Cold plasma waves

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(This version is of 18 February 2019)

1. Introduction

In previous notes we have derived models for magnetized plasmas. We had to assume that the characteristic lengths and time scales of the phenomena of interest were much longer than the typical gyroradius and the inverse of the typical gyrofrequency. The models that we obtained were non-linear, but in many of the examples presented, we linearized the equations to study the response of the plasma to small perturbations. If instead of searching for a nonlinear model of the plasma, we simply want to understand its response to very small perturbations, we can directly linearize the Vlasov equation and we need not assume anything about the time and length scales of the perturbations. Plasma waves are the linear response of the plasma to small perturbations.

The study of plasma waves is interesting by itself because it reveals how ions, electrons and electromagnetic fields respond differently to the same frequency. Moreover, some of the waves that we will discuss can be used to probe plasmas, and even to heat them or to impart momentum to them.

We start by studying the simplest waves: cold plasma waves in a homogeneous plasma. We consider electromagnetic waves of the form

$$\delta\mathbf{E} = \tilde{\mathbf{E}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \mathrm{i}\omega t),$$
$$\delta\mathbf{B} = \tilde{\mathbf{B}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \mathrm{i}\omega t), \tag{1.1}$$

where $\delta\mathbf{E}$ and $\delta\mathbf{B}$ are the electric and magnetic field of the wave, $\mathbf{k}$ is the wavevector and ω the wave frequency. The wave is a cold plasma wave when its phase velocity, ω/k, is much larger than the characteristic velocity of the particles in the plasma, the thermal speed v_{ts}, that is,

$$\frac{\omega}{k} \gg v_{ts} \tag{1.2}$$

for all s. When condition (1.2) is satisfied, all the particles in the system see the same electromagnetic fields: an electromagnetic wave that moves at a large speed ω/k. Conversely, when the phase velocity of the wave and the thermal speed are comparable, some particles can leave the wave behind, so particles with different velocities experience different electromagnetic fields. We will treat this latter case in the notes about hot plasma waves.

2. Equations

We consider a system composed of

(a) A homogeneous, steady state, collisionless plasma with several species s (we will eventually particularize to a plasma with only one ion species and electrons). Each species s has a distribution function f_s constant in time and uniform in space. Only the density

$n_s = \int f_s \, \mathrm{d}^3v$ will be important for cold plasma waves. The densities satisfy quasineutrality,

$$\sum_s Z_s e n_s = 0. \tag{2.1}$$

(b) A background magnetic field $\mathbf{B}$ constant in time and uniform in space.
(c) No background electric field $\mathbf{E}$.

Note that to simplify the equations, we have neglected collisions and we have assumed that the background electric field is zero. The decision to neglect the electric field can be easily justified. In general, we can only have strong electric fields in the direction perpendicular to $\mathbf{B}$ (particles can move freely along $\mathbf{B}$, and short out electric fields in this direction). A perpendicular electric field can be eliminated by going to a frame that moves with the $\mathbf{E} \times \mathbf{B}$ velocity $\mathbf{v}_E = B^{-1}\mathbf{E} \times \hat{\mathbf{b}}$.

We do not assume quasineutrality for the wave since it can have characteristic time and length scales of the order of the plasma frequency and the Debye length. Thus, we work with the full system of Maxwell's equations. Using the wave assumption in (1.1), we write the induction equation as

$$\nabla \times \delta\mathbf{E} = -\frac{\partial \delta\mathbf{B}}{\partial t} \Rightarrow \mathrm{i}\mathbf{k} \times \tilde{\mathbf{E}} = \mathrm{i}\omega\tilde{\mathbf{B}} \tag{2.2}$$

and Ampere's law as

$$\nabla \times \delta\mathbf{B} = \frac{1}{\epsilon_0 c^2}\delta\mathbf{J} + \frac{1}{c^2}\frac{\partial \delta\mathbf{E}}{\partial t} \Rightarrow \mathrm{i}\mathbf{k} \times \tilde{\mathbf{B}} = \frac{1}{\epsilon_0 c^2}\tilde{\mathbf{J}} - \frac{\mathrm{i}\omega}{c^2}\tilde{\mathbf{E}}. \tag{2.3}$$

Note that we have included in Ampere's law the perturbation to the current density due to the response of the plasma to the electromagnetic fields,

$$\delta\mathbf{J} = \sum_s Z_s e \delta\mathbf{\Gamma}_s \Rightarrow \tilde{\mathbf{J}} = \sum_s Z_s e \tilde{\mathbf{\Gamma}}_s, \tag{2.4}$$

where $\delta\mathbf{\Gamma}_s$ is the perturbed flow of species s,

$$\delta\mathbf{\Gamma}_s = \int \delta f_s \mathbf{v} \, \mathrm{d}^3v \Rightarrow \tilde{\mathbf{\Gamma}}_s = \int \tilde{f}_s \mathbf{v} \, \mathrm{d}^3v. \tag{2.5}$$

We proceed to calculate $\delta\mathbf{J}$.

We linearize the Vlasov equation to find the equation for the perturbed distribution function δf_s, and we use assumption (1.2) to write

$$\underbrace{\frac{\partial \delta f_s}{\partial t}}_{\sim\omega\delta f_s} + \underbrace{\cancelto{\text{small}}{\mathbf{v}\cdot\nabla\delta f_s}}_{\sim k v_{ts}\delta f_s \ll \omega\delta f_s} + \frac{Z_s e}{m_s}(\mathbf{v}\times\mathbf{B})\cdot\nabla_v \delta f_s = -\frac{Z_s e}{m_s}(\delta\mathbf{E} + \underbrace{\cancelto{\text{small}}{\mathbf{v}\times\delta\mathbf{B}}}_{\sim(k v_{ts}/\omega)\delta\mathbf{E}\ll\delta\mathbf{E}})\cdot\nabla_v f_s. \tag{2.6}$$

Note that we have used equation (2.2) to estimate the size of $\delta\mathbf{B}$. To find an equation for $\delta\mathbf{\Gamma}_s$, we multiply equation (2.6) by $\mathbf{v}$ and integrate over velocity space,

$$\frac{\partial \delta\mathbf{\Gamma}_s}{\partial t} - \Omega_s \delta\mathbf{\Gamma}_s \times \hat{\mathbf{b}} = \frac{Z_s e n_s}{m_s}\delta\mathbf{E} \Rightarrow -\mathrm{i}\omega\tilde{\mathbf{\Gamma}}_s - \Omega_s\tilde{\mathbf{\Gamma}}_s \times \hat{\mathbf{b}} = \frac{Z_s e n_s}{m_s}\tilde{\mathbf{E}}, \tag{2.7}$$

where $\Omega_s = Z_s e B/m_s$ is the gyrofrequency of species s and $\hat{\mathbf{b}} = \mathbf{B}/B$ is the unit vector in the direction of the background magnetic field

To find $\tilde{\mathbf{J}}$ we need to invert (2.7). To invert (2.7), we project the equation onto the orthonormal basis $\{\hat{\mathbf{e}}_1 = \tilde{\mathbf{E}}_\perp/\tilde{E}_\perp, \hat{\mathbf{e}}_2 = \hat{\mathbf{b}} \times \hat{\mathbf{e}}_1, \hat{\mathbf{b}}\}$, where $\tilde{\mathbf{E}}_\perp$ is the component of $\tilde{\mathbf{E}}$

perpendicular to $\hat{\mathbf{b}}$. In this basis, equation (2.7) becomes

$$\begin{pmatrix} -\mathrm{i}\omega & -\Omega_s & 0 \\ \Omega_s & -\mathrm{i}\omega & 0 \\ 0 & 0 & -\mathrm{i}\omega \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{\Gamma}}_s \cdot \hat{\mathbf{e}}_1 \\ \tilde{\mathbf{\Gamma}}_s \cdot \hat{\mathbf{e}}_2 \\ \tilde{\mathbf{\Gamma}}_s \cdot \hat{\mathbf{b}} \end{pmatrix} = \frac{Z_s e n_s}{m_s} \begin{pmatrix} \tilde{E}_\perp \\ 0 \\ \tilde{E}_\parallel \end{pmatrix}. \tag{2.8}$$

The solution to this system of equations is

$$\begin{pmatrix} \tilde{\mathbf{\Gamma}}_s \cdot \hat{\mathbf{e}}_1 \\ \tilde{\mathbf{\Gamma}}_s \cdot \hat{\mathbf{e}}_2 \\ \tilde{\mathbf{\Gamma}}_s \cdot \hat{\mathbf{b}} \end{pmatrix} = \frac{Z_s e n_s}{m_s} \frac{\mathrm{i}}{\omega(\omega^2 - \Omega_s^2)} \begin{pmatrix} \omega^2 \tilde{E}_\perp \\ -\mathrm{i}\omega\Omega_s \tilde{E}_\perp \\ (\omega^2 - \Omega_s^2)\tilde{E}_\parallel \end{pmatrix}. \tag{2.9}$$

This solution can be written as

$$\tilde{\mathbf{\Gamma}}_s = \frac{Z_s e n_s}{m_s} \mathrm{i}\omega \left(\frac{1}{\omega^2 - \Omega_s^2} \tilde{\mathbf{E}}_\perp + \frac{1}{\omega^2} (\tilde{\mathbf{E}} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}} - \frac{\mathrm{i}\Omega_s}{\omega(\omega^2 - \Omega_s^2)} \hat{\mathbf{b}} \times \tilde{\mathbf{E}} \right). \tag{2.10}$$

Note that we have obtained a particle flow linear in $\tilde{\mathbf{E}}$. By summing over species, we obtain a plasma current linear in $\tilde{\mathbf{E}}$,

$$\tilde{\mathbf{J}} = \sum_s Z_s e \tilde{\mathbf{\Gamma}}_s = \boldsymbol{\sigma} \cdot \tilde{\mathbf{E}}. \tag{2.11}$$

The matrix

$$\boldsymbol{\sigma} = \mathrm{i}\epsilon_0\omega \sum_s \left[\frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + \frac{\omega_{ps}^2}{\omega^2} \hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{\mathrm{i}\omega_{ps}^2 \Omega_s}{\omega(\omega^2 - \Omega_s^2)} \hat{\mathbf{b}} \times \mathbf{I} \right] \tag{2.12}$$

is the <u>conductivity tensor</u> that relates current density and electric field. Here $\omega_{ps} = \sqrt{Z_s^2 e^2 n_s / \epsilon_0 m_s}$ is the plasma frequency of species s, $\mathbf{I}$ is the unit matrix, and the tensor $\hat{\mathbf{b}} \times \mathbf{I}$ can be written in Einstein's index notation as

$$(\hat{\mathbf{b}} \times \mathbf{I})_{ij} = \epsilon_{ikl} \hat{b}_k \delta_{lj} = \epsilon_{ikj} \hat{b}_k, \tag{2.13}$$

where ϵ_{ikl} is the Levi-Civita symbol, and δ_{lj} is the Kronecker delta.

3. Cold plasma dispersion relation

To find an equation for the wave, we first express all quantities of interest as functions of $\tilde{\mathbf{E}}$. We use (2.11) for $\tilde{\mathbf{J}}$, and eliminate $\tilde{\mathbf{B}}$ by using (2.2) to find

$$\tilde{\mathbf{B}} = \frac{1}{\omega} \mathbf{k} \times \tilde{\mathbf{E}}. \tag{3.1}$$

With $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{J}}$ known as functions of $\tilde{\mathbf{E}}$, we can replace them into Ampere's law (2.3) to find

$$\frac{c^2}{\omega^2} \mathbf{k} \times (\mathbf{k} \times \tilde{\mathbf{E}}) = -\tilde{\mathbf{E}} - \frac{\mathrm{i}}{\epsilon_0 \omega} \tilde{\mathbf{J}} = -\boldsymbol{\epsilon} \cdot \tilde{\mathbf{E}}, \tag{3.2}$$

where

$$\boldsymbol{\epsilon} = \mathbf{I} + \frac{\mathrm{i}}{\epsilon_0\omega} \boldsymbol{\sigma} = \mathbf{I} - \sum_s \left[\frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + \frac{\omega_{ps}^2}{\omega^2} \hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{\mathrm{i}\omega_{ps}^2 \Omega_s}{\omega(\omega^2 - \Omega_s^2)} \hat{\mathbf{b}} \times \mathbf{I} \right] \tag{3.3}$$

is the <u>dielectric tensor</u>. It is the equivalent of the relative permittivity ϵ_r in dielectrics. The displacement vector of electromagnetic theory, which in vacuum is $\mathbf{D} = \epsilon_0 \mathbf{E}$, is

$$\mathbf{D} = \epsilon_0 \boldsymbol{\epsilon} \cdot \mathbf{E} \tag{3.4}$$

for a plasma. The plasma produces currents and charge displacements that will reinforce or reduce the size of the electric field. Because of the anisotropy introduced by the background magnetic field, the dielectric permittivity is not a scalar but a tensor.

Using $\mathbf{k} \times (\mathbf{k} \times \tilde{\mathbf{E}}) = (\mathbf{k} \cdot \tilde{\mathbf{E}})\mathbf{k} - k^2 \tilde{\mathbf{E}}$, equation (3.2) becomes

$$\left[\frac{c^2 k^2}{\omega^2} (\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon} \right] \cdot \tilde{\mathbf{E}} = 0, \tag{3.5}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$ is the unit vector in the direction of $\mathbf{k}$. This equation is linear in $\tilde{\mathbf{E}}$, and it has the trivial solution $\tilde{\mathbf{E}} = 0$ that is uninteresting. To obtain a non-trivial solution, the linear system in (3.5) must be singular, that is,

$$\det \left[\frac{c^2 k^2}{\omega^2} (\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon} \right] = 0. \tag{3.6}$$

This equation can be thought of as the relation that gives the magnitude k as a function of the frequency ω, the direction $\hat{\mathbf{k}}$ and the plasma characteristics contained in $\boldsymbol{\epsilon}$. Given an antenna with certain frequency that emits waves in a certain direction, the plasma response will decide k and hence the phase velocity of the wave. It is typical to use the index of refraction

$$n = \frac{kc}{\omega} \tag{3.7}$$

instead of k. Note that then the phase velocity is given by c/n.

Once k is obtained from (3.6), we go back to (3.5) to calculate the polarization of the wave, that is, the direction of $\tilde{\mathbf{E}}$. Note that the magnitude of $\tilde{\mathbf{E}}$ cannot be found from (3.5), and it is determined by the boundary condition at the antenna.

We proceed to discuss (3.5) and (3.6).

• The cold plasma dielectric tensor $\boldsymbol{\epsilon}$ is the term that contains most of the information. It is useful to split it into three components,

$$\boldsymbol{\epsilon} = \epsilon_\perp (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + \epsilon_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}} - \mathrm{i} g \hat{\mathbf{b}} \times \mathbf{I}, \tag{3.8}$$

where

$$\epsilon_\perp = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2},\ \epsilon_\parallel = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2},\ g = -\sum_s \frac{\omega_{ps}^2 \Omega_s}{\omega(\omega^2 - \Omega_s^2)}. \tag{3.9}$$

It is convetional to use the orthonormal basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{b}}\}$, with $\hat{\mathbf{x}} = \mathbf{k}_\perp / k_\perp$ the unit vector in the direction of $\mathbf{k}_\perp = \mathbf{k} - (\mathbf{k} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}}$, and $\hat{\mathbf{y}} = \hat{\mathbf{b}} \times \hat{\mathbf{x}}$ (see figure 1). In this basis, the dielectric tensor is

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_\perp & ig & 0 \\ -ig & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix}. \tag{3.10}$$

• The tensor $\boldsymbol{\epsilon}$ is Hermitian. As a result, n^2 is real. To prove it, we transpose and conjugate (3.5) to find

$$\tilde{\mathbf{E}}^* \cdot \left[(n^2)^* (\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon} \right] = 0. \tag{3.11}$$

Post-multiplying this equation by $\tilde{\mathbf{E}}$ and pre-multiplying (3.5) by $\tilde{\mathbf{E}}^*$, we obtain

$$\tilde{\mathbf{E}}^* \cdot \left[(n^2)^* (\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon} \right] \cdot \tilde{\mathbf{E}} = 0 \tag{3.12}$$

and

$$\tilde{\mathbf{E}}^* \cdot \left[n^2 (\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon} \right] \cdot \tilde{\mathbf{E}} = 0. \tag{3.13}$$

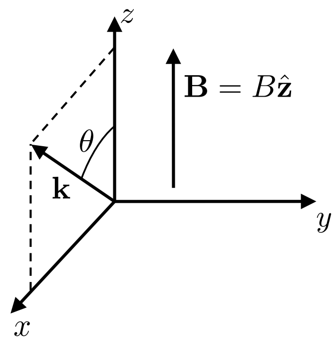

FIGURE 1. Basis used to solve (3.6).

Subtracting (3.13) from (3.12), we find

$$[(n^2)^* - n^2](|\tilde{\mathbf{E}}|^2 - |\hat{\mathbf{k}} \cdot \tilde{\mathbf{E}}|^2) = 0. \tag{3.14}$$

Thus, either n^2 is real, or $\tilde{\mathbf{E}}$ is parallel to $\mathbf{k}$. When $\tilde{\mathbf{E}}$ is parallel to $\mathbf{k}$, the wave is electrostatic and we cannot prove that n^2 is real, but in fact, we cannot determine n^2, as we discuss below.

• When $\tilde{\mathbf{E}}$ is parallel to $\mathbf{k}$, the wave is electrostatic because it can be written as

$$\tilde{\mathbf{E}} = -\mathrm{i}\mathbf{k}\tilde{\phi} \Rightarrow \delta\mathbf{E} = -\nabla\delta\phi. \tag{3.15}$$

In this limiting case, equation (3.5) becomes

$$\boldsymbol{\epsilon} \cdot \hat{\mathbf{k}} = 0. \tag{3.16}$$

Since the tensor $\boldsymbol{\epsilon}$ does not depend on k, this equation cannot determine k. To find k we have to relax the cold plasma assumption (1.2).
Typically, waves with large index of refraction n are approximately electrostatic. Indeed, for $n \gg 1$, equation (3.5) is, to lowest order in $1/n^2$,

$$n^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) \cdot \tilde{\mathbf{E}} \simeq 0. \tag{3.17}$$

Thus, to lowest order, $\tilde{\mathbf{E}} \propto \hat{\mathbf{k}}$. Expanding in $1/n^2$, we can write $\tilde{\mathbf{E}} = \hat{\mathbf{k}} + \tilde{\mathbf{E}}^{(1)}$, where $\tilde{\mathbf{E}}^{(1)} \sim 1/n^2$. Then, equation (3.5) becomes

$$\boldsymbol{\epsilon} \cdot \hat{\mathbf{k}} + n^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) \cdot \tilde{\mathbf{E}}^{(1)} \simeq 0. \tag{3.18}$$

We can eliminate the higher order correction $\tilde{\mathbf{E}}^{(1)}$ by pre-multiplying by $\hat{\mathbf{k}}$, leaving

$$\hat{\mathbf{k}} \cdot \boldsymbol{\epsilon} \cdot \hat{\mathbf{k}} \simeq 0. \tag{3.19}$$

This is the approximate dispersion relation for waves with large index of refraction n. As we have noted above, this dispersion relation is independent of k and it determines the frequency of the wave as a function of the direction $\hat{\mathbf{k}}$.

• To solve for n in (3.6), it is convenient to use the orthonormal basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{b}}\}$ in figure 1. In this basis, we define the direction of $\mathbf{k}$ using

$$\hat{\mathbf{k}} = \sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{b}}. \tag{3.20}$$

Using this form of $\hat{\mathbf{k}}$, the form of $\boldsymbol{\epsilon}$ in (3.8), and projecting on the basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{b}}\}$, the tensor in (3.5) becomes

$$n^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_\perp - n^2\cos^2\theta & ig & n^2\sin\theta\cos\theta \\ -ig & \epsilon_\perp - n^2 & 0 \\ n^2\sin\theta\cos\theta & 0 & \epsilon_\parallel - n^2\sin^2\theta \end{pmatrix}. \tag{3.21}$$

As a result, equation (3.6) becomes

$$\alpha n^4 + \beta n^2 + \gamma = 0, \tag{3.22}$$

where

$$\alpha = \epsilon_\parallel \cos^2\theta + \epsilon_\perp \sin^2\theta,\ \beta = -\epsilon_\perp\epsilon_\parallel(1+\cos^2\theta) - (\epsilon_\perp^2 - g^2)\sin^2\theta,\ \gamma = \epsilon_\parallel(\epsilon_\perp^2 - g^2). \tag{3.23}$$

This is the Booker quartic. Its solution is

$$n^2 = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \tag{3.24}$$

Depending on the sign chosen for the square root, we have slow waves (larger n and hence smaller phase velocity), and fast waves (smaller n and larger phase velocity).

• Using $\cos^2\theta = (1+\tan^2\theta)^{-1}$ and $\sin^2\theta = \tan^2\theta/(1+\tan^2\theta)$ in (3.22), we can solve for $\tan^2\theta$, finding the useful expression

$$\tan^2\theta = -\frac{\epsilon_\parallel(n^2 - \epsilon_\perp + g)(n^2 - \epsilon_\perp - g)}{\epsilon_\perp(n^2 - \epsilon_\parallel)(n^2 - \epsilon_\perp + g^2/\epsilon_\perp)}. \tag{3.25}$$

4. Parallel propagation

We consider waves that propagate along $\mathbf{B}$, that is, $\hat{\mathbf{k}} = \hat{\mathbf{b}}$. Then, equation (3.5) becomes

$$[(\epsilon_\perp - n^2)(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + \epsilon_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}} - \mathrm{i}g\hat{\mathbf{b}} \times \mathbf{I}] \cdot \tilde{\mathbf{E}} = 0. \tag{4.1}$$

We could impose that this linear system of equations be singular, find n^2, and once n^2 is given, we can find the polarization of $\tilde{\mathbf{E}}$. Instead, we take another approach. Taking the scalar product of $\hat{\mathbf{b}}$ with equation (4.1), we obtain

$$\epsilon_\parallel \hat{\mathbf{b}} \cdot \tilde{\mathbf{E}} = 0. \tag{4.2}$$

Thus, for parallel propagation we have two options: either $\epsilon_\parallel = 0$ or $\hat{\mathbf{b}} \cdot \tilde{\mathbf{E}} = 0$.

(*a*) If $\epsilon_\parallel = 0$, equation (4.1) gives that $\tilde{\mathbf{E}}$ must be parallel to $\hat{\mathbf{b}}$. This wave is electrostatic because $\hat{\mathbf{b}}$ and $\mathbf{k}$ are parallel. As we have seen, electrostatic waves can have any value of n in the cold plasma approximation. To see what $\epsilon_\parallel = 0$ gives, we consider a plasma with only one ion species with charge Ze and mass m_i, and electrons with charge $-e$ and mass m_e. Considering that according to quasineutrality, $Zn_i = n_e$, $\epsilon_\parallel$ becomes

$$\epsilon_\parallel = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{Zm_e}{m_i}\right). \tag{4.3}$$

Since $m_e/m_i \ll 1$, $\epsilon_\parallel = 0$ gives

$$\omega \simeq \omega_{pe} = \sqrt{\frac{e^2 n_e}{\epsilon_0 m_e}}. \tag{4.4}$$

This mode represents electrostatic oscillations along the magnetic field line. Since the

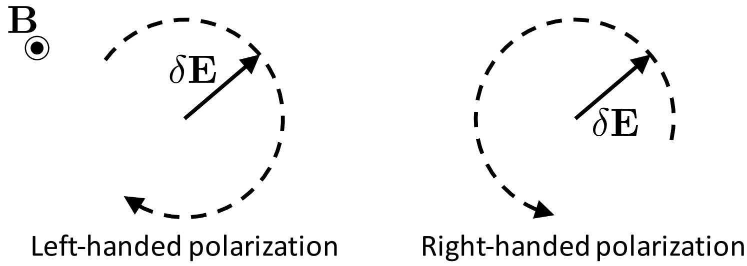

FIGURE 2. Left- and right-handed circular polarization. The magnetic field **B** is pointing out of the page.

electric field points along **B**, particles move mostly along **B** and the magnetic force does not play any role in these oscillations.

(b) Another possibility allowed by (4.2) is that $\hat{\mathbf{b}} \cdot \tilde{\mathbf{E}} = 0$. Then, equation (4.1) gives

$$\begin{pmatrix} \epsilon_\perp - n^2 & \mathrm{i}g \\ -\mathrm{i}g & \epsilon_\perp - n^2 \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{E}} \cdot \hat{\mathbf{x}} \\ \tilde{\mathbf{E}} \cdot \hat{\mathbf{y}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.5}$$

where we are using the orthonormal basis in figure 1. The determinant of the matrix, $(\epsilon_\perp - n^2)^2 - g^2$, must be zero, giving

$$n^2 = \epsilon_\perp \pm g. \tag{4.6}$$

The polarization corresponding to the this solution is

$$\tilde{\mathbf{E}} \propto \hat{\mathbf{x}} \mp \mathrm{i}\hat{\mathbf{y}}. \tag{4.7}$$

Taking into account that the time dependence in (1.1) is $\exp(-\mathrm{i}\omega t)$, we find that

- the top sign in (4.6) and (4.7) corresponds to left-handed circular polarization, and
- the bottom sign in (4.6) and (4.7) corresponds to right-handed circular polarization.

See figure 2 for the two types of polarization.

The dependence on plasma parameters of the index of refraction in equation (4.6) is not explicit. Considering a plasma with only one ion species again, we find

$$n^2 = \epsilon_\perp \pm g = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega \mp \Omega_s)} = 1 - \frac{\omega_{pe}^2}{\omega(\omega \pm \Omega_e)} - \frac{\omega_{pi}^2}{\omega(\omega \mp \Omega_i)}. \tag{4.8}$$

Importantly, **the electron gyrofrequency $\Omega_e = eB/m_e$ is defined to be positive, and as a result, when $s \to e$, $\Omega_s \to -\Omega_e$.** Using $\omega_{pe}^2 \Omega_i - \omega_{pi}^2 \Omega_e = 0$, equation (4.8) becomes

$$n^2 = \frac{\omega^2 \pm (\Omega_e - \Omega_i)\omega - \Omega_e \Omega_i - \omega_{pe}^2 - \omega_{pi}^2}{(\omega \pm \Omega_e)(\omega \mp \Omega_i)}, \tag{4.9}$$

where the top sign corresponds to left-handed circular polarization, and the bottom sign to right-handed circular polarization. Using the frequencies

$$\omega_L = \sqrt{\left(\frac{\Omega_e + \Omega_i}{2}\right)^2 + \omega_{pe}^2 + \omega_{pi}^2} - \frac{\Omega_e - \Omega_i}{2}, \tag{4.10}$$

$$\omega_R = \sqrt{\left(\frac{\Omega_e + \Omega_i}{2}\right)^2 + \omega_{pe}^2 + \omega_{pi}^2} + \frac{\Omega_e - \Omega_i}{2}, \tag{4.11}$$

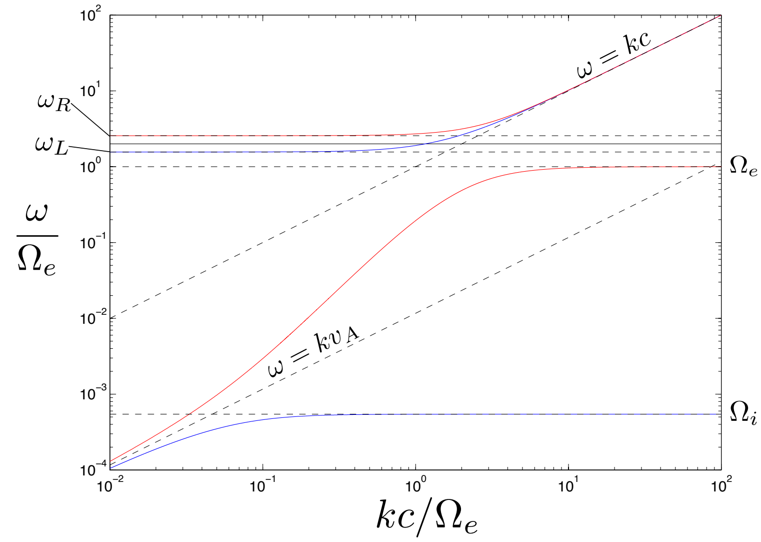

FIGURE 3. Cold plasma waves with parallel propagation for an electron-proton plasma ($Z = 1$, $m_i/m_e = 1836$) with $\omega_{pe}/\Omega_e = 2$. The plasma oscillation ($\omega = \omega_{pe}$) is represented as a black line, the waves with left-handed polarization as blue lines, and the waves with right-handed polarization as red lines. For comparison, we plot the frequencies ω_L, ω_R, Ω_e and Ω_i and the dispersion relations for light ($\omega = kc$) and Alfven waves ($\omega = kv_A$) as dashed lines.

we can rewrite equation (4.9) as

$$n^2 = \frac{(\omega \mp \omega_L)(\omega \pm \omega_R)}{(\omega \pm \Omega_e)(\omega \mp \Omega_i)}. \tag{4.12}$$

Using $\omega_{pi}^2/\omega_{pe}^2 = \Omega_i/\Omega_e = Zm_e/m_i \ll 1$, the relative size of the frequencies ω_{pe}, ω_L, ω_R, Ω_e and Ω_i is determined by the non-dimensional parameter ω_{pe}/Ω_e. Most plasmas of interest in astrophysics and magnetic confinement fusion satisfy $\omega_{pe}/\Omega_e \gtrsim 1$, leading to

$$\omega_L \simeq \sqrt{\frac{\Omega_e^2}{4} + \omega_{pe}^2} - \frac{\Omega_e}{2}, \quad \omega_R \simeq \sqrt{\frac{\Omega_e^2}{4} + \omega_{pe}^2} + \frac{\Omega_e}{2}. \tag{4.13}$$

In this regime, $\omega_R > \omega_{pe} > \omega_L \gg \Omega_i$. The relative size of Ω_e with respect to ω_{pe} and ω_L depends on the exact value of ω_{pe}/Ω_e. For example, for $\omega_{pe}/\Omega_e > \sqrt{2}$, we find

$$\omega_R > \omega_{pe} > \omega_L > \Omega_e \gg \Omega_i. \tag{4.14}$$

In figure 3 we represent the dispersion relation of the waves with parallel propagation for the assumptions in (4.14). The relations in (4.14) and the fact that n^2 in (4.12) must be positive for wave propagation indicate that waves with left-handed polarization exist for $\omega > \omega_L$ and $\omega < \Omega_i$. Similarly, waves with right-handed polarization exist for $\omega > \omega_R$ and $\omega < \Omega_e$. Under the assumptions (4.14), we find several interesting limits:

(*a*) For $\omega \gg \omega_R, \omega_L, \Omega_e, \Omega_i$, we find light,

$$\omega \simeq kc. \tag{4.15}$$

Light can have both left- and right-handed polarization.

(*b*) For $\omega \simeq \Omega_e$, we find the electron cyclotron wave,

$$\Omega_e - \omega \simeq \frac{\omega_{pe}^2 \Omega_e}{k^2 c^2} \ll 1. \tag{4.16}$$

This wave has right-handed polarization.

(*c*) For $\Omega_i \ll \omega \ll \Omega_e$, we find the whistler wave,

$$\omega \simeq \frac{k^2 c^2 \Omega_e}{\omega_{pe}^2}. \tag{4.17}$$

This wave has right-handed polarization.

(*d*) For $\omega \simeq \Omega_i$, we find the ion cyclotron wave,

$$\Omega_i - \omega \simeq \frac{\Omega_i^3}{k^2 v_A^2} \ll 1, \tag{4.18}$$

where $v_A = \sqrt{\epsilon_0 c^2 B^2 / n_i m_i}$ is the Alfven speed. This wave has left-handed polarization.

(*e*) For $\omega \ll \Omega_i, \Omega_e$, we find the Alfven waves,

$$\omega \simeq k v_A. \tag{4.19}$$

These waves can have both left- and right-handed polarization.

5. Perpendicular propagation

We consider $\hat{\mathbf{k}} \cdot \hat{\mathbf{b}} = 0$. In this case, equation (3.5) becomes

$$[\epsilon_\perp (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) - n^2 (\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}}) + \epsilon_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}} - \mathrm{i} g \hat{\mathbf{b}} \times \mathbf{I}] \cdot \tilde{\mathbf{E}} = 0. \tag{5.1}$$

Taking the scalar product of this equation with $\hat{\mathbf{b}}$, we find

$$(\epsilon_\parallel - n^2) \hat{\mathbf{b}} \cdot \tilde{\mathbf{E}} = 0. \tag{5.2}$$

Thus, there are two possibilities: either $\epsilon_\parallel - n^2 = 0$ or $\hat{\mathbf{b}} \cdot \tilde{\mathbf{E}} = 0$.

(*a*) Ordinary mode. The velocity of propagation of the ordinary mode is given by $\epsilon_\parallel - n^2 = 0$. Using $\epsilon_\parallel - n^2 = 0$ in (5.1), we find that $\tilde{\mathbf{E}}$ must be parallel to $\mathbf{B}$. This electric field only moves particles along $\mathbf{B}$, and as a result, the magnetic force does not play a role. For a plasma composed of a single ion species and electrons, the equation $\epsilon_\parallel - n^2 = 0$ gives

$$\omega^2 = \omega_{pe}^2 + k^2 c^2, \tag{5.3}$$

where we have used (4.3) and $Z m_e / m_i \ll 1$.

(*b*) Extraordinary mode. Assuming $\hat{\mathbf{b}} \cdot \tilde{\mathbf{E}} = 0$, equation (5.1) becomes

$$[\epsilon_\perp \mathbf{I} - n^2 (\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}}) - \mathrm{i} g \hat{\mathbf{b}} \times \mathbf{I}] \cdot \tilde{\mathbf{E}} = 0. \tag{5.4}$$

To solve this equation, we project onto the basis in figure 1 to find

$$\begin{pmatrix} \epsilon_\perp & \mathrm{i} g \\ -\mathrm{i} g & \epsilon_\perp - n^2 \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{E}} \cdot \hat{\mathbf{x}} \\ \tilde{\mathbf{E}} \cdot \hat{\mathbf{y}} \end{pmatrix} = 0. \tag{5.5}$$

Imposing that the determinant of the matrix be zero, we obtain

$$n^2 = \epsilon_\perp - \frac{g^2}{\epsilon_\perp}. \tag{5.6}$$

The polarization is given by

$$\tilde{\mathbf{E}} \propto \mathrm{i} g \hat{\mathbf{x}} - \epsilon_\perp \hat{\mathbf{y}}. \tag{5.7}$$

Thus, these waves have elliptical polarization.

Equation (5.6) can be written in terms of the plasma properties. We assume again a plasma with a single ion species. The index of refraction in (5.6) is

$$n^2 = \frac{\epsilon_\perp^2 - g^2}{\epsilon_\perp}, \tag{5.8}$$

where $\epsilon_\perp^2 - g^2$ and $\epsilon_\perp$ can be written in a convenient way. Using equations (4.6) and (4.12) to write

$$\epsilon_\perp \pm g = \frac{(\omega \mp \omega_L)(\omega \pm \omega_R)}{(\omega \pm \Omega_e)(\omega \mp \Omega_i)}, \tag{5.9}$$

we find

$$\epsilon_\perp^2 - g^2 = (\epsilon_\perp + g)(\epsilon_\perp - g) = \frac{(\omega^2 - \omega_L^2)(\omega^2 - \omega_R^2)}{(\omega^2 - \Omega_e^2)(\omega^2 - \Omega_i^2)}. \tag{5.10}$$

We also have

$$\epsilon_\perp = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}. \tag{5.11}$$

Substituting (5.10) and (5.11) into (5.8), and using $\omega_{pi}^2 \Omega_e = \omega_{pe}^2 \Omega_i$, we find

$$n^2 = \frac{(\omega^2 - \omega_L^2)(\omega^2 - \omega_R^2)}{\omega^4 - (\Omega_e^2 + \Omega_i^2 + \omega_{pe}^2 + \omega_{pi}^2)\omega^2 + \Omega_e^2\Omega_i^2 + \omega_{pe}^2 \Omega_i (\Omega_e + \Omega_i)}. \tag{5.12}$$

Finally, using the upper hybrid frequency

$$\omega_{UH}^2 = \frac{\Omega_e^2 + \Omega_i^2 + \omega_{pe}^2 + \omega_{pi}^2}{2} + \sqrt{\left(\frac{\Omega_e^2 + \Omega_i^2 + \omega_{pe}^2 + \omega_{pi}^2}{2}\right)^2 - \Omega_e^2\Omega_i^2 - \omega_{pe}^2\Omega_i(\Omega_e + \Omega_i)} \tag{5.13}$$

and the lower hybrid frequency

$$\omega_{LH}^2 = \frac{\Omega_e^2 + \Omega_i^2 + \omega_{pe}^2 + \omega_{pi}^2}{2} - \sqrt{\left(\frac{\Omega_e^2 + \Omega_i^2 + \omega_{pe}^2 + \omega_{pi}^2}{2}\right)^2 - \Omega_e^2\Omega_i^2 - \omega_{pe}^2\Omega_i(\Omega_e + \Omega_i)}, \tag{5.14}$$

equation (5.12) can be written as

$$n^2 = \frac{(\omega^2 - \omega_L^2)(\omega^2 - \omega_R^2)}{(\omega^2 - \omega_{UH}^2)(\omega^2 - \omega_{LH}^2)}. \tag{5.15}$$

Again, using $\omega_{pi}^2/\omega_{pe}^2 = \Omega_i/\Omega_e = Zm_e/m_i \ll 1$, the relative size of ω_{pe}, ω_R, ω_L, ω_{UH} and ω_{LH} is determined by the non-dimensional number ω_{pe}/Ω_e. For $\omega_{pe}/\Omega_e \gtrsim 1$, we find

$$\omega_{UH} \simeq \sqrt{\omega_{pe}^2 + \Omega_e^2}, \quad \omega_{LH} \simeq \sqrt{\frac{\Omega_i\Omega_e}{1 + \Omega_e^2/\omega_{pe}^2}}, \tag{5.16}$$

leading to

$$\omega_R > \omega_{UH} > \omega_{pe} > \omega_L \gg \omega_{LH}. \tag{5.17}$$

Thus, according to (5.12), there are extraordinary waves for $\omega > \omega_R$, $\omega_L < \omega < \omega_{UH}$ and $\omega < \omega_{LH}$. In figure 4 we give the possible waves with perpendicular propagation for the assumptions in (5.17).

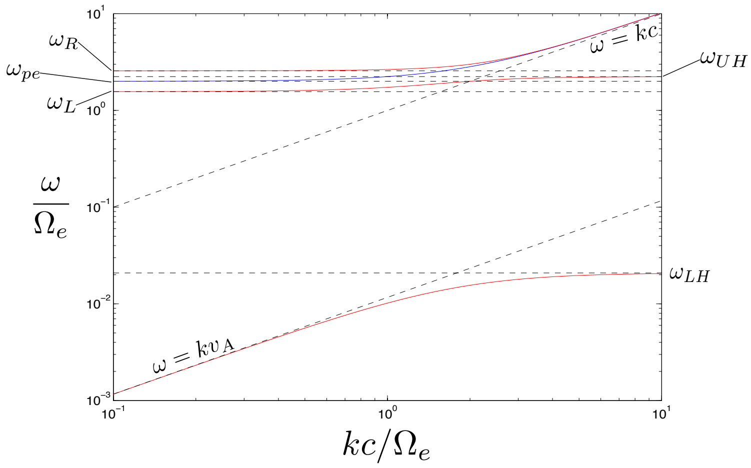

FIGURE 4. Cold plasma waves with perpendicular propagation for an electron-proton plasma ($Z = 1$, $m_i/m_e = 1836$) with $\omega_{pe}/\Omega_e = 2$. The ordinary mode is represented as a blue line, and the extraordinary mode as red lines. For comparison, we plot the frequencies ω_L, ω_R, ω_{UH} and ω_{LH} and the dispersion relations for light ($\omega = kc$) and Alfven waves ($\omega = kv_A$) as dashed lines.