Hot plasma waves

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(This version is of 4 March 2019)

1. Introduction

In these notes, we study the behavior of waves that satisfy

$$\frac{\omega}{k} \sim v_{ts} \tag{1.1}$$

for at least one species s. Under these conditions, plasma waves can grow due to instabilities, or decrease by transferring their energy to the background plasma. The waves must be treated kinetically.

2. Background plasma

We consider a system composed of

(a) A homogeneous, steady state, collisionless plasma with several species s. Each species s follows the distribution function $f_s(\mathbf{v})$.

(b) A background magnetic field $\mathbf{B}$ constant in time and uniform in space.

(c) No background electric field $\mathbf{E}$.

The distribution functions must satisfy the collisionless Vlasov equations,

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{Z_s e}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_s = 0. \tag{2.1}$$

Since the plasma is homogeneous and in steady state, and $\mathbf{E} = 0$, this equation becomes

$$\frac{Z_s e}{m_s}(\mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_s = 0. \tag{2.2}$$

To see what this equation implies for f_s, we describe the velocity space using the component of the velocity parallel to $\mathbf{B}$, $v_\|$, the perpendicular velocity, $v_\perp$, and the gyrophase, φ. The velocity is then

$$\mathbf{v} = v_\perp(\cos\varphi\, \hat{\mathbf{x}} - \sin\varphi\, \hat{\mathbf{y}}) + v_\| \hat{\mathbf{b}}, \tag{2.3}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are two orthogonal unit vectors that satisfy $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{b}}$. The velocity is sketched in figure 1 (note the sign of φ). Using $\{v_\|, v_\perp, \varphi\}$, we find

$$\nabla_v = \nabla_v v_\| \frac{\partial}{\partial v_\|} + \nabla_v v_\perp \frac{\partial}{\partial v_\perp} + \nabla_v \varphi \frac{\partial}{\partial \varphi} = \hat{\mathbf{b}} \frac{\partial}{\partial v_\|} + \frac{\mathbf{v}_\perp}{v_\perp} \frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp^2} \mathbf{v} \times \hat{\mathbf{b}} \frac{\partial}{\partial \varphi}, \tag{2.4}$$

and as a result, equation (2.2) becomes

$$\Omega_s \frac{\partial f_s}{\partial \varphi} = 0. \tag{2.5}$$

Thus, f_s does not depend on φ,

$$f_s(\mathbf{v}) = f_s(v_\|, v_\perp). \tag{2.6}$$

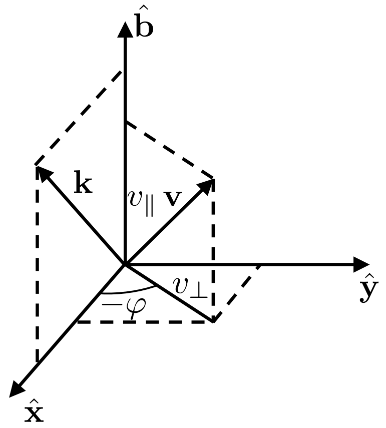

FIGURE 1. Basis for velocity. Note that the gyrophase φ is defined to be the opposite of the usual cylindrical angle.

The distribution function does not depend on the direction of $\mathbf{v}_\perp$, given by φ, because particles in a magnetic field gyrate around magnetic field lines rapidly. As a result, all possible directions of $\mathbf{v}_\perp$ seem equally probable.

Finally, we assume that the background plasma satisfies quasineutrality,

$$\sum_s Z_s e n_s = 0, \tag{2.7}$$

where the background density of species s is

$$n_s = \int f_s \,\mathrm{d}^3 v = \int f_s \left| \det\left(\frac{\partial \mathbf{v}}{\partial(v_\parallel, v_\perp, \varphi)} \right) \right| \mathrm{d}v_\parallel \,\mathrm{d}v_\perp \,\mathrm{d}\varphi = 2\pi \int f_s v_\perp \,\mathrm{d}v_\parallel \,\mathrm{d}v_\perp. \tag{2.8}$$

Here we have used the determinant of the Jacobian of the transformation $\mathbf{v}(v_\parallel, v_\perp, \varphi)$,

$$\det\left(\frac{\partial \mathbf{v}}{\partial(v_\parallel, v_\perp, \varphi)} \right) = \frac{1}{\nabla_v v_\parallel \cdot (\nabla_v v_\perp \times \nabla_v \varphi)} = -v_\perp. \tag{2.9}$$

3. Hot plasma waves

We assume that the electromagnetic fields of the waves are of the form

$$\delta\mathbf{E} = \tilde{\mathbf{E}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \mathrm{i}\omega t),$$
$$\delta\mathbf{B} = \tilde{\mathbf{B}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \mathrm{i}\omega t). \tag{3.1}$$

As a result, the induction equation and Ampere's law become

$$-\mathrm{i}\omega\tilde{\mathbf{B}} = -\mathrm{i}\mathbf{k}\times\tilde{\mathbf{E}} \tag{3.2}$$

and

$$\mathrm{i}\mathbf{k}\times\tilde{\mathbf{B}} = \frac{1}{\epsilon_0 c^2}\tilde{\mathbf{J}} - \frac{\mathrm{i}\omega}{c^2}\tilde{\mathbf{E}}. \tag{3.3}$$

Equations (3.2) and (3.3) can be combined in the same way as we combined them to find the cold plasma dispersion relation,

$$[n^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon}] \cdot \tilde{\mathbf{E}} = 0. \tag{3.4}$$

The only difference with the cold plasma dispersion relation is the dielectric tensor

$$\boldsymbol{\epsilon} = \mathbf{I} + \frac{\mathrm{i}\boldsymbol{\sigma}}{\epsilon_0 \omega}, \tag{3.5}$$

where the conductivity tensor, defined by

$$\tilde{\mathbf{J}} = \boldsymbol{\sigma} \cdot \tilde{\mathbf{E}}, \tag{3.6}$$

is different from the conductivity tensor in the cold plasma dispersion relation. We calculate the new dielectric and conductivity tensors in the following section.

4. Dielectric and conductivity tensors

The perturbed plasma current is given by

$$\delta\mathbf{J} = \sum_s Z_s e \int \delta f_s \mathbf{v}\, \mathrm{d}^3 v \Rightarrow \tilde{\mathbf{J}} = \sum_s Z_s e \int \tilde{f}_s \mathbf{v}\, \mathrm{d}^3 v. \tag{4.1}$$

Thus, to obtain the conductivity tensor in (3.6), we need to first find $\tilde{f}_s$ as a function of $\tilde{\mathbf{E}}$, and then integrate over velocity space to find $\tilde{\mathbf{J}}$.

4.1. *Perturbed distribution function*

The equation for the perturbation to the distribution function δf_s is obtained by linearizing the Vlasov equation (2.1),

$$\frac{\partial \delta f_s}{\partial t} + \mathbf{v} \cdot \nabla \delta f_s + \frac{Z_s e}{m_s} (\mathbf{v} \times \mathbf{B}) \cdot \nabla_v \delta f_s = -\frac{Z_s e}{m_s} (\delta \mathbf{E} + \mathbf{v} \times \delta \mathbf{B}) \cdot \nabla_v f_s. \tag{4.2}$$

Fourier analyzing and using (3.1), equation (4.2) becomes

$$-\mathrm{i}\omega \tilde{f}_s + \mathrm{i}\mathbf{k} \cdot \mathbf{v} \tilde{f}_s + \frac{Z_s e}{m_s} (\mathbf{v} \times \mathbf{B}) \cdot \nabla_v \tilde{f}_s = -\frac{Z_s e}{m_s} (\tilde{\mathbf{E}} + \mathbf{v} \times \tilde{\mathbf{B}}) \cdot \nabla_v f_s. \tag{4.3}$$

Since we want to obtain the conductivity tensor in (3.6), it is convenient to express equation (4.3) in terms of $\tilde{\mathbf{E}}$ only, and not in terms of both $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$. Using equation (3.2), equation (4.3) becomes

$$(-\mathrm{i}\omega + \mathrm{i}\mathbf{k} \cdot \mathbf{v}) \tilde{f}_s + \frac{Z_s e}{m_s} (\mathbf{v} \times \mathbf{B}) \cdot \nabla_v \tilde{f}_s = -\frac{Z_s e}{m_s} \left[\tilde{\mathbf{E}} + \frac{1}{\omega} \mathbf{v} \times (\mathbf{k} \times \tilde{\mathbf{E}}) \right] \cdot \nabla_v f_s. \tag{4.4}$$

To solve equation (4.4), we describe the velocity space using the coordinates $\{v_\parallel, v_\perp, \varphi\}$ in figure 1. Recalling equations (2.4) and (2.6), we find

$$\nabla_v f_s = \hat{\mathbf{b}} \frac{\partial f_s}{\partial v_\parallel} + \frac{\mathbf{v}_\perp}{v_\perp} \frac{\partial f_s}{\partial v_\perp} = \frac{\mathbf{v}}{v_\perp} \frac{\partial f_s}{\partial v_\perp} + \hat{\mathbf{b}} \left(\frac{\partial f_s}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp} \frac{\partial f_s}{\partial v_\perp} \right). \tag{4.5}$$

Using this result, equation (2.4) and $\mathbf{v} \times (\mathbf{k} \times \tilde{\mathbf{E}}) = (\mathbf{v} \cdot \tilde{\mathbf{E}})\mathbf{k} - (\mathbf{k} \cdot \mathbf{v})\tilde{\mathbf{E}}$, equation (4.4) becomes

$$\begin{aligned} (-\mathrm{i}\omega + \mathrm{i}\mathbf{k} \cdot \mathbf{v}) \tilde{f}_s + \Omega_s \frac{\partial \tilde{f}_s}{\partial \varphi} = &-\frac{Z_s e}{m_s v_\perp} \frac{\partial f_s}{\partial v_\perp} \left[\tilde{\mathbf{E}} \cdot \mathbf{v} + \cancelto{0}{\frac{1}{\omega} [\mathbf{v} \times (\mathbf{k} \times \tilde{\mathbf{E}})] \cdot \mathbf{v}} \right] \\ &- \frac{Z_s e}{m_s \omega} \left(\frac{\partial f_s}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp} \frac{\partial f_s}{\partial v_\perp} \right) \left[(\omega - \mathbf{k} \cdot \mathbf{v}) \tilde{\mathbf{E}} \cdot \hat{\mathbf{b}} + k_\parallel (\tilde{\mathbf{E}} \cdot \mathbf{v}) \right]. \end{aligned} \tag{4.6}$$

Using $\mathbf{v} = v_\parallel \hat{\mathbf{b}} + \mathbf{v}_\perp$ and $\mathbf{k} = k_\parallel \hat{\mathbf{b}} + \mathbf{k}_\perp$, this equation can be rewritten as

$$\left(-\mathrm{i}\omega + \mathrm{i}k_\parallel v_\parallel + \mathrm{i}\mathbf{k}_\perp \cdot \mathbf{v}_\perp + \Omega_s \frac{\partial}{\partial \varphi} \right) \left[\tilde{f}_s + \frac{Z_s e \mathrm{i}}{m_s \omega} \left(\frac{\partial f_s}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp} \frac{\partial f_s}{\partial v_\perp} \right) \tilde{\mathbf{E}} \cdot \hat{\mathbf{b}} \right]$$
$$= -\frac{Z_s e}{m_s} \left[\frac{\partial f_s}{\partial v_\perp} + \frac{k_\parallel}{\omega} \left(v_\perp \frac{\partial f_s}{\partial v_\parallel} - v_\parallel \frac{\partial f_s}{\partial v_\perp} \right) \right] \left(\frac{\tilde{\mathbf{E}} \cdot \mathbf{v}_\perp}{v_\perp} + \frac{v_\parallel}{v_\perp} \tilde{\mathbf{E}} \cdot \hat{\mathbf{b}} \right). \tag{4.7}$$

To integrate equation (4.7), we write it in the orthonormal basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{b}}\}$, where $\hat{\mathbf{x}} = \mathbf{k}_\perp / k_\perp$ and $\hat{\mathbf{y}} = \hat{\mathbf{b}} \times \mathbf{k}_\perp / k_\perp$ (see figure 1). In this basis,

$$\left(-\mathrm{i}\omega + \mathrm{i}k_\parallel v_\parallel + \mathrm{i}k_\perp v_\perp \cos\varphi + \Omega_s \frac{\partial}{\partial \varphi} \right) \left[\tilde{f}_s + \frac{Z_s e \mathrm{i}}{m_s \omega} \left(\frac{\partial f_s}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp} \frac{\partial f_s}{\partial v_\perp} \right) \hat{\mathbf{b}} \cdot \tilde{\mathbf{E}} \right]$$
$$= -\frac{Z_s e}{m_s} \left[\frac{\partial f_s}{\partial v_\perp} + \frac{k_\parallel}{\omega} \left(v_\perp \frac{\partial f_s}{\partial v_\parallel} - v_\parallel \frac{\partial f_s}{\partial v_\perp} \right) \right] \left(\frac{\mathbf{k}_\perp}{k_\perp} \cos\varphi - \frac{\hat{\mathbf{b}} \times \mathbf{k}_\perp}{k_\perp} \sin\varphi + \frac{v_\parallel}{v_\perp} \hat{\mathbf{b}} \right) \cdot \tilde{\mathbf{E}}. \tag{4.8}$$

Equation (4.8) has an integrating factor: $\exp(\mathrm{i}\lambda_s \sin\varphi)$, where

$$\lambda_s = \frac{k_\perp v_\perp}{\Omega_s}. \tag{4.9}$$

Multiplying by this factor, we find

$$\left(-\mathrm{i}\omega + \mathrm{i}k_\parallel v_\parallel + \Omega_s \frac{\partial}{\partial \varphi} \right) \left\{ \left[\tilde{f}_s + \frac{Z_s e \mathrm{i}}{m_s \omega} \left(\frac{\partial f_s}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp} \frac{\partial f_s}{\partial v_\perp} \right) \hat{\mathbf{b}} \cdot \tilde{\mathbf{E}} \right] \exp(\mathrm{i}\lambda_s \sin\varphi) \right\}$$
$$= -\frac{Z_s e}{m_s} \left[\frac{\partial f_s}{\partial v_\perp} + \frac{k_\parallel}{\omega} \left(v_\perp \frac{\partial f_s}{\partial v_\parallel} - v_\parallel \frac{\partial f_s}{\partial v_\perp} \right) \right] \exp\left(\mathrm{i}\lambda_s \sin\varphi\right)$$
$$\times \left(\frac{\mathbf{k}_\perp}{k_\perp} \cos\varphi - \frac{\hat{\mathbf{b}} \times \mathbf{k}_\perp}{k_\perp} \sin\varphi + \frac{v_\parallel}{v_\perp} \hat{\mathbf{b}} \right) \cdot \tilde{\mathbf{E}}. \tag{4.10}$$

To solve equation (4.10), we Fourier analyze in the gyrophase. In particular, we Fourier analyze the function

$$\left[\tilde{f}_s + \frac{Z_s e \mathrm{i}}{m_s \omega} \left(\frac{\partial f_s}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp} \frac{\partial f_s}{\partial v_\perp} \right) \hat{\mathbf{b}} \cdot \tilde{\mathbf{E}} \right] \exp\left(\mathrm{i}\lambda_s \sin\varphi\right) = \sum_{m=-\infty}^{\infty} \tilde{F}_{s,m} \exp(\mathrm{i}m\varphi). \tag{4.11}$$

To Fourier analyze equation (4.10), we use that

$$\exp(\mathrm{i}\lambda_s \sin\varphi) = \sum_{m=-\infty}^{\infty} J_m(\lambda_s) \exp(\mathrm{i}m\varphi), \tag{4.12}$$

$$\cos\varphi \exp(\mathrm{i}\lambda_s \sin\varphi) = \frac{1}{\mathrm{i}\lambda_s} \frac{\partial}{\partial \varphi} [\exp(\mathrm{i}\lambda_s \sin\varphi)] = \sum_{m=-\infty}^{\infty} \frac{m J_m(\lambda_s)}{\lambda_s} \exp(\mathrm{i}m\varphi) \tag{4.13}$$

and

$$\sin\varphi \exp(\mathrm{i}\lambda_s \sin\varphi) = \frac{1}{\mathrm{i}} \frac{\partial}{\partial \lambda_s} [\exp(\mathrm{i}\lambda_s \sin\varphi)] = -\mathrm{i} \sum_{m=-\infty}^{\infty} J_m'(\lambda_s) \exp(\mathrm{i}m\varphi), \tag{4.14}$$

where

$$J_m(\lambda_s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\, \mathrm{d}\varphi \tag{4.15}$$

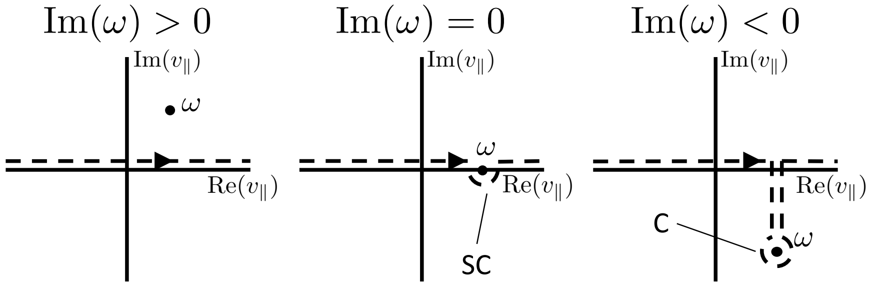

FIGURE 2. Landau contours for integration of the resonant denominators in (4.18). For $\mathrm{Im}(\omega) > 0$, the contour is the real line, as expected. For $\mathrm{Im}(\omega) \leqslant 0$, the contour surrounds ω to ensure that the integral is an analytic continuation of the integral with $\mathrm{Im}(\omega) > 0$. For $\mathrm{Im}(\omega) = 0$, the part of the contour that surrounds ω is a semi-circumference, SC, whereas for $\mathrm{Im}(\omega) < 0$, the piece of the contour that surrounds ω is a a complete circumference, C.

is the m-th order Bessel function of the first kind, and $J_m' = \mathrm{d}J_m/\mathrm{d}\lambda_s$ is its derivative. Using (4.11), (4.12), (4.13) and (4.14) in (4.10), we can solve for the Fourier coefficients $\tilde{F}_{s,m}$,

$$\tilde{F}_{s,m} = -\frac{Z_s e \mathrm{i}}{m_s} \frac{1}{\omega - k_\| v_\| - m\Omega_s} \left[\frac{\partial f_s}{\partial v_\perp} + \frac{k_\|}{\omega} \left(v_\perp \frac{\partial f_s}{\partial v_\|} - v_\| \frac{\partial f_s}{\partial v_\perp} \right) \right] \mathbf{u}_m^* \cdot \tilde{\mathbf{E}}, \quad (4.16)$$

where the complex vector $\mathbf{u}_m$ is

$$\mathbf{u}_m = \frac{m J_m(\lambda_s)}{\lambda_s} \frac{\mathbf{k}_\perp}{k_\perp} - \mathrm{i} J_m'(\lambda_s) \frac{\hat{\mathbf{b}} \times \mathbf{k}_\perp}{k_\perp} + \frac{v_\|}{v_\perp} J_m(\lambda_s) \hat{\mathbf{b}}. \quad (4.17)$$

4.2. *Conductivity tensor*

Equations (4.11) and (4.16) give the distribution function $\tilde{f}_s$. From (4.1), we obtain

$$\begin{aligned} \tilde{\mathbf{J}} = & -\mathrm{i}\epsilon_0 \omega \sum_s \frac{Z_s^2 e^2}{\epsilon_0 m_s \omega^2} \int_{-\infty}^{\infty} \mathrm{d}v_\| \int_0^\infty \mathrm{d}v_\perp \int_{-\pi}^{\pi} \mathrm{d}\varphi\, v_\perp \left(\frac{\partial f_s}{\partial v_\|} - \frac{v_\|}{v_\perp} \frac{\partial f_s}{\partial v_\perp} \right) \mathbf{v}\, (\hat{\mathbf{b}} \cdot \tilde{\mathbf{E}}) \\ & + \sum_s Z_s e \int_{C_L} \mathrm{d}v_\| \int_0^\infty \mathrm{d}v_\perp \int_{-\pi}^{\pi} \mathrm{d}\varphi\, v_\perp \exp(-\mathrm{i}\lambda_s \sin\varphi) \sum_{m=-\infty}^{\infty} \tilde{F}_{s,m} \exp(\mathrm{i}m\varphi) \mathbf{v}, \quad (4.18) \end{aligned}$$

where

$$\mathbf{v} = v_\perp \left(\cos\varphi \frac{\mathbf{k}_\perp}{k_\perp} - \sin\varphi \frac{\hat{\mathbf{b}} \times \mathbf{k}_\perp}{k_\perp} \right) + v_\| \hat{\mathbf{b}}. \quad (4.19)$$

The resonant denominator $\omega - k_\| v_\| - m\Omega_s$ in $\tilde{F}_{s,m}$ produces damping or growth. It also indicates that we should have Laplace transformed in time instead of using a Fourier transform. The final result obtained using the Laplace transform is the same except for the fact that we need to use the Landau contour C_L (shown in figure 2) to take the integrals over parallel velocity. See (Schekochihin 2015) for more details.

Using equations (4.12), (4.13) and (4.14) (and changing φ to $-\varphi$), we find

$$\int_{-\pi}^{\pi} \exp(-\mathrm{i}\lambda_s \sin\varphi) \sum_{m=-\infty}^{\infty} \tilde{F}_{s,m} \exp(\mathrm{i}m\varphi) \mathbf{v}\, \mathrm{d}\varphi = 2\pi \sum_{m=-\infty}^{\infty} \tilde{F}_{s,m} \mathbf{u}_m, \quad (4.20)$$

where the complex vector $\mathbf{u}_m$ is defined in (4.17). With this result and equation (4.16),

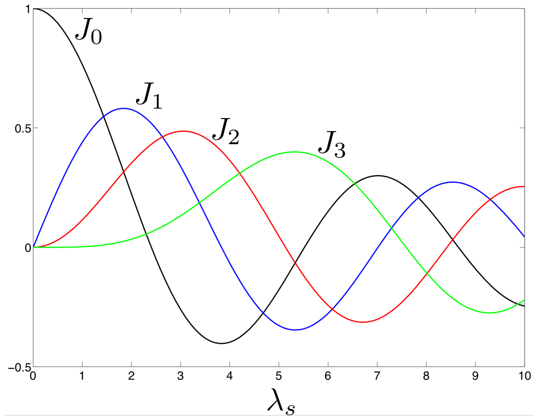

FIGURE 3. The first four Bessel functions of the first kind (J_0 in black, J_1 in blue, J_2 in red, and J_3 in green) as a function of their argument.

equation (4.18) gives the conductivity tensor (recall (3.6))

$$\boldsymbol{\sigma} = -\mathrm{i}\epsilon_0\omega \sum_s \frac{2\pi\omega_{ps}^2}{n_s\omega^2} \int_{C_L} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \left\{ \left(v_\perp \frac{\partial f_s}{\partial v_\parallel} - v_\parallel \frac{\partial f_s}{\partial v_\perp} \right) v_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}} \right.$$
$$\left. + v_\perp^2 \left[\frac{\partial f_s}{\partial v_\perp} + \frac{k_\parallel}{\omega} \left(v_\perp \frac{\partial f_s}{\partial v_\parallel} - v_\parallel \frac{\partial f_s}{\partial v_\perp} \right) \right] \sum_{m=-\infty}^{\infty} \frac{\omega \mathbf{u}_m \mathbf{u}_m^*}{\omega - k_\parallel v_\parallel - m\Omega_s} \right\}. \tag{4.21}$$

4.3. *Dielectric tensor*

From (4.21), we find that the dielectric tensor is

$$\boldsymbol{\epsilon} = \mathbf{I} + \frac{\mathrm{i}\boldsymbol{\sigma}}{\epsilon_0\omega} = \mathbf{I} + \sum_s \frac{2\pi\omega_{ps}^2}{n_s\omega^2} \int_{C_L} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \left\{ \left(v_\perp \frac{\partial f_s}{\partial v_\parallel} - v_\parallel \frac{\partial f_s}{\partial v_\perp} \right) v_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}} \right.$$
$$\left. + v_\perp^2 \left[\frac{\partial f_s}{\partial v_\perp} + \frac{k_\parallel}{\omega} \left(v_\perp \frac{\partial f_s}{\partial v_\parallel} - v_\parallel \frac{\partial f_s}{\partial v_\perp} \right) \right] \sum_{m=-\infty}^{\infty} \frac{\omega \mathbf{u}_m \mathbf{u}_m^*}{\omega - k_\parallel v_\parallel - m\Omega_s} \right\}. \tag{4.22}$$

The dielectric tensor can be simplified in various limits. For the Bessel functions, it is very useful to know several properties.

• The Bessel functions with negative m can be deduced from the ones with positive m by using the change of variables $\varphi' = \pi - \varphi$, giving

$$J_{-m}(\lambda_s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(\mathrm{i}\lambda_s \sin\varphi + \mathrm{i}m\varphi)\,\mathrm{d}\varphi$$
$$= \frac{\exp(\mathrm{i}m\pi)}{2\pi} \int_{-\pi}^{\pi} \exp(\mathrm{i}\lambda_s \sin\varphi' - \mathrm{i}m\varphi')\,\mathrm{d}\varphi' = (-1)^m J_m(\lambda_s), \tag{4.23}$$

where $m = 1, 2, 3, \ldots$

• The values of the Bessel function for negative arguments can be deduced from its values for positive arguments by using the change of variables $\varphi' = \varphi - \pi$, giving

$$J_m(-\lambda_s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi$$
$$= \frac{\exp(-\mathrm{i}m\pi)}{2\pi} \int_{-\pi}^{\pi} \exp(\mathrm{i}\lambda_s \sin\varphi' - \mathrm{i}m\varphi')\,\mathrm{d}\varphi' = (-1)^m J_m(\lambda_s). \tag{4.24}$$

Hence, for m even, $J_m(\lambda_s)$ is even, and for m odd, $J_m(\lambda)$ is odd.

• The Bessel functions are bounded oscillatory functions with a decreasing amplitude. We show a few of them in figure 3.

• The derivatives of Bessel functions can also be written in terms of Bessel functions by using $\sin\varphi = [\exp(\mathrm{i}\varphi) - \exp(-\mathrm{i}\varphi)]/2\mathrm{i}$,

$$J'_m(\lambda_s) = \frac{\mathrm{i}}{2\pi}\int_{-\pi}^{\pi} \sin\varphi \exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi = \frac{1}{2}\left[J_{m-1}(\lambda_s) - J_{m+1}(\lambda_s)\right]. \tag{4.25}$$

• The Bessel functions can be expanded for small and large λ_s. For $\lambda_s \ll 1$, the Bessel functions with positive m become (see Appendix A)

$$J_m(\lambda_s) = \left(\frac{\lambda_s}{2}\right)^m \sum_{p=0}^{\infty} \frac{(-1)^p}{(m+p)!\,p!}\left(\frac{\lambda_s}{2}\right)^{2p}. \tag{4.26}$$

For $\lambda_s \gg 1$, the Bessel functions with positive m become (see Appendix B)

$$J_m(\lambda_s) \simeq \sqrt{\frac{2}{\pi\lambda_s}} \cos\left(\lambda_s - \frac{m\pi}{2} - \frac{\pi}{4}\right). \tag{4.27}$$

The resonant denominator $\omega - k_\parallel v_\parallel - m\Omega_s$ can be expanded as well (Schekochihin 2015).

5. Cold plasma limit

The cold plasma limit is recovered when we assume $kv_{ts}/\omega \ll 1$ and $\lambda_s \sim k_\perp v_{ts}/\Omega_s \ll 1$. In this limit, the Bessel functions are given by (4.23) and (4.26). Thus, to lowest order in $\lambda_s \ll 1$, $J_0(\lambda_s) \simeq 1$, $J_1(\lambda_s)/\lambda_s \simeq 1/2$, $J_{-1}(\lambda_s)/\lambda_s \simeq -1/2$, $J'_1(\lambda_s) \simeq 1/2$ and $J'_{-1}(\lambda_s) \simeq -1/2$ are the only non-zero contributions. As a result, of all the vectors $\mathbf{u}_m$ defined in (4.17), only the vectors

$$\mathbf{u}_{-1} = \frac{1}{2}\frac{\mathbf{k}_\perp}{k_\perp} + \frac{\mathrm{i}}{2}\frac{\hat{\mathbf{b}}\times\mathbf{k}_\perp}{k_\perp},\ \mathbf{u}_0 = \frac{v_\parallel}{v_\perp}\hat{\mathbf{b}},\ \mathbf{u}_1 = \frac{1}{2}\frac{\mathbf{k}_\perp}{k_\perp} - \frac{\mathrm{i}}{2}\frac{\hat{\mathbf{b}}\times\mathbf{k}_\perp}{k_\perp} \tag{5.1}$$

contribute to lowest order in $\lambda_s \ll 1$. Using this result and $kv_{ts}/\omega \ll 1$, equation (4.22) becomes

$$\boldsymbol{\epsilon} \simeq \mathbf{I} + \sum_s \frac{2\pi\omega_{ps}^2}{n_s\omega^2}\int_{-\infty}^{\infty}\mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \left\{ v_\perp v_\parallel \frac{\partial f_s}{\partial v_\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}} + v_\perp^2 \frac{\partial f_s}{\partial v_\perp}\left(\frac{\omega\mathbf{u}_{-1}\mathbf{u}_{-1}^*}{\omega+\Omega_s} + \frac{\omega\mathbf{u}_1\mathbf{u}_1^*}{\omega-\Omega_s}\right)\right\}. \tag{5.2}$$

Since

$$\begin{aligned}\mathbf{u}_{-1}\mathbf{u}_{-1}^* = \frac{1}{4k_\perp^2}[\mathbf{k}_\perp\mathbf{k}_\perp + (\hat{\mathbf{b}}\times\mathbf{k}_\perp)(\hat{\mathbf{b}}\times\mathbf{k}_\perp)] + \frac{\mathrm{i}}{4k_\perp^2}[(\hat{\mathbf{b}}\times\mathbf{k}_\perp)\mathbf{k}_\perp - \mathbf{k}_\perp(\hat{\mathbf{b}}\times\mathbf{k}_\perp)] \\ = \frac{1}{4}(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + \frac{\mathrm{i}}{4}\hat{\mathbf{b}}\times\mathbf{I}\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\mathbf{u}_1\mathbf{u}_1^* = \frac{1}{4k_\perp^2}[\mathbf{k}_\perp\mathbf{k}_\perp + (\hat{\mathbf{b}}\times\mathbf{k}_\perp)(\hat{\mathbf{b}}\times\mathbf{k}_\perp)] - \frac{\mathrm{i}}{4k_\perp^2}[(\hat{\mathbf{b}}\times\mathbf{k}_\perp)\mathbf{k}_\perp - \mathbf{k}_\perp(\hat{\mathbf{b}}\times\mathbf{k}_\perp)] \\ = \frac{1}{4}(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) - \frac{\mathrm{i}}{4}\hat{\mathbf{b}}\times\mathbf{I},\end{aligned} \tag{5.4}$$

equation (5.2) can be rewritten as

$$\boldsymbol{\epsilon} \simeq \mathbf{I} + \sum_s \frac{2\pi\omega_{ps}^2}{n_s\omega^2} \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \left\{ v_\perp v_\parallel \frac{\partial f_s}{\partial v_\parallel} \hat{\mathbf{b}}\hat{\mathbf{b}} + \frac{v_\perp^2}{2} \frac{\partial f_s}{\partial v_\perp} \left[\frac{\omega^2}{\omega^2 - \Omega_s^2} (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) - \frac{\mathrm{i}\omega\Omega_s}{\omega^2 - \Omega_s^2} \hat{\mathbf{b}} \times \mathbf{I} \right] \right\}. \tag{5.5}$$

Finally, integrating by parts in $v_\parallel$ and $v_\perp$, and using

$$2\pi \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp \, v_\perp f_s = n_s, \tag{5.6}$$

the dielectric tensor becomes the cold plasma dielectric tensor

$$\boldsymbol{\epsilon} = \epsilon_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}} + \epsilon_\perp (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) - \mathrm{i}g\hat{\mathbf{b}} \times \mathbf{I}, \tag{5.7}$$

where

$$\epsilon_\perp = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}, \; \epsilon_\parallel = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}, \; g = -\sum_s \frac{\omega_{ps}^2 \Omega_s}{\omega(\omega^2 - \Omega_s^2)}. \tag{5.8}$$

6. Dispersion relation for Maxwellian distribution functions

In this section, we consider waves for stationary Maxwellian distribution functions,

$$f_s = f_{Ms} \equiv n_s \left(\frac{m_s}{2\pi T_s} \right)^{3/2} \exp\left(- \frac{m_s(v_\parallel^2 + v_\perp^2)}{2T_s} \right). \tag{6.1}$$

With these distribution functions and the integration variables $w = v_\perp/v_{ts}$ and $u = (k_\parallel/|k_\parallel|)(v_\parallel/v_{ts})$, where $v_{ts} = \sqrt{2T_s/m_s}$, the dielectric tensor in (4.22) becomes

$$\boldsymbol{\epsilon} = \mathbf{I} + \sum_s \frac{\omega_{ps}^2}{\omega|k_\parallel|v_{ts}} \frac{4}{\sqrt{\pi}} \int_{C_L} \mathrm{d}u \, \exp(-u^2) \int_0^\infty \mathrm{d}w \, w^3 \exp(-w^2) \sum_{m=-\infty}^{\infty} \frac{\mathbf{u}_m \mathbf{u}_m^*}{u - \zeta_{s,m}}, \tag{6.2}$$

where

$$\zeta_{s,m} = \frac{\omega - m\Omega_s}{|k_\parallel| v_{ts}}, \tag{6.3}$$

$$\mathbf{u}_m = \left(\frac{Z_s}{|Z_s|} \right)^{m-1} \frac{m J_m(w\sqrt{2b_s})}{w\sqrt{2b_s}} \frac{\mathbf{k}_\perp}{k_\perp} - \mathrm{i} \left(\frac{Z_s}{|Z_s|} \right)^{m-1} J_m'(w\sqrt{2b_s}) \frac{\hat{\mathbf{b}} \times \mathbf{k}_\perp}{k_\perp} + \frac{k_\parallel}{|k_\parallel|} \left(\frac{Z_s}{|Z_s|} \right)^m \frac{u}{w} J_m(w\sqrt{2b_s}) \hat{\mathbf{b}} \tag{6.4}$$

and

$$b_s = \frac{k_\perp^2 v_{ts}^2}{2\Omega_s^2} = \frac{k_\perp^2 T_s}{m_s \Omega_s^2}. \tag{6.5}$$

Note that we had to keep track of the sign of Ω_s, which is the same as the sign of Z_s, because $\sqrt{2b_s}$ is alway positive.

Equation (6.2) can be expressed in terms of known special functions. Using the plasma dispersion function $\mathcal{Z}(\zeta_{s,m})$, we find

$$\frac{1}{\sqrt{\pi}} \int_{C_L} \frac{\exp(-u^2)}{u - \zeta_{s,m}} \mathrm{d}u = \mathcal{Z}(\zeta_{s,m}), \tag{6.6}$$

$$\frac{1}{\sqrt{\pi}}\int_{C_L}\frac{u\exp(-u^2)}{u-\zeta_{s,m}}\,\mathrm{d}u = 1+\zeta_{s,m}\mathcal{Z}(\zeta_{s,m}) \tag{6.7}$$

and

$$\frac{1}{\sqrt{\pi}}\int_{C_L}\frac{u^2\exp(-u^2)}{u-\zeta_{s,m}}\,\mathrm{d}u = \zeta_{s,m}[1+\zeta_{s,m}\mathcal{Z}(\zeta_{s,m})]. \tag{6.8}$$

We will also need several integrals of the Bessel functions of the first kind. We calculate these integrals in Appendix C, where we obtain

$$2\int_0^\infty J_m^2(w\sqrt{2b_s})w\exp(-w^2)\,\mathrm{d}w = I_m(b_s)\exp(-b_s), \tag{6.9}$$

$$4\int_0^\infty J_m(w\sqrt{2b_s})J_m'(w\sqrt{2b_s})w^2\exp(-w^2)\,\mathrm{d}w = \sqrt{2b_s}\,[I_m'(b_s)-I_m(b_s)]\exp(-b_s) \tag{6.10}$$

and

$$4\int_0^\infty [J_m'(w\sqrt{2b_s})]^2w^3\exp(-w^2)\,\mathrm{d}w = \left[\frac{m^2I_m(b_s)}{b_s}+2b_s\left(I_m(b_s)-I_m'(b_s)\right)\right]\exp(-b_s), \tag{6.11}$$

where

$$I_m(b_s) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(b_s\cos\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi. \tag{6.12}$$

are the m-th order modified Bessel function of the first kind and $I_m'(b_s) = \mathrm{d}I_m/\mathrm{d}b_s$ is the derivative of the modified Bessel function with respect to its argument.

Using all the results above, the dielectric tensor in (6.2) in the basis shown in figure 1 is

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{xx} & \mathrm{i}\epsilon_{xy} & \epsilon_{xz} \\ -\mathrm{i}\epsilon_{xy} & \epsilon_{yy} & \mathrm{i}\epsilon_{yz} \\ \epsilon_{xz} & -\mathrm{i}\epsilon_{yz} & \epsilon_{zz} \end{pmatrix}, \tag{6.13}$$

where

$$\epsilon_{xx} = 1+\sum_s\sum_{m=-\infty}^{\infty}\frac{\omega_{ps}^2}{\omega|k_\parallel|v_{ts}}\frac{m^2I_m(b_s)}{b_s}\exp(-b_s)\mathcal{Z}(\zeta_{s,m}), \tag{6.14}$$

$$\epsilon_{yy} = 1+\sum_s\sum_{m=-\infty}^{\infty}\frac{\omega_{ps}^2}{\omega|k_\parallel|v_{ts}}\left[\frac{m^2I_m(b_s)}{b_s}+2b_s\left(I_m(b_s)-I_m'(b_s)\right)\right]\exp(-b_s)\mathcal{Z}(\zeta_{s,m}), \tag{6.15}$$

$$\epsilon_{zz} = 1+\sum_s\sum_{m=-\infty}^{\infty}\frac{2\omega_{ps}^2}{\omega|k_\parallel|v_{ts}}I_m(b_s)\exp(-b_s)\,\zeta_{s,m}\left[1+\zeta_{s,m}\mathcal{Z}(\zeta_{s,m})\right], \tag{6.16}$$

$$\epsilon_{xy} = \sum_s\sum_{m=-\infty}^{\infty}\frac{\omega_{ps}^2}{\omega|k_\parallel|v_{ts}}m\left[I_m'(b_s)-I_m(b_s)\right]\exp(-b_s)\mathcal{Z}(\zeta_{s,m}), \tag{6.17}$$

$$\epsilon_{xz} = \sum_s\sum_{m=-\infty}^{\infty}\frac{\omega_{ps}^2}{\omega k_\parallel v_{ts}}\frac{Z_s}{|Z_s|}\sqrt{\frac{2}{b_s}}mI_m(b_s)\exp(-b_s)\left[1+\zeta_{s,m}\mathcal{Z}(\zeta_{s,m})\right] \tag{6.18}$$

and

$$\epsilon_{yz} = -\sum_s\sum_{m=-\infty}^{\infty}\frac{\omega_{ps}^2}{\omega k_\parallel v_{ts}}\frac{Z_s}{|Z_s|}\sqrt{2b_s}\left[I_m'(b_s)-I_m(b_s)\right]\exp(-b_s)\left[1+\zeta_{s,m}\mathcal{Z}(\zeta_{s,m})\right]. \tag{6.19}$$

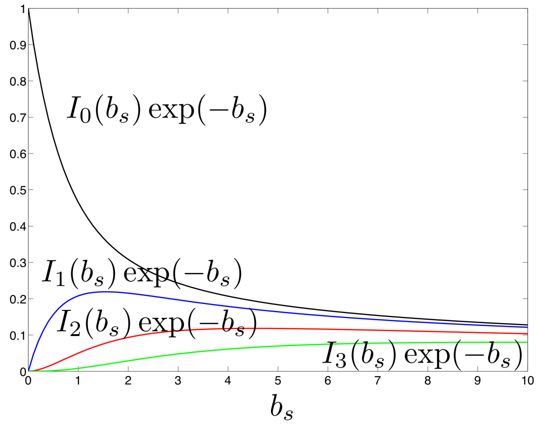

FIGURE 4. Functions $I_m(b_s)\exp(-b_s)$ as a function of their argument b_s for $m = 0$ (black), $m = 1$ (blue), $m = 2$ (red) and $m = 3$ (green).

As in the case of a general distribution function $f_s(v_\|, v_\perp)$, this dielectric tensor can be simplified in various limits. For the modified Bessel functions, it is very useful to know several properties.

• The modified Bessel functions with negative m can be deduced from the ones with positive m by using the change of variables $\varphi' = -\varphi$, giving

$$\begin{aligned} I_{-m}(b_s) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(b_s\cos\varphi + \mathrm{i}m\varphi)\,\mathrm{d}\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(b_s\cos\varphi' - \mathrm{i}m\varphi')\,\mathrm{d}\varphi' = I_m(b_s), \end{aligned} \tag{6.20}$$

where $m = 1, 2, 3, \ldots$

• The modified Bessel functions are functions that diverge exponentially for large arguments. In figure 4, we show $I_m(b_s)\exp(-b_s)$ for a few m. We multiply $I_m(b_s)$ by the exponential because this is the combination in which the modified Bessel functions appear in the dispersion relation.

• The derivatives of the modified Bessel functions can also be written in terms of modified Bessel functions by using $\cos\varphi = [\exp(\mathrm{i}\varphi) + \exp(-\mathrm{i}\varphi)]/2$,

$$I'_m(b_s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\varphi\exp(b_s\cos\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi = \frac{1}{2}\left[I_{m-1}(b_s) + I_{m+1}(b_s)\right]. \tag{6.21}$$

• The modified Bessel functions of the first kind are the Fourier coefficients of the function $\exp(b_s\cos\varphi)$,

$$\exp(b_s\cos\varphi) = \sum_{m=-\infty}^{\infty} I_m(b_s)\exp(\mathrm{i}m\varphi). \tag{6.22}$$

This Fourier series is useful to calculate infinite sums of modified Bessel functions.

• The modified Bessel functions can be expanded for small and large b_s. For $b_s \ll 1$, the Bessel functions with positive m become (see Appendix A)

$$I_m(b_s) = \left(\frac{b_s}{2}\right)^m \sum_{p=0}^{\infty} \frac{1}{(m+p)!\,p!}\left(\frac{b_s}{2}\right)^{2p}. \tag{6.23}$$

For $b_s \gg 1$, the modified Bessel functions become (see Appendix D)

$$I_m(b_s) = \frac{\exp(b_s)}{\sqrt{2\pi b_s}} \left(1 - \frac{4m^2 - 1}{8b_s} + O(b_s^{-2}) \right). \tag{6.24}$$

The plasma dispersion function can also be expanded for $\zeta_{s,m}$ small and large.

We proceed to solve three examples in which we use the hot plasma dispersion relation for Maxwellian distribution functions. In these examples, the plasma is composed of an ion species with charge Ze and mass m_i, and electrons with charge $-e$ and mass m_e. The electron gyrofrequency $\Omega_e = eB/m_e$ is defined to be positive, that is, $\Omega_s \to -\Omega_e$ when $s = e$. We also assume that $\omega_{pe} \sim \Omega_e$ and that the temperatures of electrons and ions are of the same order, $T_i \sim T_e$.

6.1. *Electron cyclotron damping*

We study the damping of the the electron cyclotron wave. We discussed this wave in the notes on cold plasma waves. This wave propagates parallel to the magnetic field line ($\hat{\mathbf{k}} = \hat{\mathbf{b}}$, $k = k_\parallel$), its polarization is right-handed circular, and its dispersion relation is

$$\Omega_e - \omega \simeq \frac{\omega_{pe}^2 \Omega_e}{k_\parallel^2 c^2} \ll 1. \tag{6.25}$$

To study how this wave damps, we consider the limit

$$\omega \simeq \Omega_e \sim \omega_{pe} \gg |\omega - \Omega_e| \gtrsim |k_\parallel| v_{te}, \tag{6.26}$$

and we ignore the ions because they do not respond to high frequencies. In the limit (6.26), the wave damping is small, and the wave is very close to the cold plasma limit. To simplify (6.13), we use that terms that contain $\mathcal{Z}(\zeta_{e,-1})$, with $\zeta_{e,-1} = (\omega - \Omega_e)/|k_\parallel| v_{te} \gtrsim 1$, are much larger than the rest of the terms because for $m \neq -1$, $\zeta_{e,m} \gg 1$ and hence $\mathcal{Z}(\zeta_{e,m}) \simeq -1/\zeta_{e,m} \ll 1$. Moreover, $b_e = 0$ because $k_\perp = 0$, giving

- $I_m(b_e) = 0$ for $m \neq 0$ and $I_0(b_e) = 1$,
- $m^2 I_m(b_e)/b_e = 0$ for $m \neq \pm 1$ and $I_{-1}(b_e)/b_e = 1/2 = I_1(b_e)/b_e$, and
- $I'_m(b_e) = 0$ for $m \neq \pm 1$ and $I'_{-1}(b_e) = 1/2 = I'_1(b_e)$.

Using these approximations, the dielectric tensor (6.13) simplifies to

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{xx} & \mathrm{i}\epsilon_{xy} & 0 \\ -\mathrm{i}\epsilon_{xy} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}, \tag{6.27}$$

where

$$\epsilon_{xx} \simeq \epsilon_{yy} \simeq -\epsilon_{xy} \simeq \frac{\omega_{pe}^2}{2\Omega_e |k_\parallel| v_{te}} \mathcal{Z}(\zeta_{e,-1}) + O\left(\frac{|\omega - \Omega_e|}{\Omega_e} \right). \tag{6.28}$$

In the subsidiary limit $(\omega - \Omega_e)/|k_\parallel| v_{te} = \zeta_{e,-1} \gg 1$,

$$\mathcal{Z}(\zeta_{e,-1}) \simeq -\frac{1}{\zeta_{e,-1}} - \frac{1}{2\zeta_{e,-1}^3} + \mathrm{i}\sqrt{\pi} \exp(-\zeta_{e,-1}^2). \tag{6.29}$$

Thus, equation (6.28) becomes

$$\epsilon_{xx} \simeq \epsilon_{yy} \simeq -\epsilon_{xy} \simeq -\frac{\omega_{pe}^2}{2\Omega_e(\omega - \Omega_e)} \left(1 + \frac{1}{2\zeta_{e,-1}^2} - \mathrm{i}\sqrt{\pi}\zeta_{e,-1} \exp(-\zeta_{e,-1}^2) \right). \tag{6.30}$$

Substituting (6.27) into (3.4) and assuming that $\tilde{\mathbf{E}}\cdot\hat{\mathbf{b}} = \tilde{E}_z = 0$, we obtain the solutions

$$\frac{k_\parallel^2 c^2}{\omega^2} = \frac{\epsilon_{xx} + \epsilon_{yy}}{2} \pm \sqrt{\left(\frac{\epsilon_{xx} - \epsilon_{yy}}{2}\right)^2 + \epsilon_{xy}^2}. \tag{6.31}$$

The electron cyclotron wave is the solution corresponding to the plus sign,

$$\frac{k_\parallel^2 c^2}{\Omega_e^2} \simeq 2\epsilon_{xx} = -\frac{\omega_{pe}^2}{\Omega_e(\omega - \Omega_e)}\left(1 + \frac{1}{2\zeta_{e,-1}^2} - \mathrm{i}\sqrt{\pi}\zeta_{e,-1}\exp(-\zeta_{e,-1}^2)\right). \tag{6.32}$$

The last term in equation (6.32) is exponentially small when $|\zeta_{e,-1}| \gg 1$, but we keep it because it is fundamentally different from other terms: it is imaginary and it will give wave damping. The term small by $\zeta_{e,-1}^{-2} \ll 1$ is kept because it is needed to obtain the correct factor of order unity for the exponentially small correction. To lowest order we find the cold plasma wave solution

$$\omega^{(0)} \simeq \Omega_e - \frac{\omega_{pe}^2\Omega_e}{k_\parallel^2 c^2}. \tag{6.33}$$

We can find the correction to this dispersion relation by using $\omega = \omega^{(0)} + \omega^{(1)} + \mathrm{i}\gamma$, where $\gamma \ll \omega^{(1)} \ll \omega^{(0)}$. We find

$$\frac{\omega^{(1)}}{\Omega_e - \omega^{(0)}} \simeq -\frac{1}{2(\zeta_{e,-1}^{(0)})^2} \simeq -\frac{k_\parallel^6 T_e c^4}{m_e\Omega_e^2\omega_{pe}^4}, \tag{6.34}$$

where $\zeta_{e,-1}^{(0)} = (\omega^{(0)} - \Omega_e)/|k_\parallel|v_{te}$, and

$$\frac{\gamma}{\Omega_e - \omega^{(0)}} \simeq \sqrt{\pi}\zeta_{e,-1}^{(0)}\exp\left(-\zeta_{e,-1}^2\right) \simeq -\sqrt{\pi}\frac{\Omega_e - \omega^{(0)}}{|k_\parallel|v_{te}}\exp\left(-\frac{(\Omega_e - \omega^{(0)})^2}{k_\parallel^2 v_{te}^2} + 1\right). \tag{6.35}$$

Thus, the resonance with the electron cyclotron motion damps the wave. In this case, where we have assumed $|\omega - \Omega_e| \gg |k_\parallel|v_{te}$, the damping is small, but the damping becomes large for $|\omega - \Omega_e| \sim |k_\parallel|v_{te}$.

6.2. *Electron Bernstein waves*

We now consider perpendicular propagation ($k_\parallel = 0$). To simplify the problem, we assume that the index of refraction is very large, that is, $k_\perp c/\omega \gg 1$. We showed in the cold plasma waves notes that waves with very large index of refraction are electrostatic and have to satisfy the dispersion relation $\hat{\mathbf{k}}\cdot\boldsymbol{\epsilon}\cdot\hat{\mathbf{k}} \simeq 0$. In the perpendicular propagation case, this dispersion relation becomes $\epsilon_{xx} = 0$. This condition gives the lower and upper hybrid resonance in the cold plasma dispersion relation. In the hot plasma dispersion relation, using that $k_\parallel = 0$ implies that $(|k_\parallel|v_{ti})^{-1}\mathcal{Z}(\zeta_{s,m}) = -(\omega - m\Omega_s)^{-1}$, we find

$$\epsilon_{xx} = 1 - \sum_s \sum_{m=-\infty}^{\infty} \frac{\omega_{ps}^2}{\omega(\omega - m\Omega_s)}\frac{m^2 I_m(b_s)}{b_s}\exp(-b_s) = 0. \tag{6.36}$$

This is the dispersion relation for Bernstein waves.

We consider only frequencies of the order of the electron gyrofrequency Ω_e. Then, the ion contribution can be neglected, leading to

$$\epsilon_{xx} \simeq 1 - \sum_{m=1}^{\infty} \frac{\omega_{pe}^2}{\omega^2 - m^2\Omega_e^2}\frac{2m^2 I_m(b_e)}{b_e}\exp(-b_e) = 0, \tag{6.37}$$

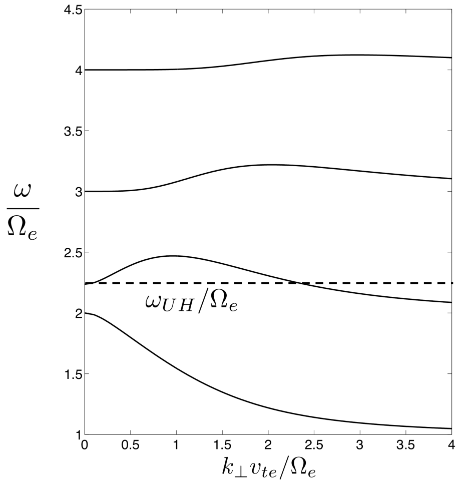

FIGURE 5. Frequency ω as a function of the perpendicular wavenumber $k_\perp$ for electron Bernstein waves with $\omega_{pe}/\Omega_e = 2$.

where we have used the fact that $I_m(b_s) = I_{-m}(b_s)$. The dispersion relation (6.37) can be solved numerically.

We show the dispersion relation for $\omega_{pe}/\Omega_e = 2$ in figure 5. The dispersion relation is multivalued, giving several possible frequencies at each value of $k_\perp$. It is easy to see why there are multiple solutions in the limits $k_\perp v_{te}/\Omega_e \ll 1$ and $k_\perp v_{te}/\Omega_e \gg 1$. For $k_\perp v_{te}/\Omega_e \ll 1$, b_e is small and the functions $I_m(b_e)/b_e$ become small for $m \geqslant 2$. Thus, one possible way to satisfy equation (6.37) is that one of the denominators become small, that is, $\omega \simeq q\Omega_e$ for some integer $q \geqslant 2$. Similarly, for $k_\perp v_{te}/\Omega_e \gg 1$, the functions $I_m(b_e)\exp(-b_e)/b_e$ become small, and the frequency must be close to one of the multiples of Ω_e to ensure that at least one of the denominators is small. Interestingly, the frequencies $\omega = q\Omega_e$ with $q \geqslant 2$ are not the only possible solutions when $k_\perp v_{te}/\Omega_e \ll 1$. In this limit, $m^2 I_m(b_e)/b_e = 0$ for $m \neq 1$, and $I_1(b_e)/b_e = 1/2$, giving

$$\epsilon_{xx} \simeq 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} = 0. \tag{6.38}$$

The solution to this dispersion relation is the upper-hybrid frequency $\omega_{UH} = \sqrt{\omega_{pe}^2 + \Omega_e^2}$ that can be seen in figure 5.

The electrostatic approximation used to obtain equation (6.37) fails for sufficiently small $k_\perp$ because the index of refraction $k_\perp c/\omega$ becomes sufficiently small. Thus, the region of small $k_\perp v_{te}/\Omega_e$ is not exact.

We finish by pointing out that these waves are not damped because of the restrictive assumption $k_\parallel = 0$.

6.3. *Low frequency modes: linear gyrokinetics*

We consider modes with frequencies much lower than the gyrofrequencies, $\omega \ll \Omega_s$. In this limit, we recover drift kinetic results when $b_s \ll 1$. By keeping finite gyroradius effects, $b_s \sim 1$, we obtain gyrokinetics. We assume $\omega \sim k_\parallel v_{ts}$ and at the same time $k_\perp v_{ts}/\Omega_s \sim 1$. These two assumptions imply that $k_\parallel/k_\perp \sim \omega/\Omega_s \ll 1$. These requirements can be summarized in the following orderings

$$b_s \sim \zeta_{s,0} \sim \frac{\omega_{ps}}{\Omega_s} \sim 1 \ll \zeta_{s,m\neq 0} \sim \frac{k_\perp}{k_\parallel} \sim \frac{\Omega_s}{\omega}. \tag{6.39}$$

Note that we are not making a distinction between electrons and ions, that is, we are assuming that $\sqrt{m_e/m_i} \ll \omega/\Omega_s$. The assumptions in (6.39) imply that, for $m \neq 0$,

$$\mathcal{Z}(\zeta_{s,m}) = -\frac{1}{\zeta_{s,m}} + O\left(\frac{1}{\zeta_{s,m}^3}\right) = \frac{|k_\parallel| v_{ts}}{m\Omega_s} \frac{1}{1 - \omega/m\Omega_s} + O\left(\frac{\omega^3}{\Omega_s^3}\right) \simeq \underbrace{\frac{|k_\parallel| v_{ts}}{m\Omega_s}}_{\sim\omega/\Omega_s} + \underbrace{\frac{\omega |k_\parallel| v_{ts}}{m^2\Omega_s^2}}_{\sim\omega^2/\Omega_s^2}, \tag{6.40}$$

$$1 + \zeta_{s,m}\mathcal{Z}(\zeta_{s,m}) = O\left(\frac{1}{\zeta_{s,m}^2}\right) = O\left(\frac{\omega^2}{\Omega_s^2}\right) \tag{6.41}$$

and

$$\zeta_{s,m}[1 + \zeta_{s,m}\mathcal{Z}(\zeta_{s,m})] = -\frac{1}{2\zeta_{s,m}} + O\left(\frac{1}{\zeta_{s,m}^3}\right) \simeq \frac{|k_\parallel| v_{ts}}{2m\Omega_s} + \frac{\omega |k_\parallel| v_{ts}}{2m^2\Omega_s^2}. \tag{6.42}$$

With these results, the coefficients of the dielectric tensor in equations (6.14) - (6.19) become

$$\epsilon_{xx} = 1 + \sum_s \sum_{m=1}^{\infty} \frac{2\omega_{ps}^2}{\Omega_s^2} \frac{I_m(b_s)}{b_s} \exp(-b_s) = O(1), \tag{6.43}$$

$$\epsilon_{yy} = \sum_s \frac{\omega_{ps}^2}{\omega |k_\parallel| v_{ts}} 2b_s \left[I_0(b_s) - I_0'(b_s)\right] \exp(-b_s) \mathcal{Z}(\zeta_{s,0}) = O\left(\frac{\Omega_s^2}{\omega^2}\right), \tag{6.44}$$

$$\epsilon_{zz} = \sum_s \frac{2\omega_{ps}^2}{\omega |k_\parallel| v_{ts}} I_0(b_s) \exp(-b_s)\, \zeta_{s,0} \left[1 + \zeta_{s,0}\mathcal{Z}(\zeta_{s,0})\right] = O\left(\frac{\Omega_s^2}{\omega^2}\right), \tag{6.45}$$

$$\epsilon_{xy} = \sum_s \sum_{m=1}^{\infty} \frac{2\omega_{ps}^2}{\omega\Omega_s} \left[I_m'(b_s) - I_m(b_s)\right] \exp(-b_s) = O\left(\frac{\Omega_s}{\omega}\right). \tag{6.46}$$

and

$$\epsilon_{yz} = -\sum_s \frac{\omega_{ps}^2}{\omega k_\parallel v_{ts}} \frac{Z_s}{|Z_s|} \sqrt{2b_s} \left[I_0'(b_s) - I_0(b_s)\right] \exp(-b_s) \left[1 + \zeta_{s,0}\mathcal{Z}(\zeta_{s,0})\right] = O\left(\frac{\Omega_s^2}{\omega^2}\right), \tag{6.47}$$

where we have used $I_{-m}(b_s) = I_m(b_s)$ in ϵ_{xx} and ϵ_{xy}. Note that the components of the dielectric tensor with $m = 0$ are the largest. We do not give the component $\epsilon_{xz} = O(1) \ll \Omega_i/\omega$ because its contribution to the determinant of $n^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon}$ is small. The elements ϵ_{xx} and ϵ_{xy} can be simplified by evaluating (6.22) and its derivative with respect to b_s at $\varphi = 0$ to obtain

$$\exp(b_s) = \sum_{m=-\infty}^{\infty} I_m(b_s) = I_0(b_s) + 2\sum_{m=1}^{\infty} I_m(b_s) \tag{6.48}$$

and

$$\exp(b_s) = \sum_{m=-\infty}^{\infty} I_m'(b_s) = I_0'(b_s) + 2\sum_{m=1}^{\infty} I_m'(b_s). \tag{6.49}$$

Thus, equations (6.43) and (6.46) become

$$\epsilon_{xx} = 1 + \sum_s \frac{\omega_{ps}^2}{\Omega_s^2} \frac{1 - I_0(b_s)\exp(-b_s)}{b_s} = O(1) \tag{6.50}$$

and

$$\epsilon_{xy} = \sum_s \frac{\omega_{ps}^2}{\omega \Omega_s} \left[I_0(b_s) - I_0'(b_s) \right] \exp(-b_s) = O\left(\frac{\Omega_s}{\omega}\right). \tag{6.51}$$

Using equations (6.44), (6.45), (6.47), (6.50) and (6.51), and assuming

$$\frac{k_\parallel c}{\omega} \sim 1 \ll \frac{k_\perp c}{\omega} \sim \frac{\Omega_s}{\omega}, \tag{6.52}$$

the dispersion relation finally becomes

$$[n^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon}] \cdot \tilde{\mathbf{E}} =$$
$$\begin{pmatrix} \epsilon_{xx} - k_\parallel^2 c^2/\omega^2 & \mathrm{i}\epsilon_{xy} & k_\parallel k_\perp c^2/\omega^2 \\ -\mathrm{i}\epsilon_{xy} & \epsilon_{yy} - k_\perp^2 c^2/\omega^2 & \mathrm{i}\epsilon_{yz} \\ k_\parallel k_\perp c^2/\omega^2 & -\mathrm{i}\epsilon_{yz} & \epsilon_{zz} - k_\perp^2 c^2/\omega^2 \end{pmatrix} \begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6.53}$$

This is the simplest linear gyrokinetic model. Note that large factors such as Ω_s/ω or $k_\perp/k_\parallel$ appear in equation (6.53). To eliminate these factors, we normalize the equations. We use the normalized electric field $(\tilde{E}_x, (k_\perp c/\omega)\tilde{E}_y, (k_\perp/k_\parallel)\tilde{E}_z)$, and we multiply the second equation by $(\omega/k_\perp c)$ and the third equation by $k_\parallel/k_\perp$. With these operations, we obtain

$$\begin{pmatrix} D_{xx} - k_\parallel^2 c^2/\omega^2 & \mathrm{i}D_{xy} & k_\parallel^2 c^2/\omega^2 \\ -\mathrm{i}D_{xy} & D_{yy} - 1 & \mathrm{i}D_{yz} \\ k_\parallel^2 c^2/\omega^2 & -\mathrm{i}D_{yz} & D_{zz} - k_\parallel^2 c^2/\omega^2 \end{pmatrix} \begin{pmatrix} \tilde{E}_x \\ (k_\perp c/\omega)\tilde{E}_y \\ (k_\perp/k_\parallel)\tilde{E}_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{6.54}$$

where

$$D_{xx} = \epsilon_{xx} = 1 + \sum_s \frac{c^2 \beta_s}{v_{ts}^2} \frac{1 - I_0(b_s)\exp(-b_s)}{b_s}, \tag{6.55}$$

$$D_{yy} = \frac{\omega^2}{k_\perp^2 c^2} \epsilon_{yy} = \sum_s \beta_s \left[I_0(b_s) - I_0'(b_s) \right] \exp(-b_s) \zeta_{s,0} \mathcal{Z}(\zeta_{s,0}), \tag{6.56}$$

$$D_{zz} = \frac{k_\parallel^2}{k_\perp^2} \epsilon_{zz} = \sum_s \frac{c^2 \beta_s}{v_{ts}^2} \frac{1}{b_s} I_0(b_s) \exp(-b_s) \left[1 + \zeta_{s,0} \mathcal{Z}(\zeta_{s,0}) \right], \tag{6.57}$$

$$D_{xy} = \frac{\omega}{k_\perp c} \epsilon_{xy} = \sum_s \frac{c \beta_s}{v_{ts}} \frac{1}{\sqrt{2 b_s}} \left[I_0(b_s) - I_0'(b_s) \right] \exp(-b_s). \tag{6.58}$$

and

$$D_{yz} = \frac{k_\parallel}{k_\perp} \frac{\omega}{k_\perp c} \epsilon_{yz} = D_{xy} + \sum_s \frac{c \beta_s}{v_{ts}} \frac{1}{\sqrt{2 b_s}} \left[I_0(b_s) - I_0'(b_s) \right] \exp(-b_s) \zeta_{s,0} \mathcal{Z}(\zeta_{s,0}), \tag{6.59}$$

Here $\beta_s = 2\mu_0 n_s T_s / B^2$ is the β parameter for species s.

As an example of the use of these equations, we study kinetic Alfven waves. We first expand in $\sqrt{m_e/m_i} \ll 1$ assuming that $v_{ti} \sim v_A \ll v_{te} \ll c$, i.e. $\beta_i = 2\mu_0 n_i T_i / B^2$ is of order unity. Here $v_A = v_{ti}/\sqrt{\beta_i} = \sqrt{B^2/\mu_0 n_i m_i}$ is the Alfven velocity. We use the ordering

$$b_e \sim \frac{m_e}{m_i} \ll \zeta_{e,0} \sim \sqrt{\frac{m_e}{m_i}} \ll \zeta_{i,0} \sim b_i \sim \beta_i \sim \beta_e \sim \frac{c}{v_{te}} \sim 1 \ll \frac{c}{v_{ti}} \sim \frac{k_\parallel c}{\omega} \sim \sqrt{\frac{m_i}{m_e}}. \tag{6.60}$$

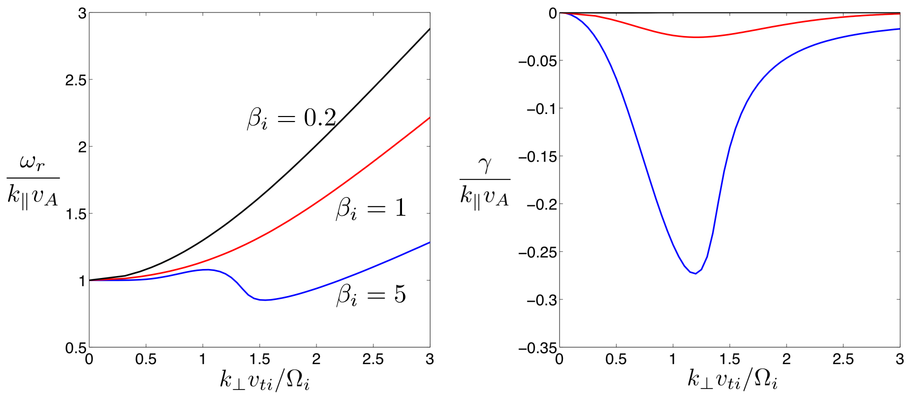

FIGURE 6. Real frequency ω_r and growth rate γ as a function of the perpendicular wavenumber $k_\perp$ for kinetic Alfven waves with $ZT_e/T_i = 1$ and $\beta_i = 0.2$(black), 1(red), 5(blue). The growth rate of the case with $\beta_i = 0.2$ is barely visible.

Then, $\mathcal{Z}(\zeta_{e,0}) \simeq \mathrm{i}\sqrt{\pi}$, $I_0(b_e) \simeq 1$, $[1 - I_0(b_e)\exp(-b_e)]/b_e \simeq 1$ and $I_0'(b_e) \simeq 0$. With these results, equations (6.55)-(6.59) become

$$D_{xx} \simeq \frac{c^2}{v_A^2} \frac{1 - I_0(b_i)\exp(-b_i)}{b_i}, \tag{6.61}$$

$$D_{yy} \simeq \beta_i \left[I_0(b_i) - I_0'(b_i)\right] \exp(-b_i) \zeta_{i,0} \mathcal{Z}(\zeta_{i,0}), \tag{6.62}$$

$$D_{zz} \simeq \frac{c^2}{v_A^2} \frac{1}{b_i} \left[\frac{T_i}{ZT_e} + I_0(b_i)\exp(-b_i)\left(1 + \zeta_{i,0}\mathcal{Z}(\zeta_{i,0})\right) \right], \tag{6.63}$$

$$D_{xy} \simeq \frac{c}{v_A} \sqrt{\frac{\beta_i}{2b_i}} \left[\left(I_0(b_i) - I_0'(b_i)\right)\exp(-b_i) - 1 \right] \tag{6.64}$$

and

$$D_{yz} \simeq D_{xy} + \frac{c}{v_A}\sqrt{\frac{\beta_i}{2b_i}} \left[I_0(b_i) - I_0'(b_i)\right]\exp(-b_i)\zeta_{i,0}\mathcal{Z}(\zeta_{i,0}). \tag{6.65}$$

Substituting these results into (6.54) and setting the determinant of the matrix equal to zero, one obtains the frequency of the kinetic Alfven wave. In figure 6, the complex frequency $\omega = \omega_r + \mathrm{i}\gamma$ of the kinetic Alfven wave is shown as a function of b_i for $ZT_e/T_i = 1$ and several values of β_i.

The dispersion relation (6.61)-(6.65) gives the shear Alfven wave in the limit $b_i \ll 1$ because in this limit, the dispersion relation reduces to $D_{xx} - k_\parallel^2 c^2/\omega^2 = 0$, where $D_{xx} \simeq c^2/v_A^2$. Thus, $\omega = k_\parallel v_A$. Keeping higher order terms in $b_i \ll 1$ gives the small damping of the shear Alfven wave.

In the limit $b_i \gg 1$, the dispersion relation (6.61)-(6.65) becomes

$$D_{xx} \simeq \frac{c^2}{v_A^2}\frac{1}{b_i},\ D_{zz} \simeq \frac{c^2}{v_A^2}\frac{T_i}{ZT_e}\frac{1}{b_i},\ D_{xy} \simeq D_{yz} \simeq -\frac{c}{v_A}\sqrt{\frac{\beta_i}{2b_i}}. \tag{6.66}$$

The coefficient $D_{yy} \sim \beta_i/b_i^{3/2}$ can be neglected for $\beta_i \sim 1$. Thus, the dispersion relation (6.54) becomes

$$\begin{pmatrix} D_{xx} - k_\parallel^2 c^2/\omega^2 & \mathrm{i}D_{xy} & k_\parallel^2 c^2/\omega^2 \\ -\mathrm{i}D_{xy} & -1 & \mathrm{i}D_{xy} \\ k_\parallel^2 c^2/\omega^2 & -\mathrm{i}D_{xy} & D_{zz} - k_\parallel^2 c^2/\omega^2 \end{pmatrix} \begin{pmatrix} \tilde{E}_x \\ (k_\perp c/\omega)\tilde{E}_y \\ (k_\perp/k_\parallel)\tilde{E}_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6.67}$$

Setting the determinant of the matrix equal to zero gives

$$\omega = \sqrt{\frac{1 + ZT_e/T_i}{1 + (1 + ZT_e/T_i)\beta_i/2}} k_\| v_A \sqrt{b_i}. \tag{6.68}$$

The polarization corresponding to this solution is

$$\frac{\tilde{E}_y}{\tilde{E}_x} = \frac{\mathrm{i}k_\|}{k_\perp}\left(1 + \frac{ZT_e}{T_i}\right)^{3/2} \sqrt{\frac{\beta_i}{2 + (1 + ZT_e/T_i)\beta_i}}, \quad \frac{\tilde{E}_z}{\tilde{E}_x} = -\frac{k_\|}{k_\perp}\frac{ZT_e}{T_i}. \tag{6.69}$$

Note that this wave, like the shear Alfven wave for $b_i \ll 1$, is not strongly damped even though we have assumed that $\zeta_{i,0} \sim 1$. Unlike in the shear Alfven wave, this wave induces a parallel electric field and a perturbation to the magnitude of the magnetic field, $\tilde{B} = \tilde{B}_z = k_\perp \tilde{E}_y/\omega$. However, due to finite ion gyroradius effects, the combination of parallel electric field and perturbation to the magnetic field magnitude in (6.69) does not cause damping.

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Appendix A. Bessel functions and modified Bessel functions for small argument

For $\lambda_s \ll 1$,

$$J_m(\lambda_s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi = \sum_{q=0}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{q!}\mathrm{i}^q \lambda_s^q \sin^q\varphi \exp(-\mathrm{i}m\varphi)\,\mathrm{d}\varphi. \tag{A 1}$$

Using

$$\mathrm{i}^q \sin^q\varphi = \frac{1}{2^q}[\exp(\mathrm{i}\varphi) - \exp(-\mathrm{i}\varphi)]^q = \frac{1}{2^q}\sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}\exp(\mathrm{i}(2r-q)\varphi), \tag{A 2}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(\mathrm{i}m\varphi)\,\mathrm{d}\varphi = \delta_{0m}, \tag{A 3}$$

where δ_{ij} is the Kronecker delta, equation (A 1) with $m \geqslant 0$ becomes equation (4.26).

A similar calculation for the modified Bessel functions gives (6.23).

Appendix B. Bessel functions for large argument

For $\lambda_s \gg 1$, the integral in (4.15) is dominated by the values of φ around the maxima and minima of the phase $\lambda_s \sin\varphi$, that is, by the values of φ around $\pi/2$ and $-\pi/2$. To obtain the integral, we follow the stationary phase method (Bender & Orszag 1999). The integrand of (4.15) is

$$\exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi) \simeq \exp\left(\mathrm{i}\lambda_s - \frac{\mathrm{i}m\pi}{2} - \frac{\mathrm{i}\lambda_s(\varphi-\pi/2)^2}{2}\right) \tag{B 1}$$

around $\varphi = \pi/2$ and

$$\exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi) \simeq \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}m\pi}{2} + \frac{\mathrm{i}\lambda_s(\varphi+\pi/2)^2}{2}\right) \tag{B 2}$$

around $\varphi = -\pi/2$. For $|\varphi - \pi/2| \gg 1/\sqrt{\lambda_s}$ and $|\varphi + \pi/2| \gg 1/\sqrt{\lambda_s}$, the integrand is highly oscillatory and it does not contribute much to the integral in (4.15), as we will show below.

For $\lambda_s \gg 1$, we write

$$\begin{aligned}
J_m(\lambda_s) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi \\
&\simeq \frac{1}{2\pi}\int_{-\pi/2-A/\sqrt{\lambda_s}}^{-\pi/2+A/\sqrt{\lambda_s}} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}m\pi}{2} + \frac{\mathrm{i}\lambda_s(\varphi+\pi/2)^2}{2}\right)\mathrm{d}\varphi \\
&+ \frac{1}{2\pi}\int_{\pi/2-A/\sqrt{\lambda_s}}^{\pi/2+A/\sqrt{\lambda_s}} \exp\left(\mathrm{i}\lambda_s - \frac{\mathrm{i}m\pi}{2} - \frac{\mathrm{i}\lambda_s(\varphi-\pi/2)^2}{2}\right)\mathrm{d}\varphi \\
&+ \frac{1}{2\pi}\int_{\mathrm{rest}}\exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi,
\end{aligned} \tag{B 3}$$

where A is a large positive number that satisfies

$$1 \ll A \ll \sqrt{\lambda_s}. \tag{B 4}$$

We will show that the exact value of A is not important. The last integral in (B 3) (the integral over the "rest") is the integral over what is left of the interval $[-\pi, \pi]$ after subtracting the intervals $[-\pi/2 - A/\sqrt{\lambda_s}, -\pi/2 + A/\sqrt{\lambda_s}]$ and $[\pi/2 - A/\sqrt{\lambda_s}, \pi/2 + A/\sqrt{\lambda_s}]$,

$$\int_{\text{rest}} \exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi = \int_{-\pi}^{-\pi/2 - A/\sqrt{\lambda_s}} \exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi \\ + \int_{-\pi/2 + A/\sqrt{\lambda_s}}^{\pi/2 - A/\sqrt{\lambda_s}} \exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi + \int_{\pi/2 + A/\sqrt{\lambda_s}}^{\pi} \exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi. \tag{B 5}$$

We will show at the end of this appendix that these integrals are negligible.

To take the first two integrals in equation (B 3), we use the complex plane. The first integral in (B 3) is equal to the integrals over the paths shown in figure 7(a): C (the straight line through $\varphi = -\pi/2$ at a $\pi/4$ angle with respect to the real axis), $C_{-\infty}$ and C_∞ (the two circumference sectors at large $|\varphi + \pi/2|$). Thus,

$$\frac{1}{2\pi}\int_{-\pi/2 - A/\sqrt{\lambda_s}}^{-\pi/2 + A/\sqrt{\lambda_s}} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}m\pi}{2} + \frac{\mathrm{i}\lambda_s(\varphi + \pi/2)^2}{2}\right)\mathrm{d}\varphi = \\ \frac{1}{2\pi}\int_{C} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}m\pi}{2} + \frac{\mathrm{i}\lambda_s(\varphi + \pi/2)^2}{2}\right)\mathrm{d}\varphi \\ + \frac{1}{2\pi}\int_{C_{-\infty}} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}m\pi}{2} + \frac{\mathrm{i}\lambda_s(\varphi + \pi/2)^2}{2}\right)\mathrm{d}\varphi \\ + \frac{1}{2\pi}\int_{C_{\infty}} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}m\pi}{2} + \frac{\mathrm{i}\lambda_s(\varphi + \pi/2)^2}{2}\right)\mathrm{d}\varphi. \tag{B 6}$$

The integral over C dominates. We take this integral using $\varphi = -\pi/2 + t\sqrt{2/\lambda_s}\exp(\mathrm{i}\pi/4)$

$$\frac{1}{2\pi}\int_{C} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}m\pi}{2} + \frac{\mathrm{i}\lambda_s(\varphi + \pi/2)^2}{2}\right)\mathrm{d}\varphi \\ = \frac{1}{\pi\sqrt{2\lambda_s}} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}m\pi}{2} + \frac{\mathrm{i}\pi}{4}\right)\int_{-A/\sqrt{2}}^{A/\sqrt{2}} \exp(-t^2)\,\mathrm{d}t. \tag{B 7}$$

Since we have chosen $A \gg 1$, we find $\int_{-A/\sqrt{2}}^{A/\sqrt{2}} \exp(-t^2)\,\mathrm{d}t \simeq \int_{-\infty}^{\infty} \exp(-t^2)\,\mathrm{d}t = \sqrt{\pi}$, leading to

$$\frac{1}{2\pi}\int_{C} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}n\pi}{2} + \frac{\mathrm{i}\lambda_s(\varphi + \pi/2)^2}{2}\right)\mathrm{d}\varphi \simeq \frac{1}{\sqrt{2\pi\lambda_s}} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}n\pi}{2} + \frac{\mathrm{i}\pi}{4}\right). \tag{B 8}$$

We proceed to show that the integrals over $C_{-\infty}$ and C_∞ are negligible. For the integral over $C_{-\infty}$ we use $\varphi = -\pi/2 + (A/\sqrt{\lambda_s})\exp(\mathrm{i}(\theta - \pi))$,

$$\frac{1}{2\pi}\int_{C_{-\infty}} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}m\pi}{2} + \frac{\mathrm{i}\lambda_s(\varphi + \pi/2)^2}{2}\right)\mathrm{d}\varphi \\ = \frac{A\mathrm{i}}{2\pi\sqrt{\lambda_s}} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}m\pi}{2}\right)\int_0^{\pi/4} \exp\left(\frac{A^2\mathrm{i}}{2}\exp(2\mathrm{i}\theta) + \mathrm{i}(\theta - \pi)\right)\mathrm{d}\theta. \tag{B 9}$$

Using that in the interval $0 < \theta < \pi/4$,

$$\left|\exp\left(\frac{A^2\mathrm{i}}{2}\exp(2\mathrm{i}\theta) + \mathrm{i}(\theta - \pi)\right)\right| = \exp\left(-\frac{A^2}{2}\sin 2\theta\right) \leqslant \exp\left(-\frac{2A^2}{\pi}\theta\right), \tag{B 10}$$

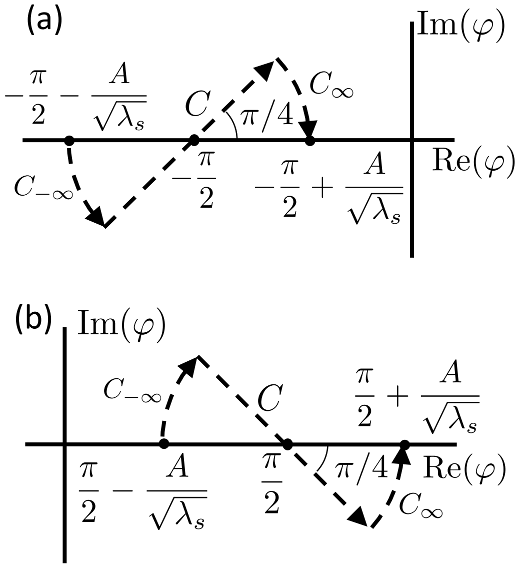

FIGURE 7. Contours in the complex plane used to take the integrals in equation (B 3).

we find

$$\frac{1}{2\pi}\left|\int_{C_{-\infty}} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}n\pi}{2} + \frac{\mathrm{i}\lambda_s(\varphi+\pi/2)^2}{2}\right)\,\mathrm{d}\varphi\right|$$
$$\leqslant \frac{A}{2\pi\sqrt{\lambda_s}}\int_0^{\pi/4}\exp\left(-\frac{2A^2}{\pi}\theta\right)\,\mathrm{d}\theta = O\left(\frac{1}{A\sqrt{\lambda_s}}\right) \ll \frac{1}{\sqrt{\lambda_s}}. \tag{B 11}$$

Thus, the integral over the path $C_{-\infty}$ is negligible compared to (B 8). Using a similar method, we can show that the integral over C_∞ is negligible as well, leaving

$$\frac{1}{2\pi}\int_{-\pi/2-A/\sqrt{\lambda_s}}^{-\pi/2+A/\sqrt{\lambda_s}} \exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}m\pi}{2} + \frac{\mathrm{i}\lambda_s(\varphi+\pi/2)^2}{2}\right)\,\mathrm{d}\varphi$$
$$\simeq \frac{1}{\sqrt{2\pi\lambda_s}}\exp\left(-\mathrm{i}\lambda_s + \frac{\mathrm{i}m\pi}{2} + \frac{\mathrm{i}\pi}{4}\right) \tag{B 12}$$

We can take the second integral in (B 3) using the path shown in figure 7(b). Following the procedure that we used to obtain (B 12), we find

$$\frac{1}{2\pi}\int_{\pi/2-A/\sqrt{\lambda_s}}^{\pi/2+A/\sqrt{\lambda_s}} \exp\left(\mathrm{i}\lambda_s - \frac{\mathrm{i}m\pi}{2} - \frac{\mathrm{i}\lambda_s(\varphi-\pi/2)^2}{2}\right)\,\mathrm{d}\varphi$$
$$\simeq \frac{1}{\sqrt{2\pi\lambda_s}}\exp\left(\mathrm{i}\lambda_s - \frac{\mathrm{i}m\pi}{2} - \frac{\mathrm{i}\pi}{4}\right) \tag{B 13}$$

Adding the integrals in (B 12) and (B 13), we find equation (4.27).

We finish by arguing that the integrals in (B 5) are negligible. We can prove it by integrating by parts. We show the procedure for the first integral in the right side of

(B 5). Integrating by parts this integral, we find

$$\int_{-\pi}^{-\pi/2-A/\sqrt{\lambda_s}} \exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi = -\frac{\mathrm{i}}{\lambda_s}\left[\frac{\exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)}{\cos\varphi}\right]_{-\pi}^{-\pi/2-A/\sqrt{\lambda_s}} + \frac{\mathrm{i}}{\lambda_s}\int_{-\pi}^{-\pi/2-A/\sqrt{\lambda_s}} \exp(\mathrm{i}\lambda_s \sin\varphi)\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{\exp(-\mathrm{i}m\varphi)}{\cos\varphi}\right)\mathrm{d}\varphi. \tag{B 14}$$

In the first term of this equation, the limit $\varphi = -\pi/2 - A/\sqrt{\lambda_s}$ dominates because $\cos\varphi \simeq \varphi + \pi/2$ around $\varphi = -\pi/2$, giving

$$-\frac{\mathrm{i}}{\lambda_s}\left[\frac{\mathrm{i}\exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)}{\cos\varphi}\right]_{-\pi}^{-\pi/2-A/\sqrt{\lambda_s}} = O\left(\frac{1}{A\sqrt{\lambda_s}}\right). \tag{B 15}$$

The second integral in the right side of (B 14) can be bounded. The integrand of the second integral in the right side of (B 14) goes as $1/(\varphi+\pi/2)^2$ for φ near $-\pi/2$. Thus, we will find its maximum value in this region. Taking this into consideration, in the interval $[-\pi, -\pi/2 - A/\sqrt{\lambda_s}]$, there is a constant $K \sim 1$ such that

$$\left|\exp(\mathrm{i}\lambda_s \sin\varphi)\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{\exp(-\mathrm{i}m\varphi)}{\cos\varphi}\right)\right| \leqslant \frac{K}{(\varphi+\pi/2)^2}, \tag{B 16}$$

leading to

$$\left|\frac{\mathrm{i}}{\lambda_s}\int_{-\pi}^{-\pi/2-A/\sqrt{\lambda_s}} \exp(\mathrm{i}\lambda_s \sin\varphi)\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{\exp(-\mathrm{i}m\varphi)}{\cos\varphi}\right)\mathrm{d}\varphi\right| \leqslant \frac{1}{\lambda_s}\int_{-\pi}^{-\pi/2-A/\sqrt{\lambda_s}} \frac{K}{(\varphi+\pi/2)^2}\,\mathrm{d}\varphi = O\left(\frac{1}{A\sqrt{\lambda_s}}\right). \tag{B 17}$$

This bound is not very accurate, and it can be made better by integrating by parts again. However, to prove that the integral is negligible, this bound is sufficient. Estimates (B 15) and (B 17) give

$$\int_{-\pi}^{-\pi/2-A/\sqrt{\lambda_s}} \exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi = O\left(\frac{1}{A\sqrt{\lambda_s}}\right) \ll \frac{1}{\sqrt{\lambda_s}}. \tag{B 18}$$

Thus, this integral is much smaller than the main contribution (4.27). All the integrals in (B 5) are of the same order and hence the integrals in (B 5) are negligible,

$$\int_{\mathrm{rest}} \exp(\mathrm{i}\lambda_s \sin\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi = O\left(\frac{1}{A\sqrt{\lambda_s}}\right) \ll \frac{1}{\sqrt{\lambda_s}}. \tag{B 19}$$

Appendix C. Useful integrals of Bessel functions

In this Appendix, we calculate the integrals in (6.9), (6.10) and (6.11). To calculate these integrals, we will use the auxiliary function

$$F(\xi,\eta) = 2\int_0^\infty J_m(\xi w)J_m(\eta w)w\exp(-w^2)\,\mathrm{d}w = \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}w\int_{-\pi}^{\pi}\mathrm{d}\varphi\int_{-\pi}^{\pi}\mathrm{d}\varphi'\,w \times \exp(-w^2 + \mathrm{i}\xi w\sin\varphi + \mathrm{i}\eta w\sin\varphi' - \mathrm{i}m(\varphi+\varphi')). \tag{C 1}$$

Using the integration variables $w_x = w\cos\varphi$, $w_y = w\sin\varphi$ and $\theta = \varphi + \varphi'$, we find

$$F(\xi,\eta) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\mathrm{d}w_x\int_{-\infty}^{\infty}\mathrm{d}w_y\int_{-\pi}^{\pi}\mathrm{d}\theta \\ \times\exp(-w_x^2 - w_y^2 + \mathrm{i}\eta w_x\sin\theta + \mathrm{i}w_y(\xi - \eta\cos\theta) - \mathrm{i}m\theta). \tag{C 2}$$

Integrating over w_x by completing the square, we obtain

$$F(\xi,\eta) = \frac{1}{2\pi^{3/2}}\int_{-\infty}^{\infty}\mathrm{d}w_y\int_{-\pi}^{\pi}\mathrm{d}\theta\,\exp\left(-\frac{\eta^2}{4}\sin^2\theta - w_y^2 + \mathrm{i}w_y(\xi - \eta\cos\theta) - \mathrm{i}m\theta\right). \tag{C 3}$$

Integrating over w_y gives

$$F(\xi,\eta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(-\frac{\eta^2}{4}\sin^2\theta - \frac{(\xi-\eta\cos\theta)^2}{4} - \mathrm{i}m\theta\right)\mathrm{d}\theta. \tag{C 4}$$

After a few manipulations, this integral becomes

$$F(\xi,\eta) = I_m(\xi\eta/2)\exp\left(-\frac{\xi^2+\eta^2}{4}\right). \tag{C 5}$$

Noting that $2\int_0^\infty J_m^2(w\sqrt{2b_s})w\exp(-w^2)\,\mathrm{d}w = F(\sqrt{2b_s},\sqrt{2b_s})$, we obtain (6.9). Using that

$$4\int_0^\infty J_m(w\sqrt{2b_s})J_m'(w\sqrt{2b_s})w^2\exp(-w^2)\,\mathrm{d}w = 2\left.\frac{\partial F}{\partial\eta}\right|_{\xi=\sqrt{2b_s},\eta=\sqrt{2b_s}}, \tag{C 6}$$

we find (6.10). Finally, employing that

$$4\int_0^\infty [J_m'(w\sqrt{2b_s})]^2 w^3\exp(-w^2)\,\mathrm{d}w = 2\left.\frac{\partial^2 F}{\partial\xi\partial\eta}\right|_{\xi=\sqrt{2b_s},\eta=\sqrt{2b_s}}, \tag{C 7}$$

we obtain

$$4\int_0^\infty [J_m'(w\sqrt{2b_s})]^2 w^3\exp(-w^2)\,\mathrm{d}w = \\ [I_m'(b_s) + b_s\left(I_m''(b_s) - 2I_m'(b_s) + I_m(b_s)\right)]\exp(-b_s). \tag{C 8}$$

Since the modified Bessel functions satisfy the differential equation

$$b_s^2 I_m'' + b_s I_m' - \left(b_s^2 + n^2\right)I_m = 0, \tag{C 9}$$

equation (C 8) can be written as (6.11).

Appendix D. Modified Bessel functions for large argument

For $b_s \gg 1$, the integral

$$I_m(b_s) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(b_s\cos\varphi - \mathrm{i}m\varphi)\,\mathrm{d}\varphi. \tag{D 1}$$

is dominated by a small region around $\varphi = 0$. Indeed, at $\varphi = 0$ the argument of the exponential is maximum and it can be approximated by

$$b_s\cos\varphi - \mathrm{i}m\varphi = b_s - \frac{b_s}{2}\varphi^2 + \frac{b_s}{24}\varphi^4 + O(b_s\varphi^6) + \mathrm{i}m\varphi. \tag{D 2}$$

Thus, since the argument of the exponential $\exp(b_s \cos\varphi - \mathrm{i}m\varphi)$ is very negative for $\varphi \gg 1/\sqrt{b_s}$, the integrand of the integral in (D 1) is only significantly different from zero in an interval of size $1/\sqrt{b_s}$ around $\varphi = 0$.

To calculate the integral, we change to the integration variable $\alpha = \varphi\sqrt{b_s/2}$ and we use the approximation (D 2) to write

$$I_m(b_s) = \frac{\exp(b_s)}{\pi\sqrt{2b_s}} \int_{-\pi\sqrt{b_s/2}}^{\pi\sqrt{b_s/2}} \exp(-\alpha^2) \exp\left(-\sqrt{\frac{2}{b_s}}\mathrm{i}m\alpha + \frac{1}{6b_s}\alpha^4 + O(b_s^{-2})\right) \mathrm{d}\alpha. \quad \text{(D 3)}$$

Since $b_s \gg 1$, we can approximate the the limits of the integral to be $-\infty$ and ∞, and we can Taylor expand one of the exponentials to find

$$\begin{aligned} \exp\left(-\sqrt{\frac{2}{b_s}}\mathrm{i}m\alpha + \frac{1}{6b_s}\alpha^4 + O(b_s^{-2})\right) =& 1 - \sqrt{\frac{2}{b_s}}\mathrm{i}m\alpha - \frac{m^2}{b_s}\alpha^2 + \frac{1}{6b_s}\alpha^4 \\ &+ \frac{\mathrm{i}m^3\sqrt{2}}{3b_s^{3/2}}\alpha^3 + O(b_s^{-2}). \end{aligned} \quad \text{(D 4)}$$

With these approximations, equation (D 3) gives

$$\begin{aligned} I_m(b_s) =& \frac{\exp(b_s)}{\pi\sqrt{2b_s}} \int_{-\infty}^{\infty} \exp(-\alpha^2) \left(1 - \sqrt{\frac{2}{b_s}}\mathrm{i}m\alpha - \frac{m^2}{b_s}\alpha^2 + \frac{1}{6b_s}\alpha^4 \right. \\ &\left. + \frac{\mathrm{i}m^3\sqrt{2}}{3b_s^{3/2}}\alpha^3 + O(b_s^{-2})\right) \mathrm{d}\alpha \\ =& \frac{\exp(b_s)}{\pi\sqrt{2b_s}} \left[\Gamma(1/2) - \frac{6m^2\Gamma(3/2) - \Gamma(5/2)}{6b_s} + O(b_s^{-2})\right], \end{aligned} \quad \text{(D 5)}$$

where $\Gamma(\nu) = \int_0^\infty x^{\nu-1}\exp(-x)\,\mathrm{d}x$ is the Euler Gamma function. Equation (D 5) leads to expression (6.24).