Collisionless Plasma Physics
Problem Set I
Due: Wednesday 3 February 2021

1.1 (10 points) Find the general solution to the equation

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} = \Omega \mathbf{w} \times \hat{\mathbf{b}}. \tag{1}$$

The scalar Ω and the unit vector $\hat{\mathbf{b}}$ are constants. Use your preferred method to solve systems of linear ordinary differential equations with constant coefficients.

1.2 (10 points) A particle with charge Ze and mass m moves in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. Its position and velocity at $t = 0$ are $\mathbf{r}(t = 0) = 0$ and $\mathbf{v}(t = 0) = v_0\hat{\mathbf{x}}$. At $t = 0$, the electric field $\mathbf{E} = E\hat{\mathbf{y}}\sin(\omega t)$ is switched on.

(a) Calculate the exact particle's position $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$ for $t > 0$. What happens for $\omega = \Omega = ZeB/m$?

(b) Expand the solution for $v_0 \sim E/B$ and $\omega \ll \Omega$, keeping only the leading order terms.

(c) Calculate the initial values of the guiding center parallel velocity $v_\parallel$ and magnetic moment μ. Using these values, integrate the guiding center equations for this system assuming that $\omega \ll \Omega$. Compare to the result in (b).

1.3 (15 points) A particle of charge Ze and mass m moves in the magnetic field

$$\mathbf{B} = B_r(r, z)\hat{\mathbf{r}} + B_z(z)\hat{\mathbf{z}}, \tag{2}$$

where

$$B_z(z) = \begin{cases} B_1 & \text{for } z < 0 \\ B_2 & \text{for } z \geq 0 \end{cases} \tag{3}$$

Here $\{r, \theta, z\}$ are the usual cylindrical coordinates, and B_1 and B_2 are two constants that satisfy $B_1 < B_2$.

(a) Using $\nabla \cdot \mathbf{B} = 0$, and assuming that B_r at $r = 0$ is regular, prove that

$$B_r(r, z) = -\frac{(B_2 - B_1)r}{2}\delta(z), \tag{4}$$

where $\delta(z)$ is the Dirac delta function.

(b) The particle approaches the point $z = 0$ from negative z. When it reaches $z = 0^-$, its velocity is $\mathbf{v} = -v_\perp\hat{\boldsymbol{\theta}} + v_\parallel\hat{\mathbf{z}}$ and its position is $r = mv_\perp/ZeB_1$ and $\theta = 0$. Show that from $z = 0^-$ to $z = 0^+$, the particle's radial velocity v_r does not change appreciably, but the particle's azimuthal velocity v_θ changes by the amount

$$\Delta v_\theta = -\frac{1}{2}\left(\frac{B_2}{B_1} - 1\right)v_\perp. \tag{5}$$

(c) Argue that the kinetic energy of the particle must be conserved, and use this fact to calculate the change in axial velocity v_z from $z = 0^-$ to $z = 0^+$. For which values of $v_\parallel/v_\perp$ does the particle bounce back?

(d) Compare your results with the particle motion along a magnetic field line in which the magnitude of the magnetic field changes gradually from B_1 to B_2 over a length $L \gg mv_\perp/ZeB_1$. Sketch Δv_θ vs. $(B_2/B_1 - 1)$ and Δv_z vs. $(B_2/B_1 - 1)$ for both situations.

(e) What happens to Δv_z if the particle reaches $z = 0^-$ with velocity $\mathbf{v} = v_\perp\hat{\boldsymbol{\theta}} + v_\parallel\hat{\mathbf{z}}$ instead of $\mathbf{v} = -v_\perp\hat{\boldsymbol{\theta}} + v_\parallel\hat{\mathbf{z}}$?

1.4 (25 points) Along a magnetic field line, the magnitude of the magnetic field is

$$B(l) = B_0\left[1 - b\cos\left(\frac{\pi l}{L}\right)\right], \tag{6}$$

where $0 < b < 1$ is a constant. The electric field is negligible and the magnetic field is in steady state. Consider the motion of a magnetized particles of mass m and charge Ze in this magnetic configuration. To describe this motion we use the magnitude of the velocity $v = \sqrt{2\mathcal{E}/m}$ and the coordinate $\lambda = m\mu B_0/\mathcal{E}$. Here $\mathcal{E} = mv_\parallel^2/2 + m\mu B(l)$ is the kinetic energy and $\mu = w_\perp^2/2B$ is the magnetic moment.

(a) Show that the parallel velocity can be written as $v_\parallel = \pm v\sqrt{1 - \lambda B/B_0}$.

(b) Show that the maximum value that λ can take at each position l is $\lambda_{\max}(l) = B_0/B(l)$. What is the maximum value of $\lambda_{\max}$?

(c) We call particles that cannot leave the region $|l| < L$ "trapped particles". Show that the minimum value of λ for which particles are trapped is $\lambda_t = (1 + b)^{-1}$. Show that trapped particles satisfy

$$|l(t)| < l_b = \frac{L}{\pi}\arccos\left(\frac{1 - \lambda^{-1}}{b}\right).$$

Configurations with a magnetic field as in equation (6) are known as *magnetic mirrors* because trapped particles are reflected by the regions of high B.

(d) Sketch the phase space trajectories $v_\parallel$ vs. l of particles in the magnetic field (6). In the sketch, mark the most important features.

(e) Calculate the trajectory of the "deeply trapped" particles, that is, the particles with $0 < \lambda^{-1} - (1 - b) \ll b$. To solve this problem, prove that for deeply trapped particles, $|l(t)| \ll L$ and hence we can use $B(l)/B_0 \simeq 1 - b + b\pi^2 l^2/2L^2$. Use this approximation to solve for $l(t)$. What is the bounce period?

Consider the steady state distribution of particles f.

(f) Using the steady state, lowest order, low flow, drift kinetic equation, show that f is constant along the phase space trajectories of part (c), that is, the phase space trajectories are the *characteristics* of the drift kinetic equation.

(g) Assume that all non-trapped particles are lost to absorbing walls at $l = -L$ and $l = L$. Argue that $f = 0$ for $0 < \lambda < \lambda_t$.

(h) Assume that

$$f(l=0,v,\lambda) = \begin{cases} n_0\sqrt{\frac{1+b}{2b}}\left(\frac{m}{2\pi T_0}\right)^{3/2}\exp\left(-\frac{mv^2}{2T_0}\right) & \text{for } \lambda_t < \lambda < \lambda_{\max}(l=0) \\ 0 & \text{for } 0 < \lambda < \lambda_t. \end{cases}$$

Using the results in (f) and (g), calculate the density $n(l) = \int f\,\mathrm{d}^3v$ for every position l. [Hint: write the distribution function in the variables $\{v_\|, \mu, \varphi\}$ for which you already know the differential velocity space volume, or calculate the differential velocity space volume for $\{v, \lambda, \mu\}$.]

1.5 (15 points) The θ-pinch is a magnetic confinement cylindrical configuration in which the magnetic field is aligned with the axis of the cylinder (see Figure 1). The magnitude of the magnetic field depends on radius, that is, $\mathbf{B} = B(r)\hat{\mathbf{z}}$.

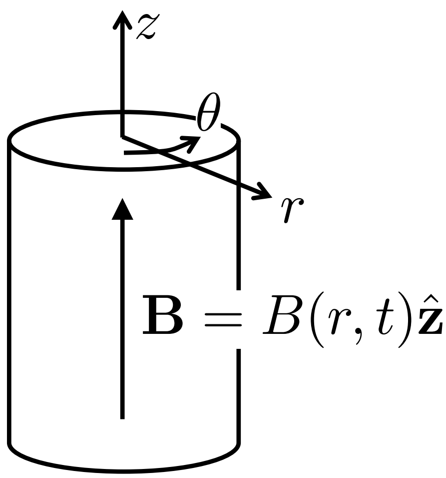

Figure 1: θ-pinch.

At $t = 0$, the magnetic field starts increasing in time, $\mathbf{B}(r,t) = B(r,t)\hat{\mathbf{z}}$, with $\partial \ln B/\partial t \sim v/L$ and v the characteristic speed of the particles.

(a) Calculate the azimuthal electric field generated by the change in magnetic field. Neglect the electric field in the radial and axial directions.

(b) Give the guiding center equations of motion to lowest order in $\rho_* \ll 1$ during the magnetic field change. What happens to the parallel velocity? And to the perpendicular velocity?

(c) Show that during the change in the magnetic field, the quantity

$$\Psi(r,t) = \int_0^r B(r',t)r'\,\mathrm{d}r' \tag{7}$$

is a conserved quantity for the particle, that is, $\Psi(r(t),t) = \text{constant}$.

(d) Using the result in (c), calculate $r(t)$ for $B(r,t) = B_0\exp(t/t_0)$.

(e) Assume that the initial distribution function is

$$f(\mathbf{r}, \mathbf{v}, t=0) = \begin{cases} n_0 \left(\frac{m}{2\pi T_0}\right)^{3/2} \exp\left(-\frac{m(v_\parallel^2/2 + \mu B(r,t=0))}{T_0}\right) & \text{for } r < R_0 \\ 0 & \text{for } r \geq R_0 \end{cases}, \tag{8}$$

where n_0 and T_0 are constants. Using the drift kinetic equation, calculate the distribution function for $B(r,t) = B_0 \exp(t/t_0)$. Calculate the density $n = \int f \, \mathrm{d}^3 v$, the parallel average velocity $u_\parallel = n^{-1} \int f \, v_\parallel \, \mathrm{d}^3 v$, the parallel pressure $p_\parallel = \int f \, m(v_\parallel - u_\parallel)^2 \, \mathrm{d}^3 v$ and the perpendicular pressure $p_\perp = \int f (m w_\perp^2/2) \, \mathrm{d}^3 v$. Check that the total number of particles is conserved.

1.6 (25 points) The z-pinch is a magnetic confinement cylindrical configuration in which the magnetic field closes in azimuthal loops (see Figure 2). The magnitude of the magnetic field depends on radius, that is, $\mathbf{B} = B_\theta(r)\hat{\boldsymbol{\theta}}$. The plasma contained in the z-pinch is composed of electrons with charge $-e$ and mass m_e, and one ion species with charge Ze and mass m_i. We assume that the equilibrium gyroaveraged distribution functions for both ions and electrons are

$$\langle f_s \rangle_\varphi (r, v_\parallel, \mu) = f_{Ms}(r, v_\parallel, \mu) \equiv n_s(r) \left(\frac{m_s}{2\pi T_s(r)}\right)^{3/2} \exp\left(-\frac{m_s(v_\parallel^2/2 + \mu B_\theta(r))}{T_s(r)}\right). \tag{9}$$

We also assume that the equilibrium electric field is zero.

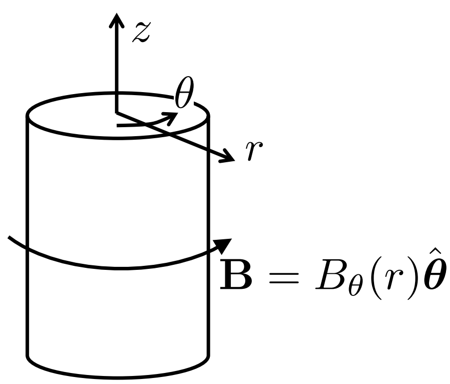

Figure 2: z-pinch.

Consider an electrostatic perturbation to the z-pinch of the form $\tilde{\phi}(r) \exp(-\mathrm{i}\omega t + \mathrm{i}M\theta + \mathrm{i}k_z z)$.

(a) If the perturbations to the gyroaveraged distribution functions are of the form $\tilde{g}_s(r, v_\parallel, \mu) \exp(-\mathrm{i}\omega t + \mathrm{i}M\theta + \mathrm{i}k_z z)$, show that the perturbed, electrostatic drift kinetic equation gives to lowest order

$$\left[-\mathrm{i}\omega + \frac{\mathrm{i}M v_\parallel}{r} + \frac{\mathrm{i}k_z}{\Omega_s}\left(\frac{v_\parallel^2}{r} - \mu \frac{\mathrm{d}B_\theta}{\mathrm{d}r}\right)\right] \tilde{g}_s = \left\{\mathrm{i}\omega_{*s}\left[1 + \left(\frac{m_s(v_\parallel^2/2 + \mu B_\theta)}{T_s} - \frac{3}{2}\right)\eta_s\right] - \frac{\mathrm{i}M v_\parallel}{r} - \frac{\mathrm{i}k_z}{\Omega_s}\left(\frac{v_\parallel^2}{r} - \mu \frac{\mathrm{d}B_\theta}{\mathrm{d}r}\right)\right\} \frac{Z_s e \tilde{\phi}}{T_s} f_{Ms}, \tag{10}$$

where $\Omega_s = Z_s e B_\theta / m_s$, $\omega_{*s} = k_z T_s / Z_s e B_\theta L_{n_s}$, $\eta_s = L_{n_s}/L_{T_s}$, and the length scale of any quantity $Q(r)$ is given by $L_Q = -(\mathrm{d}\ln Q/\mathrm{d}r)^{-1}$.

To simplify the problem, we are going to expand in $\eta_i \sim \eta_e \gg 1$ assuming

$$\frac{M v_{ti}}{\omega_{*i} r} \sim \frac{L_n}{r} \sim \frac{1}{\eta_i} \ll Z \sim \frac{T_i}{T_e} \sim \frac{\omega}{\omega_{*i}} \sim \frac{\omega}{\omega_{*e}} \sim 1 \ll \eta_i \ll \frac{M v_{te}}{\omega_{*e} r}, \tag{11}$$

(b) For the ions, expand keeping zeroth and first order terms in the expansion in $\eta_i^{-1} \ll 1$ to find

$$\tilde{g}_i \simeq -\frac{\omega_{*i}}{\omega}\left\{\eta_i\left(\frac{m_i(v_\parallel^2/2 + \mu B_\theta)}{T_i} - \frac{3}{2}\right)\left[1 + \frac{M v_\parallel}{\omega r} + \frac{k_z}{\omega\Omega_s}\left(\frac{v_\parallel^2}{r} - \mu\frac{\mathrm{d}B_\theta}{\mathrm{d}r}\right)\right] + 1\right\}\frac{Ze\tilde{\phi}}{T_i} f_{Mi} \tag{12}$$

For the electrons, use the lowest order approximation in the limit $\eta_e^{-1} \ll 1$ to obtain

$$\tilde{g}_e \simeq \frac{e\tilde{\phi}}{T_e} f_{Me}. \tag{13}$$

(c) Using the quasineutrality condition $Z\int g_i \,\mathrm{d}^3v = \int g_e \,\mathrm{d}^3v$, obtain the dispersion relation

$$1 - \frac{\omega_{*e}}{\omega} + \frac{\omega_{*e}^2 \eta_i}{\omega^2}\frac{T_i}{ZT_e}\left(\frac{L_n}{r} - \frac{L_n}{B_\theta}\frac{\mathrm{d}B_\theta}{\mathrm{d}r}\right) = 0. \tag{14}$$

What is the condition for this mode to be unstable? Why did we keep the zeroth and first order terms in the expansion in $\eta_i^{-1} \ll 1$ for the ions but only the lowest order for the electrons?

(d) We have assumed that $M \neq 0$. What happens to the electron distribution function for $M = 0$?