt{\pi}\frac{q^{2(q-1)}(q-1)^2}{2^{2q}(q-1)!}\frac{\Omega_i \beta_i^{q-1}}{|k_\parallel| v_{ti}} \exp\left(-\frac{(\omega - q\Omega_i)^2}{k_\parallel^2 v_{ti}^2}\right)\right],$$

where $\beta_i = 2\mu_0 n_i T_i/B^2 \ll 1$. Thus the wave is damped by the resonance with harmonic $q\Omega_i$, with $q \geq 2$, but the damping is small in $\beta_i^{q-1} \ll 1$. Note that $k_\parallel$ must be different from zero to get damping.

(g) The resonance with Ω_i is not an efficient method to heat the ions, as demonstrated by the fact that the expression in (f) particularized to $q = 1$ gives zero damping (why is the expression in (f) valid when $q = 1$ even though we derived it assuming $q \geq 2$?). Find the polarization of the wave for $\omega \simeq \Omega_i$ and use it to explain why there is no wave damping.