# Kinetic Magneto-Hydro-Dynamics (MHD)

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## 1. Introduction

In these notes, we consider a magnetized plasma ($\rho_{s*} \ll 1$ and $\omega/\Omega_s \ll 1$ for all species $s$) with magnetic energy of the order of the thermal energy,

$$\beta = \frac{2\mu_0 p}{B^2} \sim 1. \tag{1.1}$$

Thus, the plasma has sufficient energy to modify the background magnetic field.

To simplify the derivation as much as possible, we know consider the high flow regime, where

$$\mathbf{E}_\perp \sim v_{ti} B \sim \frac{1}{\rho_{i*}} \frac{T}{eL} \tag{1.2}$$

and

$$E_\| \sim \frac{T}{eL}. \tag{1.3}$$

In this regime, we will obtain a model that is the kinetic equivalent of Magneto-Hydro-Dynamics (MHD).

## 2. Kinetic Magneto-Hydro-Dynamics

To determine the state of the plasma, we need to evolve in time the distribution functions $f_s(\mathbf{r}, \mathbf{v}, t)$ for all species $s$, the magnetic field $\mathbf{B}(\mathbf{r}, t)$ and the electric field $\mathbf{E}(\mathbf{r}, t)$. To evolve the electric field, we split it into its parallel component, $E_\|$, small by a factor of $\rho_{i*} \ll 1$, and $\mathbf{E}_\perp$. The perpendicular component $\mathbf{E}_\perp$ determines the $\mathbf{E} \times \mathbf{B}$ drift,

$$\mathbf{v}_E = \frac{1}{B} \mathbf{E} \times \hat{\mathbf{b}}, \tag{2.1}$$

so instead of $\mathbf{E}_\perp$, we can evolve in time $\mathbf{v}_E$. Thus, we look for equations to determine $f_s$, $\mathbf{B}$, $E_\|$ and $\mathbf{v}_E$. These equations are

- the high flow drift kinetic equation for $f_s$,
- Faraday's induction law for $\mathbf{B}$,
- the quasineutrality equation for $E_\|$, and
- the perpendicular momentum conservation equation for $\mathbf{v}_E$.

### 2.1. *High flow drift kinetic equation*

For $f_s(\mathbf{r}, \mathbf{v}, t)$ we can use the drift kinetic equation. To lowest order in the expansion in $\rho_{s*} \ll 1$, we know that the distribution function is approximately independent of gyrophase, $f_s(\mathbf{r}, v_\|, \mu, \varphi, t) \simeq \langle f_s \rangle_\varphi (\mathbf{r}, v_\|, \mu, t)$. The equation for the gyrophase independent piece of the distribution function is

$$\frac{\partial \langle f_s \rangle_\varphi}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \langle f_s \rangle_\varphi + \left[ \frac{Z_s e}{m_s} \left( \hat{\mathbf{b}} + \frac{1}{\Omega_s} \hat{\mathbf{b}} \times \frac{\mathrm{D}\hat{\mathbf{b}}}{\mathrm{D}t} \right) \cdot \mathbf{E} - \mu \hat{\mathbf{b}} \cdot \nabla B \right] \frac{\partial \langle f_s \rangle_\varphi}{\partial v_\parallel} = 0, \tag{2.2}$$

where we have defined the operator

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla. \tag{2.3}$$

We can rewrite equation (2.2) such that $E_\parallel$ and $\mathbf{v}_E$ appear instead of $\mathbf{E}$,

$$\boxed{\frac{\partial \langle f_s \rangle_\varphi}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \langle f_s \rangle_\varphi + \left( \frac{Z_s e}{m_s} E_\parallel + \frac{\mathrm{D}\hat{\mathbf{b}}}{\mathrm{D}t} \cdot \mathbf{v}_E - \mu \hat{\mathbf{b}} \cdot \nabla B \right) \frac{\partial \langle f_s \rangle_\varphi}{\partial v_\parallel} = 0.} \tag{2.4}$$

We will use this equation instead of (2.2).

The approximation $f_s(\mathbf{r}, v_\parallel, \mu, \varphi, t) \simeq \langle f_s \rangle_\varphi (\mathbf{r}, v_\parallel, \mu, t)$ implies that the average flow is

$$\mathbf{u}_s \simeq u_{s\parallel} \hat{\mathbf{b}} + \mathbf{v}_E, \tag{2.5}$$

where

$$u_{s\parallel} = \frac{1}{n_s} \int f_s v_\parallel \, \mathrm{d}^3 v = \frac{2\pi}{n_s} \int B \langle f_s \rangle_\varphi v_\parallel \, \mathrm{d}v_\parallel \, \mathrm{d}\mu. \tag{2.6}$$

### 2.2. *Faraday's induction law*

Due to the orderings (1.2) and (1.3), we find that

$$\mathbf{E} + \mathbf{v}_E \times \mathbf{B} = E_\parallel \hat{\mathbf{b}} \simeq 0. \tag{2.7}$$

Note that according to (2.5), this equation is equivalent to $\mathbf{E} + \mathbf{u}_s \times \mathbf{B} \simeq 0$, which is one of the MHD equations.

Taking the curl of (2.7) and using Faraday's induction law $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, we obtain

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v}_E \times \mathbf{B}).} \tag{2.8}$$

The initial condition for $\mathbf{B}$, $\mathbf{B}(\mathbf{r}, t = 0) = \mathbf{B}_0(\mathbf{r})$, must satisfy $\nabla \cdot \mathbf{B}_0 = 0$. Equation (2.8) ensures that if $\nabla \cdot \mathbf{B} = 0$ is satisfied at $t = 0$, it is satisfied at all times.

### 2.3. *Quasineutrality equation*

We start from Gauss' law

$$\epsilon_0 \nabla \cdot \mathbf{E} = \sum_s Z_s e n_s. \tag{2.9}$$

Using (1.2), we find that the electric field term is of order

$$\frac{\epsilon_0 \nabla \cdot \mathbf{E}}{e n_e} \sim \frac{v_{ti}}{c\sqrt{\beta}} \frac{\lambda_D}{L}, \tag{2.10}$$

where $\lambda_D = \sqrt{\epsilon_0 T_e / e^2 n_e}$ is the Debye length. Thus, the electric field term is negligible if the Debye length is smaller than $L$ and the velocity $v_{ti}/\sqrt{\beta}$ is not relativistic (we will see that the Alfven speed $v_A \sim v_{ti}/\sqrt{\beta}$ is the speed of propagation of perturbation in the magnetic field). With these assumptions, we finally obtain the quasineutrality equation

$$\boxed{\sum_s Z_s e n_s = \sum_s Z_s e \int 2\pi B \langle f_s \rangle_\varphi \, \mathrm{d}v_\parallel \, \mathrm{d}\mu = 0.} \tag{2.11}$$

### 2.4. *Perpendicular momentum conservation equation*

According to (2.5), the perpendicular component of the average velocity of species $s$ is $\mathbf{v}_E$. Thus, all species move at the same perpendicular velocity $\mathbf{v}_E$. We can calculate that velocity using the perpendicular momentum equation. We multiply the Vlasov equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s + \frac{Z_s e}{m_s}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_v f_s = 0 \tag{2.12}$$

by $m_s\mathbf{v}$ and we integrate over velocity space to find the momentum conservation equation for species $s$,

$$n_s m_s\left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s\cdot\nabla\mathbf{u}_s\right) = -\nabla\cdot\mathbf{P}_s + Z_s e n_s\mathbf{E} + Z_s e n_s\mathbf{u}_s\times\mathbf{B}. \tag{2.13}$$

Here the pressure tensor is

$$\mathbf{P}_s = \int f_s\, m_s(\mathbf{v}-\mathbf{u}_s)(\mathbf{v}-\mathbf{u}_s)\,\mathrm{d}^3v. \tag{2.14}$$

So far, we have not used the assumption $\rho_{s*} \ll 1$. We use it to simplify $\mathbf{P}_s$. Since $\mathbf{u}_s$ is given by (2.5) to lowest order, and $f_s \simeq \langle f_s\rangle_\varphi$, we can write

$$\mathbf{P}_s \simeq \int B\langle f_s\rangle_\varphi\, m_s[(v_\| - u_{s\|})\hat{\mathbf{b}} + \mathbf{w}_\perp][(v_\| - u_{s\|})\hat{\mathbf{b}} + \mathbf{w}_\perp]\,\mathrm{d}v_\|\,\mathrm{d}\mu\,\mathrm{d}\varphi, \tag{2.15}$$

where

$$\mathbf{w}_\perp = \sqrt{2\mu B}(\cos\varphi\,\hat{\mathbf{e}}_1 - \sin\varphi\,\hat{\mathbf{e}}_2). \tag{2.16}$$

Integrating first over $\varphi$, we obtain

$$\mathbf{P}_s \simeq 2\pi\int B\langle f_s\rangle_\varphi\, m_s[(v_\| - u_{s\|})^2\hat{\mathbf{b}}\hat{\mathbf{b}} + \mu B(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}})] = p_{s\|}\hat{\mathbf{b}}\hat{\mathbf{b}} + p_{s\perp}(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}), \tag{2.17}$$

where we have used that $\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 = \mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}$. The quantities $p_{s\|} = \int 2\pi B\langle f_s\rangle_\varphi m_s(v_\| - u_{s\|})^2\,\mathrm{d}v_\|\,\mathrm{d}\mu$ and $p_{s\perp} = \int 2\pi B^2\langle f_s\rangle_\varphi m_s\mu\,\mathrm{d}v_\|\,\mathrm{d}\mu$ are known as parallel and perpendicular pressures.

Using the result for $\mathbf{P}_s$ in (2.17), and employing

$$\begin{aligned}\nabla\cdot\mathbf{P}_s = \nabla\cdot[p_{s\perp}\mathbf{I} + (p_{s\|} - p_{s\perp})\hat{\mathbf{b}}\hat{\mathbf{b}}] = \nabla p_{s\perp} + \hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla(p_{s\|} - p_{s\perp})\\ + (p_{s\|} - p_{s\perp})(\nabla\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} + (p_{s\|} - p_{s\perp})\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}},\end{aligned} \tag{2.18}$$

we find that the parallel and perpendicular components of (2.13) are

$$n_s m_s\left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s\cdot\nabla\mathbf{u}_s\right)\cdot\hat{\mathbf{b}} = -\hat{\mathbf{b}}\cdot\nabla p_{s\|} + (p_{s\perp} - p_{s\|})\nabla\cdot\hat{\mathbf{b}} + Z_s e n_s E_\| \tag{2.19}$$

and

$$n_s m_s\left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s\cdot\nabla\mathbf{u}_s\right)_\perp = -\nabla_\perp p_{s\perp} + (p_{s\perp} - p_{s\|})\boldsymbol{\kappa} + Z_s e n_s\mathbf{E}_\perp + Z_s e n_s\mathbf{u}_s\times\mathbf{B}. \tag{2.20}$$

Equation (2.19) is an identity because it can be deduced from the conservative form of the drift kinetic equation (2.4), as we showed in the notes about the drift kinetic equation. Thus, only (2.20) contains new information. In fact, the sum over all species of (2.20) gives the equation for $\mathbf{v}_E$. Summing over species, we obtain

$$\sum_s n_s m_s\left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s\cdot\nabla\mathbf{u}_s\right)_\perp = \overbrace{\cancel{\sum_s Z_s e n_s}}^{0\text{ due to quasineutrality}}\mathbf{E}_\perp - \nabla_\perp P_\perp + (P_\perp - P_\|)\boldsymbol{\kappa} + \mathbf{J}\times\mathbf{B}, \tag{2.21}$$

where $P_\parallel = \sum_s p_{s\parallel}$, $P_\perp = \sum_s p_{s\perp}$ and $\mathbf{J} = \sum_s Z_s e n_s \mathbf{u}_s$ is the current density. To see that equation (2.21) gives the time evolution of $\mathbf{v}_E$, we use $\mathbf{u}_s \simeq u_{s\parallel}\hat{\mathbf{b}} + \mathbf{v}_E$ and $\hat{\mathbf{b}} \cdot \partial \mathbf{v}_E/\partial t + \mathbf{v}_E \cdot \partial \hat{\mathbf{b}}/\partial t = 0 = \nabla \mathbf{v}_E \cdot \hat{\mathbf{b}} + \nabla \hat{\mathbf{b}} \cdot \mathbf{v}_E$ to write

$$\left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s\right)_\perp = \frac{\partial \mathbf{v}_E}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{v}_E + \mathbf{v}_E \cdot \left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + \mathbf{u}_s \cdot \nabla \hat{\mathbf{b}}\right)\hat{\mathbf{b}} + u_{s\parallel}\left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + \mathbf{u}_s \cdot \nabla \hat{\mathbf{b}}\right), \tag{2.22}$$

Equation (2.21) for the total perpendicular momentum of the plasma can also be obtained using the drift kinetic formalism. Equation (2.21) can be understood as an equation for the perpendicular component of the current density,

$$\mathbf{J}_\perp = \frac{1}{B}\hat{\mathbf{b}} \times \left[\sum_s n_s m_s \left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s\right)\right] + \frac{1}{B}\hat{\mathbf{b}} \times \nabla P_\perp + \frac{P_\parallel - P_\perp}{B}\hat{\mathbf{b}} \times \boldsymbol{\kappa}. \tag{2.23}$$

This same result could be obtained from drift kinetics,

$$\begin{aligned}
\mathbf{J}_\perp &= \sum_s Z_s e \int f_s \mathbf{v}_\perp \, \mathrm{d}^3 v = \sum_s Z_s e \int B f_s (\mathbf{w}_\perp + \mathbf{v}_E) \, \mathrm{d}v_\parallel \, \mathrm{d}\mu \, \mathrm{d}\varphi \\
&= \sum_s Z_s e \int B f_s \mathbf{w}_\perp \, \mathrm{d}v_\parallel \, \mathrm{d}\mu \, \mathrm{d}\varphi + \sum_s \cancelto{0 \text{ due to quasineutrality}}{Z_s e n_s} \mathbf{v}_E \\
&= \sum_s Z_s e \int 2\pi B \langle \tilde{f}_s \mathbf{w}_\perp \rangle_\varphi \, \mathrm{d}v_\parallel \, \mathrm{d}\mu \simeq \sum_s Z_s e \int 2\pi B \langle \tilde{f}_{s,1} \mathbf{w}_\perp \rangle_\varphi \, \mathrm{d}v_\parallel \, \mathrm{d}\mu.
\end{aligned} \tag{2.24}$$

Using the lowest order gyrophase dependent piece $\tilde{f}_{s,1}$, we find (2.23).

We need to close (2.21) by finding an equation for $\mathbf{J}$. We use Ampere's law,

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right). \tag{2.25}$$

Here $\epsilon_0$ is the vacuum permittivity and $\mu_0$ is the vacuum permeability. From Ampere's law we can solve for $\mathbf{J}$, $\mathbf{J} = (\nabla \times \mathbf{B})/\mu_0 - \epsilon_0(\partial \mathbf{E}/\partial t)$. Using this expression for $\mathbf{J}$ in (2.21), and employing the usual manipulation to find Maxwell's stress,

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{B} - \nabla \mathbf{B} \cdot \mathbf{B} = B^2 \boldsymbol{\kappa} - \nabla_\perp \left(\frac{B^2}{2}\right), \tag{2.26}$$

we finally obtain

$$\underbrace{\sum_s n_s m_s \left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s\right)_\perp}_{\sim \frac{p}{L}} = \underbrace{-\nabla_\perp \left(\frac{B^2}{2\mu_0} + P_\perp\right) + \left(\frac{B^2}{\mu_0} + P_\perp - P_\parallel\right)\boldsymbol{\kappa}}_{\sim \frac{B^2}{\mu_0 L} \sim \frac{p}{L}} + \underbrace{\epsilon_0 \mathbf{B} \times \frac{\partial \mathbf{E}}{\partial t}}_{\sim \frac{v_{ti}^2}{c^2} \frac{B^2}{\mu_0 L}}. \tag{2.27}$$

For the order of magnitude estimates, we have used $\beta \sim 1$. Note that for non-relativistic plasmas ($v_{ti}/c \ll 1$), equation (2.27) becomes

$$\boxed{\sum_s n_s m_s \left( \frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right)_\perp = -\nabla_\perp \left( \frac{B^2}{2\mu_0} + P_\perp \right) + \left( \frac{B^2}{\mu_0} + P_\perp - P_\parallel \right) \boldsymbol{\kappa},} \tag{2.28}$$

where the term in the left side of the equation can be evaluated using equation (2.22).

Note that the pressure force due to the plasma takes the same form as the $\mathbf{J}\times\mathbf{B}$ force: part of it is in the direction of a perpendicular gradient, and the other is in the direction of the curvature $\boldsymbol{\kappa}$. In figure 1 we consider a flux tube around a magnetic field line. The flux tube is parallel to the magnetic field in most of its surface, and it is perpendicular to it at its bases. Integrating equation (2.13) over the volume of the flux tube and summing over all species, we find

$$\begin{aligned} \frac{\partial}{\partial t} \left( \sum_s \int_V n_s m_s \mathbf{u}_s \, \mathrm{d}^3 r \right) + \sum_s \int_A n_s m_s (\mathbf{u}_s \cdot \hat{\mathbf{n}}) \mathbf{u}_s \, \mathrm{d}^2 A = \\ - \sum_s \int_A \mathbf{P}_s \cdot \hat{\mathbf{n}} \, \mathrm{d}^2 A + \int_V \mathbf{J} \times \mathbf{B} \, \mathrm{d}^3 r, \end{aligned} \tag{2.29}$$

where $V$ is the volume of the flux tube, $A$ is the area of its boundary surface, and $\hat{\mathbf{n}}$ is the normal to that surface pointing outwards of the flux tube. The left side of equation (2.29) is the change of total momentum in the flux tube, and the right side is the force on the flux tube, $\mathbf{F}$. We use (2.17) and (2.26) to write $\sum_s \mathbf{P}_s = P_\perp \mathbf{I} + (P_\parallel - P_\perp)\hat{\mathbf{b}}\hat{\mathbf{b}}$ and $\mathbf{J} \times \mathbf{B} = \nabla \cdot [(B^2/\mu_0)\hat{\mathbf{b}}\hat{\mathbf{b}} - (B^2/2\mu_0)\mathbf{I}]$. We use these results and the fact that the bases of the flux tube, $A_b$, are perpendicular to the magnetic field, $(\hat{\mathbf{b}}\cdot\hat{\mathbf{n}})\hat{\mathbf{b}} = \hat{\mathbf{n}}$, to rewrite the right side of equation (2.29) as

$$\mathbf{F} = \int_A \Bigg( \underbrace{\frac{B^2}{2\mu_0} + P_\perp}_{\text{Effective pressure}} \Bigg) (-\hat{\mathbf{n}}) \, \mathrm{d}^2 A + \int_{A_b} \Bigg( \underbrace{\frac{B^2}{\mu_0} + P_\perp - P_\parallel}_{\text{Tension}} \Bigg) \hat{\mathbf{n}} \, \mathrm{d}^2 A. \tag{2.30}$$

Thus, we find that the forces are the ones depicted in figure 1: an effective pressure against the whole of the surface of the flux tube, and a tension at the bases. In the limit of the flux tube becoming infinitely short, the tension force becomes the term proportional to $\boldsymbol{\kappa}$ in (2.28).

## 3. Alfven waves and compressional modes in kinetic MHD

Low frequency ($\omega \ll \Omega_s$) perturbations to the magnetic field (generated, for example, by an antenna) propagate as Alfven waves and compressional modes. We study this propagation in a simple system: a uniform, constant plasma composed of an ion species of charge $Ze$ and mass $m_i$ and electrons of charge $-e$ and mass $m_e$ in a uniform, constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The gyroaveraged distribution function of both ions and electrons is assumed to be a Maxwellian,

$$\langle f_s \rangle_\varphi (v_\parallel, \mu, B) = f_{Ms}(v_\parallel, \mu) \equiv n_s \left( \frac{m_s}{2\pi T_s} \right)^{3/2} \exp \left( - \frac{m_s (v_\parallel^2/2 + \mu B)}{T_s} \right), \tag{3.1}$$

where the densities $n_s$ and the temperatures $T_s$ are constants. The densities satisfy quasineutrality, $Zn_i = n_e$. We assume the equilibrium electric field to be negligible, that is, $\mathbf{v}_E = 0$ and $E_\parallel = 0$.

To study the waves that propagate in this system, we first linearize and we then solve

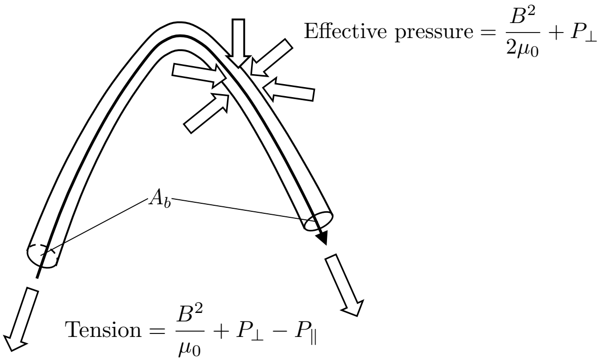


FIGURE 1. Flux tube around a magnetic field line and forces applied to it by a plasma in the kinetic MHD regime. The flux tube is parallel to the magnetic field in most of its surface, and it is perpendicular to it at its bases. Thus, the magnetic flux is conserved within the flux tube, hence its name. The forces applied on the flux tube are a total pressure on all of its surface $A$, and tension at its bases $A_b$.

the resulting linear equations using Fourier analysis in time and space. As a result we will obtain a homogeneous system of equations whose non-trivial solutions are the waves. We will consider two of the solutions: the shear Alfven waves and the compressional modes.

### 3.1. *Linearization of kinetic MHD*

We linearize the kinetic MHD equations by considering infinitesimal perturbations to the distribution function, $f_{Ms} + \delta\langle f_s\rangle_\varphi$, the magnetic field, $\mathbf{B} + \delta\mathbf{B}$, the $\mathbf{E}\times\mathbf{B}$ drift, $\delta\mathbf{v}_E$, and the parallel electric field, $\delta E_\parallel$. To linearize the equations, we need to recall that $\nabla f_{Ms} = 0 = \nabla B = \nabla\hat{\mathbf{b}}$, $u_{s\parallel} = (2\pi/n_s)\int f_{Ms} v_\parallel \, \mathrm{d}v_\parallel \, \mathrm{d}\mu = 0$ and $\delta\mathbf{v}_E \cdot \hat{\mathbf{b}} = 0$. We will also need the perturbations to the magnitude of the magnetic field $B$ and the unit vector $\hat{\mathbf{b}}$. Linearizing $B^2 = \mathbf{B}\cdot\mathbf{B}$, we find $\delta B = \hat{\mathbf{b}}\cdot\delta\mathbf{B} = \delta B_\parallel$, and linearizing $\hat{\mathbf{b}}$, we obtain $\delta\hat{\mathbf{b}} = \delta\mathbf{B}/B - (\delta B/B)\hat{\mathbf{b}} = \delta\mathbf{B}_\perp/B$. With these results, equations (2.4), (2.8), (2.28) and (2.11) give

$$\frac{\partial\delta\langle f_s\rangle_\varphi}{\partial t} + v_\parallel \hat{\mathbf{b}}\cdot\nabla\delta\langle f_s\rangle_\varphi = -\left(\frac{Z_s e}{m_s}\delta E_\parallel - \mu\hat{\mathbf{b}}\cdot\nabla\delta B_\parallel\right)\frac{\partial f_{Ms}}{\partial v_\parallel}, \tag{3.2}$$

$$\frac{\partial\delta\mathbf{B}}{\partial t} = \nabla\times(\delta\mathbf{v}_E\times\mathbf{B}), \tag{3.3}$$

$$\overset{m_e/m_i \ll 1}{(\cancel{n_e m_e} + n_i m_i)}\frac{\partial\delta\mathbf{v}_E}{\partial t} = -\nabla_\perp\left(\frac{B\delta B_\parallel}{\mu_0} + \delta p_{i\perp} + \delta p_{e\perp}\right) + \frac{B^2}{\mu_0}\hat{\mathbf{b}}\cdot\nabla\left(\frac{\delta\mathbf{B}_\perp}{B}\right), \tag{3.4}$$

$$Z\delta n_i = \delta n_e, \tag{3.5}$$

where

$$\delta n_s = \int 2\pi B\delta\langle f_s\rangle_\varphi \, \mathrm{d}v_\parallel \, \mathrm{d}\mu + \int 2\pi\delta B_\parallel f_{Ms} \, \mathrm{d}v_\parallel \, \mathrm{d}\mu, \tag{3.6}$$

$$\delta p_{s\perp} = \int 2\pi B^2\delta\langle f_s\rangle_\varphi m_s\mu \, \mathrm{d}v_\parallel \, \mathrm{d}\mu + \int 4\pi B\delta B_\parallel f_{Ms} m_s\mu \, \mathrm{d}v_\parallel \, \mathrm{d}\mu. \tag{3.7}$$

It is convenient to use the perpendicular plasma displacement $\boldsymbol{\xi}_\perp$ instead of the perturbed $\mathbf{E}\times\mathbf{B}$ drift,

$$\delta\mathbf{v}_E = \frac{\partial\boldsymbol{\xi}_\perp}{\partial t}. \tag{3.8}$$

The vector $\boldsymbol{\xi}_\perp$ indicates how far the plasma has moved in the direction perpendicular

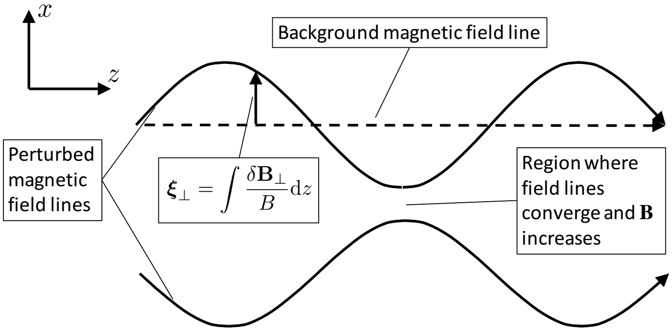


FIGURE 2. Perturbed magnetic field lines and plasma displacement $\boldsymbol{\xi}_\perp$. The perturbed magnetic field lines $\mathbf{r}(l)$ satisfy the equation $\mathrm{d}\mathbf{r}/\mathrm{d}l = \hat{\mathbf{b}}$, where $l$ is the magnetic field line arc length. Thus, the perturbed magnetic field lines $\mathbf{r}(z) = z\hat{\mathbf{z}} + \delta\mathbf{r}(z)$ must satisfy $\mathrm{d}\mathbf{r}/\mathrm{d}l = \hat{\mathbf{z}} + \delta\mathbf{B}_\perp/B$. This equation gives $z = l$ and $\mathrm{d}\delta\mathbf{r}/\mathrm{d}z = \delta\mathbf{B}_\perp/B$. The equation for $\delta\mathbf{r}$ is the same as equation (3.10) for $\boldsymbol{\xi}_\perp$. Thus, $\delta\mathbf{r} = \boldsymbol{\xi}_\perp$, proving that magnetic field lines follow the plasma displacement.

to $\mathbf{B}$. It is a very useful quantity in fluid MHD. The plasma displacement simplifies the linearized induction equation (3.3),

$$\frac{\partial \delta\mathbf{B}}{\partial t} = \frac{\partial}{\partial t}[\nabla \times (\boldsymbol{\xi}_\perp \times \mathbf{B})] = \frac{\partial}{\partial t}[\mathbf{B}\cdot\nabla\boldsymbol{\xi}_\perp - (\nabla\cdot\boldsymbol{\xi}_\perp)\mathbf{B} - \boldsymbol{\xi}_\perp \cdot \cancelto{0}{\nabla\mathbf{B}}]. \tag{3.9}$$

Integrating in time, this equation gives the components of $\delta\mathbf{B}$ parallel and perpendicular to the background magnetic field $\mathbf{B}$ as functions of $\boldsymbol{\xi}_\perp$,

$$\delta\mathbf{B}_\perp = \mathbf{B}\cdot\nabla\boldsymbol{\xi}_\perp = B\frac{\partial \boldsymbol{\xi}_\perp}{\partial z}, \tag{3.10}$$

$$\delta B_\parallel = -B(\nabla\cdot\boldsymbol{\xi}_\perp). \tag{3.11}$$

The magnetic field lines move with the plasma displacement $\boldsymbol{\xi}_\perp$, as shown in figure 2. Wherever the perturbed field lines converge, the magnitude $B$ must increase due to $\nabla\cdot\mathbf{B} = 0$. Similarly, where they diverge, $B$ decreases. This effect is the reason for $\delta B_\parallel \propto -\nabla\cdot\boldsymbol{\xi}_\perp$.

### 3.2. *Fourier analysis of the linearized equations*

To obtain an analytical solution, we can Fourier analyze in time and space,

$$\begin{aligned} \delta\langle f_s\rangle_\varphi(x, z, v_\parallel, \mu, t) &= \tilde{g}_s(v_\parallel, \mu)\exp(-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\cdot\mathbf{r}), \\ \boldsymbol{\xi}_\perp(x, z, t) &= \tilde{\boldsymbol{\xi}}_\perp \exp(-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\cdot\mathbf{r}), \\ \delta E_\parallel(x, z, t) &= \tilde{E}_\parallel \exp(-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\cdot\mathbf{r}). \end{aligned} \tag{3.12}$$

Using equations (3.1), (3.10), (3.11) and (3.12), equations (3.2), (3.4) and (3.5) become

$$(-\mathrm{i}\omega + \mathrm{i}k_\parallel v_\parallel)\tilde{g}_s = \left(\frac{Z_s e v_\parallel \tilde{E}_\parallel}{T_s} - \frac{k_\parallel v_\parallel m_s \mu B}{T_s}\mathbf{k}_\perp\cdot\tilde{\boldsymbol{\xi}}_\perp\right) f_{Ms}, \tag{3.13}$$

$$-n_i m_i \omega^2 \tilde{\boldsymbol{\xi}}_\perp = -\mathrm{i}\mathbf{k}_\perp(\tilde{p}_{i\perp} + \tilde{p}_{e\perp}) - \frac{B^2}{\mu_0}(\mathbf{k}_\perp\cdot\tilde{\boldsymbol{\xi}}_\perp)\mathbf{k}_\perp - \frac{k_\parallel^2 B^2}{\mu_0}\tilde{\boldsymbol{\xi}}_\perp, \tag{3.14}$$

$$Z\tilde{n}_i = \tilde{n}_e, \tag{3.15}$$

where

$$\tilde{n}_s = \int 2\pi B \tilde{g}_s \, \mathrm{d}v_\| \, \mathrm{d}\mu - \mathrm{i}(\mathbf{k}_\perp \cdot \tilde{\boldsymbol{\xi}}_\perp) n_s, \tag{3.16}$$

$$\tilde{p}_{s\perp} = \int 2\pi B^2 \tilde{g}_s m_s \mu \, \mathrm{d}v_\| \, \mathrm{d}\mu - 2\mathrm{i}(\mathbf{k}_\perp \cdot \tilde{\boldsymbol{\xi}}_\perp) n_s T_s. \tag{3.17}$$

To find non-trivial solutions, we first solve for $\tilde{g}_s$ using equation (3.13),

$$\tilde{g}_s = -\frac{\mathrm{i}k_\| v_\|}{k_\| v_\| - \omega} \left( \frac{Z_s e \tilde{E}_\|}{k_\| T_s} - \frac{m_s \mu B}{T_s} \mathbf{k}_\perp \cdot \tilde{\boldsymbol{\xi}}_\perp \right) f_{Ms}. \tag{3.18}$$

The distribution function $\tilde{g}_s$ is then used to calculate the ion and electron densities, and the ion and electron perpendicular pressures. The part of the integral over $\mu$ is straightforward. For the integral over $v_\|$, we change to the variable $u = (k_\|/|k_\||)(v_\|/v_{ts})$, where $v_{ts} = \sqrt{2T_s/m_s}$, and we recall that we need to use the Landau contour for the integral over $u$ (Schekochihin 2015). Using the plasma dispersion function

$$\mathcal{Z}(\zeta_s) = \frac{1}{\sqrt{\pi}} \int_{C_L} \frac{\exp(-u^2)}{u - \zeta_s} \mathrm{d}u \tag{3.19}$$

and

$$\frac{1}{\sqrt{\pi}} \int_{C_L} \frac{u \exp(-u^2)}{u - \zeta_s} \mathrm{d}u = 1 + \zeta_s \mathcal{Z}(\zeta_s), \tag{3.20}$$

the perturbed densities and pressures become

$$\tilde{n}_s = -\mathrm{i}[1 + \zeta_s \mathcal{Z}(\zeta_s)] \frac{Z_s e \tilde{E}_\|}{k_\| T_s} n_s + \mathrm{i}\zeta_s \mathcal{Z}(\zeta_s)(\mathbf{k}_\perp \cdot \tilde{\boldsymbol{\xi}}_\perp) n_s, \tag{3.21}$$

$$\tilde{p}_{s\perp} = -\mathrm{i}[1 + \zeta_s \mathcal{Z}(\zeta_s)] \frac{Z_s e \tilde{E}_\|}{k_\| T_s} n_s T_s + 2\mathrm{i}\zeta_s \mathcal{Z}(\zeta_s)(\mathbf{k}_\perp \cdot \tilde{\boldsymbol{\xi}}_\perp) n_s T_s. \tag{3.22}$$

Using (3.21) and (3.22) in (3.14) and (3.15), we find

$$\begin{aligned} &\left( \frac{k_\|^2 B^2}{\mu_0 n_e T_e} - \frac{m_i \omega^2}{Z T_e} \right) \tilde{\boldsymbol{\xi}}_\perp + [\zeta_i \mathcal{Z}(\zeta_i) - \zeta_e \mathcal{Z}(\zeta_e)] \frac{e \tilde{E}_\|}{k_\| T_e} \mathbf{k}_\perp \\ &\qquad + \left[ \frac{B^2}{\mu_0 n_e T_e} - \frac{2T_i}{Z T_e} \zeta_i \mathcal{Z}(\zeta_i) - 2\zeta_e \mathcal{Z}(\zeta_e) \right] (\mathbf{k}_\perp \cdot \tilde{\boldsymbol{\xi}}_\perp) \mathbf{k}_\perp = 0, \end{aligned} \tag{3.23}$$

$$\left\{ \frac{Z T_e}{T_i} [1 + \zeta_i \mathcal{Z}(\zeta_i)] + 1 + \zeta_e \mathcal{Z}(\zeta_e) \right\} \frac{e \tilde{E}_\|}{k_\| T_e} + [\zeta_e \mathcal{Z}(\zeta_e) - \zeta_i \mathcal{Z}(\zeta_i)] \mathbf{k}_\perp \cdot \tilde{\boldsymbol{\xi}}_\perp = 0. \tag{3.24}$$

The solutions to equations (3.23) and (3.24) become clearer if we project the vector equation (3.23) onto the directions $\mathbf{k}_\perp \times \hat{\mathbf{b}}$ and $\mathbf{k}_\perp$. Then, we find

$$\begin{pmatrix} W_{11} & 0 & 0 \\ 0 & W_{22} & W_{23} \\ 0 & -W_{23} & W_{33} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{\xi}}_\perp \cdot (\mathbf{k}_\perp \times \hat{\mathbf{b}}) \\ \mathbf{k}_\perp \cdot \tilde{\boldsymbol{\xi}}_\perp \\ e\tilde{E}_\| / k_\| T_e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.25}$$

where

$$W_{11} = \frac{B^2}{\mu_0 n_e T_e} - \frac{m_i \omega^2}{Z T_e k_\parallel^2}, \tag{3.26}$$

$$W_{22} = \frac{B^2}{\mu_0 n_e T_e}\frac{k^2}{k_\perp^2} - \frac{m_i \omega^2}{Z T_e k_\perp^2} - \frac{2T_i}{ZT_e}\zeta_i \mathcal{Z}(\zeta_i) - 2\zeta_e \mathcal{Z}(\zeta_e), \tag{3.27}$$

$$W_{23} = \zeta_i \mathcal{Z}(\zeta_i) - \zeta_e \mathcal{Z}(\zeta_e), \tag{3.28}$$

$$W_{33} = \frac{ZT_e}{T_i}[1 + \zeta_i \mathcal{Z}(\zeta_i)] + 1 + \zeta_e \mathcal{Z}(\zeta_e). \tag{3.29}$$

We proceed to find solutions to (3.25).

## 3.3. *Shear Alfven waves*

One obvious solution to (3.25) is $\mathbf{k}_\perp \cdot \tilde{\boldsymbol{\xi}}_\perp = 0 = \tilde{E}_\parallel$. Then, the wave has to satisfy $W_{11} = 0$, giving

$$\omega = k_\parallel v_A, \tag{3.30}$$

where

$$v_A = \sqrt{\frac{B^2}{\mu_0 n_i m_i}} \tag{3.31}$$

is the Alfven speed. This solution is known as the shear Alfven wave. One of its striking features is that it is not damped. The reason for this lack of damping is that the Alfven wave does not induce a parallel electric field or a perturbation to the magnitude of the magnetic field, $\tilde{B}_\parallel = -\mathrm{i}B(\mathbf{k}_\perp \cdot \tilde{\boldsymbol{\xi}}_\perp) = 0$.

## 3.4. *Compressional modes*

The other solutions to equation (3.25) satisfy $\tilde{\boldsymbol{\xi}}_\perp \cdot (\mathbf{k}_\perp \times \hat{\mathbf{b}}) = 0$. The equations for these mode are

$$\begin{pmatrix} W_{22} & W_{23} \\ -W_{23} & W_{33} \end{pmatrix} \begin{pmatrix} \mathbf{k}_\perp \cdot \tilde{\boldsymbol{\xi}}_\perp \\ e\tilde{E}_\parallel / k_\parallel T_e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.32}$$

Since the matrix must be singular, the frequency is determined by setting the determinant of the matrix to zero, that is,

$$W_{22} W_{33} + W_{23}^2 = 0. \tag{3.33}$$

The least damped solutions of equation (3.33) are known with various names. Here we call them non-propagating modes and magnetosonic waves.

We can solve the dispersion relation (3.33) numerically. In figure 3, for a hydrogen plasma ($Z = 1$, $m_i/m_e = 1836$) with $T_e = T_i$, we plot the complex frequency $\omega = \omega_r + \mathrm{i}\gamma$ of non-propagating modes and magnetosonic waves as a function of $\beta_i = 2\mu_0 n_i T_i / B^2$ for several values of $k_\parallel / k_\perp$. The real frequency of the non-propagating mode is zero, that is, the fluctuations due to this mode damp without oscillating. These modes are non-propagating because their phase velocity $\omega_r / k$ is zero. The magnetosonic wave is a damped wave that is the equivalent of a sound wave in hydrodynamics. In kinetic MHD, sound waves are due to compressions and expansions of both the plasma and the magnetic field. Note that the damping rate of the non-propagating modes and the magnetosonic waves increases with the size of $k_\parallel$.

We proceed to obtain analytical solutions for both modes in some interesting limits.

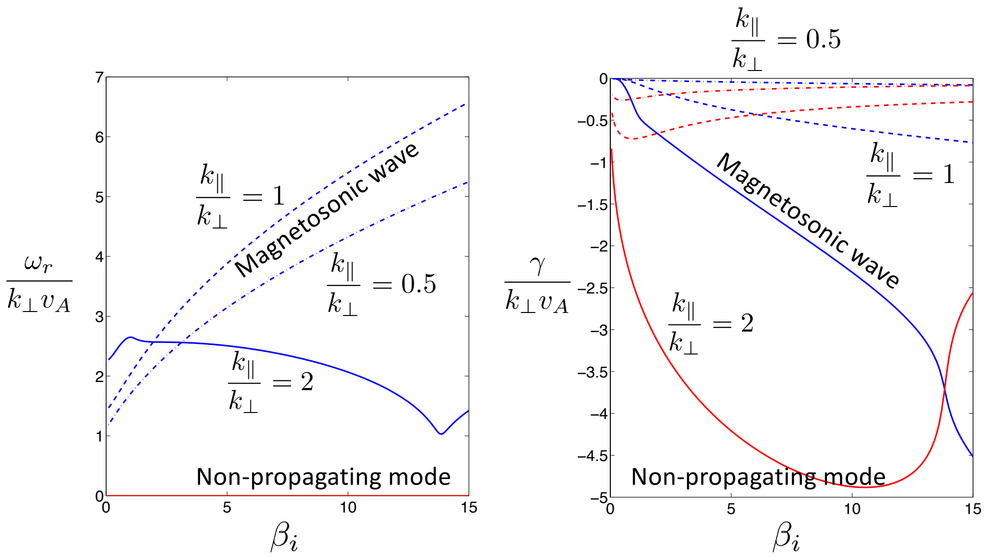


FIGURE 3. Real frequency $\omega_r$ and growth rate $\gamma$ for non-propagating modes (red) and magnetosonic waves (blue lines) with $k_\parallel/k_\perp = 1/2$(dash-dot lines), $1$(dashed lines), $2$(solid lines) as a function of $\beta_i$ in a a hydrogen plasma ($Z = 1$, $m_i/m_e = 1836$) with $T_e = T_i$.

### 3.4.1. *Non-propagating modes*

The non-propagating modes are solutions with purely imaginary frequency, $\omega = \mathrm{i}\gamma$. These modes are solutions of the dispersion relation (3.33) because the plasma dispersion function $\mathcal{Z}(\zeta_s)$ evaluated for a purely imaginary $\zeta_s$ is purely imaginary, and as a result, $\zeta_s \mathcal{Z}(\zeta_s)$ is purely real, leading to a purely real dispersion relation for the growth rate $\gamma$.

It is possible to find analytical solutions for the non-propagating modes in the limits $\beta \gg 1$ and $\beta \ll 1$. We start with $\beta \gg 1$. In this limit, and assuming that $T_i \sim T_e$ and $k_\parallel \sim k_\perp$, we look for waves that satisfy

$$\zeta_e \sim \sqrt{\frac{m_e}{m_i}}\zeta_i \ll \zeta_i \sim \frac{1}{\beta} \ll 1. \tag{3.34}$$

The plasma dispersion function in the limit $|\zeta_s| \ll 1$ is (see Appendix A)

$$\mathcal{Z}(\zeta_s) = -2\zeta_s + \frac{4}{3}\zeta_s^3 + \ldots + \mathrm{i}\sqrt{\pi}\left(1 - \zeta_s^2 + \frac{\zeta_s^4}{2} + \ldots\right). \tag{3.35}$$

Then, to lowest order, the coefficients that enter in equation (3.33) become

$$W_{22} = \underbrace{\frac{B^2}{\mu_0 n_e T_e}\frac{k^2}{k_\perp^2}}_{\sim\beta^{-1}\sim\zeta_i\ll 1} - \underbrace{\cancelto{\text{small}}{\frac{m_i\omega^2}{ZT_e k_\perp^2}}}_{\sim\zeta_i^2\ll 1} - \frac{2T_i}{ZT_e}\mathrm{i}\zeta_i\sqrt{\pi}, \tag{3.36}$$

$$W_{23} = \mathrm{i}\zeta_i\sqrt{\pi}, \tag{3.37}$$

$$W_{33} = \frac{ZT_e}{T_i} + 1. \tag{3.38}$$

With these results, equation (3.33) becomes to lowest order $W_{22} \simeq 0$, that is,

$$\omega \simeq -\frac{\mathrm{i}}{\sqrt{\pi}} \frac{B^2}{2\mu_0 n_i T_i} \frac{k^2}{k_\perp^2} |k_\parallel| v_{ti}. \tag{3.39}$$

This result satisfies the condition $\zeta_i \sim \beta^{-1} \ll 1$ that we assumed at the start of the calculation. The damping arises from the magnetic bottling force, represented in this calculation by the ion perpendicular pressure, given by equation (3.22). The perturbations in the magnetic field strength, $\delta B_\parallel$, cause a magnetic bottling force $-m_i \mu \hat{\mathbf{b}} \cdot \nabla B$ that acts as a parallel electric field, and like the parallel electric field in electrostatic waves, it can lead to Landau damping. When it is caused by the magnetic bottling force, it is called <u>Barnes damping</u>.

For $\beta \ll 1$, and assuming that $T_i \sim T_e$ and $k_\parallel \sim k_\perp$, we search for solutions of the form $\omega = \mathrm{i}\gamma$, where $\gamma$ is a negative number that satisfies $|\gamma|/|k_\parallel| v_{ti} \gg 1$. Then, $\zeta_i = \omega/|k_\parallel| v_{ti}$ is a very large imaginary number and the largest contribution to the plasma dispersion function is the exponentially large piece

$$\mathcal{Z}(\zeta_i) \simeq 2\mathrm{i}\sqrt{\pi} \exp(-\zeta_i^2) = 2\mathrm{i}\sqrt{\pi} \exp\left(\frac{\gamma^2}{k_\parallel^2 v_{ti}^2}\right). \tag{3.40}$$

The electron contribution is smaller because $\zeta_e \sim \zeta_i \sqrt{m_e/m_i} \ll \zeta_i$. Using these results, the coefficients that enter in equation (3.33) can be approximated by

$$W_{22} = \frac{B^2}{\mu_0 n_e T_e} \frac{k^2}{k_\perp^2} + \underbrace{\cancelto{\text{small}}{\frac{m_i \gamma^2}{Z T_e k_\perp^2}}}_{\sim \frac{\gamma^2}{k_\parallel^2 v_{ti}^2} \ll \exp\left(\frac{\gamma^2}{k_\parallel^2 v_{ti}^2}\right)} + \frac{4\sqrt{\pi} T_i}{Z T_e} \frac{\gamma}{|k_\parallel| v_{ti}} \exp\left(\frac{\gamma^2}{k_\parallel^2 v_{ti}^2}\right), \tag{3.41}$$

$$W_{23} = \frac{T_i}{Z T_e} W_{33} = -\frac{2\sqrt{\pi}\gamma}{|k_\parallel| v_{ti}} \exp\left(\frac{\gamma^2}{k_\parallel^2 v_{ti}^2}\right). \tag{3.42}$$

Then, equation (3.33) becomes $W_{22} + (T_i/Z T_e) W_{23} \simeq 0$,

$$\frac{|\gamma|}{|k_\parallel| v_{ti}} \exp\left(\frac{\gamma^2}{k_\parallel^2 v_{ti}^2}\right) = \frac{1}{\sqrt{\pi}} \frac{B^2}{2\mu_0 n_i T_i} \frac{k^2}{k_\perp^2}. \tag{3.43}$$

Solving this equation, we find

$$\omega = \mathrm{i}\gamma \simeq -\mathrm{i} \sqrt{\ln\left(\frac{k^2}{k_\perp^2 \sqrt{\pi} \beta_i} \frac{1}{\sqrt{\ln(k^2/k_\perp^2 \sqrt{\pi} \beta_i)}}\right)} |k_\parallel| v_{ti}. \tag{3.44}$$

Thus, our assumption $|\gamma|/|k_\parallel| v_{ti} \gg 1$ is satisfied because $|\gamma|/|k_\parallel| v_{ti} \sim \sqrt{\ln(1/\beta)}$.

### 3.4.2. *Magnetosonic waves*

To obtain an analytical solution for the magnetosonic waves, we consider the limit

$$\sqrt{\frac{m_i}{m_e}} \sqrt{\beta} \frac{k_\parallel}{k_\perp} \ll 1 \lesssim \frac{1}{\beta}. \tag{3.45}$$

Note that this limit could be achieved in systems with either $\beta \ll 1$, $k_\parallel/k_\perp \ll 1$ or both at the same time.

The magnetosonic waves will have a frequency $\omega \sim k_\perp v_A$. Thus, assuming $T_i \sim T_e$,

$$1 \ll \zeta_e \sim \sqrt{\frac{m_e}{m_i}} \frac{1}{\sqrt{\beta}} \frac{k_\perp}{k_\parallel} \sim \sqrt{\frac{m_e}{m_i}} \zeta_i \ll \zeta_i \sim \frac{1}{\sqrt{\beta}} \frac{k_\perp}{k_\parallel}. \tag{3.46}$$

We will find that the imaginary part of $\omega$ is much smaller than the real part. Taking this into account and using $|\zeta_s| \gg 1$, the plasma dispersion function becomes (see Appendix B)

$$\mathcal{Z}(\zeta_s) = -\frac{1}{\zeta_s} - \frac{1}{2\zeta_s^3} - \frac{3}{4\zeta_s^5} + \ldots + \mathrm{i}\sqrt{\pi} \exp(-\zeta_s^2). \tag{3.47}$$

Using this expansion, the coefficients that enter in equation (3.33) become

$$W_{22} \simeq \underbrace{\frac{B^2}{\mu_0 n_e T_e} \frac{k^2}{k_\perp^2} + \frac{2T_i}{ZT_e} + 2 - \frac{m_i \omega^2}{ZT_e k_\perp^2}}_{\sim \beta^{-1} \gtrsim 1} + \frac{1}{\zeta_e^2} - 2\mathrm{i}\sqrt{\pi} \zeta_e \exp(-\zeta_e^2), \tag{3.48}$$

$$W_{23} \simeq -W_{33} \simeq \frac{1}{2\zeta_e^2} - \mathrm{i}\sqrt{\pi} \zeta_e \exp(-\zeta_e^2). \tag{3.49}$$

With these results, equation (3.33) becomes to lowest order $W_{22} - W_{23} \simeq 0$,

$$\underbrace{\frac{B^2}{\mu_0 n_e T_e} \frac{k^2}{k_\perp^2}}_{\sim \beta^{-1} \gtrsim 1} + \frac{2T_i}{ZT_e} + 2 - \underbrace{\frac{m_i \omega^2}{ZT_e k_\perp^2}}_{\sim \beta^{-1} \gtrsim 1} + \frac{1}{2\zeta_e^2} - \mathrm{i}\sqrt{\pi} \zeta_e \exp(-\zeta_e^2) = 0 \tag{3.50}$$

This equation can be solved order by order in $\zeta_e^{-1} \ll 1$. The frequency is of the form $\omega \simeq \omega^{(0)} + \omega^{(1)} + \mathrm{i}\gamma$, where $\omega^{(1)}/\omega^{(0)} \sim \zeta_e^{-2} \ll 1$ and $\gamma/\omega^{(0)} \sim \exp(-\zeta_e^2) \lll 1$. To lowest order, we find

$$(\omega^{(0)})^2 = k^2 v_A^2 + \frac{2k_\perp^2 (T_i + ZT_e)}{m_i} = k^2 v_A^2 \left[ 1 + \frac{k_\perp^2}{k^2} (\beta_i + \beta_e) \right], \tag{3.51}$$

where $\beta_e = 2\mu_0 n_e T_e / B^2$. To next order, neglecting the exponentially small imaginary piece, we find

$$\frac{\omega^{(1)}}{\omega^{(0)}} = \frac{ZT_e k_\perp^2}{4m_i (\omega^{(0)})^2 (\zeta_e^{(0)})^2} = \frac{1}{8} \underbrace{\frac{k_\perp^2 k_\parallel^2}{[k^2 + k_\perp^2 (\beta_i + \beta_e)]^2} \frac{m_i}{m_e} \frac{\beta_e^2}{Z}}_{\sim \frac{k_\parallel^2}{k_\perp^2} \frac{m_i}{m_e} \beta^2 \lesssim \frac{k_\parallel^2}{k_\perp^2} \frac{m_i}{m_e} \beta \ll 1}, \tag{3.52}$$

where $\zeta_e^{(0)} = \omega^{(0)} / |k_\parallel| v_{te}$. Finally, the imaginary part is given by

$$\gamma = -\frac{\sqrt{\pi} ZT_e k_\perp^2}{2m_i \omega^{(0)}} \zeta_e^{(0)} \exp(-\zeta_e^2) \simeq -\frac{\sqrt{\pi}}{4} k v_A \sqrt{\frac{m_e}{m_i}} \frac{k_\perp^2}{k |k_\parallel|} \sqrt{Z\beta_e}$$
$$\times \exp\left( \underbrace{-\frac{\beta_e}{4} \frac{k_\perp^2}{k^2 + k_\perp^2 (\beta_i + \beta_e)}}_{\sim \beta \lesssim 1} - \underbrace{\frac{k^2 + k_\perp^2 (\beta_i + \beta_e)}{k_\parallel^2} \frac{m_e}{m_i} \frac{1}{\beta_e}}_{\sim \frac{k_\perp^2}{k_\parallel^2} \frac{m_e}{m_i} \beta^{-1} \gg 1} \right). \tag{3.53}$$

Note that we had to keep the higher order correction to $\omega$ given in (3.52) inside the exponential because it contributes a number of order unity to the damping rate.

## REFERENCES

Schekochihin, A.A. 2015 Lecture notes on Kinetic Theory and Magnetohydrodynamics of Plasmas. http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/KT/.

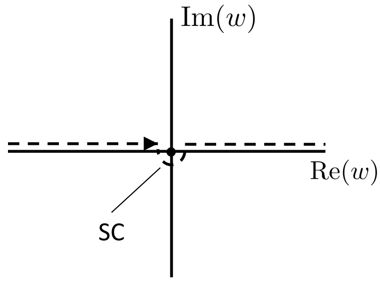


FIGURE 4. Landau contour for the integral (A 1).

## Appendix A. Plasma dispersion function in the limit $|\zeta_s| \ll 1$

For the limit $|\zeta_s| \ll 1$, we change the integration variable to $w = u - \zeta_s$. Then,

$$\mathcal{Z}(\zeta_s) = \frac{1}{\sqrt{\pi}} \int_{C_L} \frac{\exp(-u^2)}{u - \zeta_s} \, \mathrm{d}u = \frac{\exp(-\zeta_s^2)}{\sqrt{\pi}} \int_{C_L} \frac{\exp(-w^2 - 2\zeta_s w)}{w} \, \mathrm{d}w. \tag{A 1}$$

The pole for the integral in $w$ is at the origin, and for it we have to use the Landau contour shown in figure 4 with a semi-circunference $SC$ around $w = 0$. This contour gives

$$\mathcal{Z}(\zeta_s) = \frac{\exp(-\zeta_s^2)}{\sqrt{\pi}} \int_{SC} \frac{\exp(-w^2 - 2\zeta_s w)}{w} \, \mathrm{d}w + \frac{\exp(-\zeta_s^2)}{\sqrt{\pi}} \int_{-\infty}^{-r} \frac{\exp(-w^2 - 2\zeta_s w)}{w} \, \mathrm{d}w + \frac{\exp(-\zeta_s^2)}{\sqrt{\pi}} \int_{r}^{\infty} \frac{\exp(-w^2 - 2\zeta_s w)}{w} \, \mathrm{d}w. \tag{A 2}$$

The semi-circumference $SC$ is described by $w = r\exp(\mathrm{i}\theta)$. The parameter $\theta$ goes from $-\pi$ to 0 (counter-clockwise). Then, using that $\mathrm{d}w = \mathrm{i}r\exp(\mathrm{i}\theta)\,\mathrm{d}\theta$,

$$\frac{\exp(-\zeta_s^2)}{\sqrt{\pi}} \int_{SC} \frac{\exp(-w^2 - 2\zeta_s w)}{w} \, \mathrm{d}w = \frac{\mathrm{i}\exp(-\zeta_s^2)}{\sqrt{\pi}} \int_{-\pi}^{0} \exp(-r^2\exp(2\mathrm{i}\theta) - 2\zeta_s r\exp(\mathrm{i}\theta)) \, \mathrm{d}\theta. \tag{A 3}$$

We can take the limit $r \to 0$, leading to

$$\frac{\exp(-\zeta_s^2)}{\sqrt{\pi}} \int_{SC} \frac{\exp(-w^2 - 2\zeta_s w)}{w} \, \mathrm{d}w = \mathrm{i}\sqrt{\pi}\exp(-\zeta_s^2) = \mathrm{i}\sqrt{\pi}\left(1 - \zeta_s^2 + \frac{\zeta_s^4}{2} + \ldots\right). \tag{A 4}$$

For the rest of the integral in (A 2), we have

$$\lim_{r\to 0}\left(\frac{\exp(-\zeta_s^2)}{\sqrt{\pi}} \int_{-\infty}^{-r} \frac{\exp(-w^2 - 2\zeta_s w)}{w} \, \mathrm{d}w + \frac{\exp(-\zeta_s^2)}{\sqrt{\pi}} \int_{r}^{\infty} \frac{\exp(-w^2 - 2\zeta_s w)}{w} \, \mathrm{d}w\right) = \frac{\exp(-\zeta_s^2)}{\sqrt{\pi}} \mathrm{PV} \int_{-\infty}^{\infty} \frac{\exp(-w^2 - 2\zeta_s w)}{w} \, \mathrm{d}w, \tag{A 5}$$

where PV indicates that we have to take the Principal Value of the integral. We can use

$$\exp(-w^2 - 2\zeta_s w) = \exp(-w^2)\sum_{q=0}^{\infty}\frac{(-2\zeta_s w)^q}{q!}. \tag{A 6}$$

Then,

$$\frac{\exp(-\zeta_s^2)}{\sqrt{\pi}}\mathrm{PV}\int_{-\infty}^{\infty}\frac{\exp(-w^2-2\zeta_s w)}{w}\,\mathrm{d}w$$
$$= -\frac{\exp(-\zeta_s^2)}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-w^2)\sum_{p=0}^{\infty}\frac{2^{2p+1}\zeta_s^{2p+1}w^{2p}}{(2p+1)!}\,\mathrm{d}w, \tag{A 7}$$

where the integral over the odd powers of $w$ vanishes due to symmetry. Taking the integrals over $w$, we find

$$\frac{\exp(-\zeta_s^2)}{\sqrt{\pi}}\mathrm{PV}\int_{-\infty}^{\infty}\frac{\exp(-w^2-2\zeta_s w)}{w}\,\mathrm{d}w = -\exp(-\zeta_s^2)\sum_{p=0}^{\infty}\frac{2^{2p+1}\Gamma(p+1/2)\zeta_s^{2p+1}}{(2p+1)!}, \tag{A 8}$$

where $\Gamma(\nu) = \int_0^\infty x^{\nu-1}\exp(-x)\,\mathrm{d}x$ is the gamma function.

Combining equations (A 4) and (A 8), we find the approximation (3.35).

## Appendix B. Plasma dispersion function in the limit $|\zeta_s| \gg 1$

Using the Landau contour, we find that the plasma dispersion function for $\mathrm{Im}(\zeta_s) = 0$ is

$$\mathcal{Z}(\zeta_s) = \frac{1}{\sqrt{\pi}}\mathrm{PV}\int_{-\infty}^{\infty}\frac{\exp(-u^2)}{u-\zeta_s}\,\mathrm{d}u + \mathrm{i}\sqrt{\pi}\exp(-\zeta_s^2), \tag{B 1}$$

where the last term is due to the integral over the semicircle $SC$. The integral over the real axis

$$\frac{1}{\sqrt{\pi}}\mathrm{PV}\int_{-\infty}^{\infty}\frac{\exp(-u^2)}{u-\zeta_s}\,\mathrm{d}u \tag{B 2}$$

can be simplified for $|\zeta_s| \gg 1$. For most of the integration interval, $|u| \sim 1 \ll |\zeta_s|$, and the resonant denominator becomes

$$\frac{1}{u-\zeta_s} = -\frac{1}{\zeta_s}\frac{1}{1-u/\zeta_s} \simeq -\frac{1}{\zeta_s}\sum_{q=0}^{Q}\frac{u^q}{\zeta_s^q}, \tag{B 3}$$

leading to the simplified expression

$$\frac{1}{\sqrt{\pi}}\mathrm{PV}\int_{-\infty}^{\infty}\frac{\exp(-u^2)}{u-\zeta_s}\,\mathrm{d}u \simeq -\frac{1}{\zeta_s\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-u^2)\sum_{p=0}^{P}\frac{u^{2p}}{\zeta_s^{2p}}\,\mathrm{d}u, \tag{B 4}$$

where the integrals that included odd powers of $u$ have vanished due to symmetry. Integrating in $u$, we find

$$\frac{1}{\sqrt{\pi}}\mathrm{PV}\int_{-\infty}^{\infty}\frac{\exp(-u^2)}{u-\zeta_s}\,\mathrm{d}u \simeq -\frac{1}{\zeta_s\sqrt{\pi}}\sum_{p=0}^{P}\frac{\Gamma(p+1/2)}{\zeta_s^{2p}}. \tag{B 5}$$

Note that the series is an asymptotic expansion (i.e. it diverges for $P \to \infty$) and hence only a few terms must be kept. Combining the result in equation (B 5) with equation (B 1), we find the formula for the plasma dispersion function given in equation (3.47).

Even though we have deduced this formula only for $\mathrm{Im}(\zeta_s) = 0$, we expect it to be valid for sufficiently small $\mathrm{Im}(\zeta_s)$. We proceed to argue that this is the case.

The most striking feature of the expansion in equation (3.47) is that we keep the exponentially small term $\mathrm{i}\sqrt{\pi}\exp(-\zeta_s^2)$ even though we have neglected terms that are much larger (of order $1/|\zeta_s|^{2P+3}$). The apparently incongruous decision of keeping the exponentially small term $\mathrm{i}\sqrt{\pi}\exp(-\zeta_s^2)$ is justified in cases in which the imaginary part of $\zeta_s$ is also exponentially small, that is,

$$|\mathrm{Im}(\zeta_s)| \sim |\mathrm{Re}(\zeta_s)|^\alpha \exp(-[\mathrm{Re}(\zeta_s)]^2) \ll 1 \ll |\mathrm{Re}(\zeta_s)| \simeq |\zeta_s|, \tag{B 6}$$

where $\alpha$ is a constant that depends on the problem. When $\mathrm{Im}(\zeta_s)$ is exponentially small, the term $\mathrm{i}\sqrt{\pi}\exp(-\zeta_s^2)$ is comparable to the other terms in the imaginary part of the plasma dispersion function, $\mathrm{Im}(\mathcal{Z}(\zeta_s))$. Using the Landau contour, we write the imaginary part of $\mathcal{Z}(\zeta_s)$ for $\mathrm{Im}(\zeta_s) \neq 0$ as

$$\mathrm{Im}(\mathcal{Z}(\zeta_s)) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} I(u)\,\mathrm{d}u + \sigma\sqrt{\pi}\mathrm{Re}(\exp(-\zeta_s^2)), \tag{B 7}$$

where $\sigma = 0$ for $\mathrm{Im}(\zeta_s) > 0$ and $\sigma = 2$ for $\mathrm{Im}(\zeta_s) < 0$, and the integrand $I(u)$ is

$$I(u) = \frac{\mathrm{Im}(\zeta_s)\exp(-u^2)}{[u - \mathrm{Re}(\zeta_s)]^2 + [\mathrm{Im}(\zeta_s)]^2}. \tag{B 8}$$

The function $I(u)$ is sketched in figure 5 for $\mathrm{Re}(\zeta_s) = 3$ and various values of $\mathrm{Im}(\zeta_s)$. Note that for sufficiently small $\mathrm{Im}(\zeta_s)$ there are two clear peaks: one around $u = 0$ and the other around $u = \mathrm{Re}(\zeta_s)$. We can study both peaks by looking for the minima and maxima of $I(u)$, located at the points where $\mathrm{d}I/\mathrm{d}u = 0$. The equation $\mathrm{d}I/\mathrm{d}u = 0$ gives the cubic polynomial

$$u\{[u - \mathrm{Re}(\zeta_s)]^2 + [\mathrm{Im}(\zeta_s)]^2\} + [u - \mathrm{Re}(\zeta_s)] = 0. \tag{B 9}$$

We proceed to find the three roots of this polynomial when $|\mathrm{Im}(\zeta_s)| \ll 1 \ll |\mathrm{Re}(\zeta_s)|$. In the region $|u| \lesssim 1$, equation (B 9) is approximately $[\mathrm{Re}(\zeta_s)]^2 u - \mathrm{Re}(\zeta_s) = 0$, leading to an extremum at $u_{E1} \simeq 1/\mathrm{Re}(\zeta_s)$. The value of $I(u)$ at this extremum is $I(u_{E1}) \simeq \mathrm{Im}(\zeta_s)/[\mathrm{Re}(\zeta_s)]^2$, and the function decays exponentially away from it. The other two roots of the polynomial (B 9) are in the region $|u - \mathrm{Re}(\zeta_s)| \ll 1$. In this limit, equation (B 9) becomes $\mathrm{Re}(\zeta_s)\{[u - \mathrm{Re}(\zeta_s)]^2 + [\mathrm{Im}(\zeta_s)]^2\} + [u - \mathrm{Re}(\zeta_s)] = 0$, and the roots are

$$u_{E2\pm} \simeq \mathrm{Re}(\zeta_s) + \frac{\pm\sqrt{1 - 4[\mathrm{Re}(\zeta_s)\mathrm{Im}(\zeta_s)]^2} - 1}{2\mathrm{Re}(\zeta_s)}. \tag{B 10}$$

Note that for $|\mathrm{Re}(\zeta_s)\mathrm{Im}(\zeta_s)| > 1/2$ there are no extrema (see the blue curves in figure 5). However, in the limit (B 6), the parameter $|\mathrm{Re}(\zeta_s)\mathrm{Im}(\zeta_s)|$ is very small, $|\mathrm{Re}(\zeta_s)\mathrm{Im}(\zeta_s)| \ll 1$, leading to

$$u_{E2+} \simeq \mathrm{Re}(\zeta_s) - \mathrm{Re}(\zeta_s)[\mathrm{Im}(\zeta_s)]^2, \quad u_{E2-} \simeq \mathrm{Re}(\zeta_s) - \frac{1}{\mathrm{Re}(\zeta_s)}. \tag{B 11}$$

The root $u_{E2+}$ corresponds to the peak, and the value of $I(u)$ in this peak is $I(u_{E2+}) \simeq \exp(-[\mathrm{Re}(\zeta_s)]^2)/\mathrm{Im}(\zeta_s)$. The root $u_{E2-}$ corresponds to the valley between the two peaks, where the function is $I(u_{E2-}) = [\mathrm{Re}(\zeta_s)]^2\mathrm{Im}(\zeta_s)\exp(2 - [\mathrm{Re}(\zeta_s)]^2)$.

Therefore, in the limit (B 6), both peaks are well separated. The peak at $u_{E1}$ scales as $|\mathrm{Im}(\zeta_s)|/[\mathrm{Re}(\zeta_s)]^2 \sim |\mathrm{Re}(\zeta_s)|^{\alpha-2}\exp(-[\mathrm{Re}(\zeta_s)]^2)$, and it extends to the region $|u| \sim 1$.

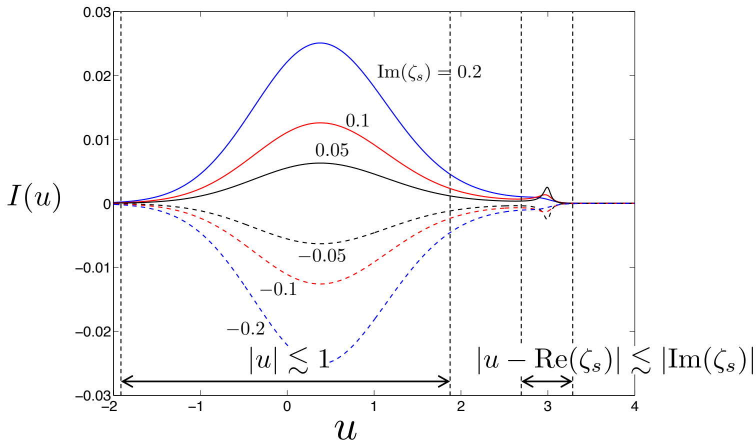


FIGURE 5. Integrand $I(u)$, given in (B 8), plotted agains the integration variable $u$ for $\mathrm{Re}(\zeta_s) = 3$ and several values of $\mathrm{Im}(\zeta_s)$.

For $|u| \sim 1$, we can Taylor expand the resonant denominator,

$$\frac{1}{[u - \mathrm{Re}(\zeta_s)]^2 + [\mathrm{Im}(\zeta_s)]^2} \simeq \frac{1}{[\mathrm{Re}(\zeta_s)]^2} \frac{1}{[1 - u/\mathrm{Re}(\zeta_s)]^2} \simeq \frac{1}{[\mathrm{Re}(\zeta_s)]^2} \sum_{q=0}^{Q} \frac{(q+1)u^q}{[\mathrm{Re}(\zeta_s)]^q}. \tag{B 12}$$

Thus, the integral around the peak at $u_{E1}$ gives

$$\frac{1}{\sqrt{\pi}} \int_{|u|\lesssim 1} I(u)\,\mathrm{d}u \simeq \frac{\mathrm{Im}(\zeta_s)}{[\mathrm{Re}(\zeta_s)]^2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-u^2) \sum_{p=0}^{P} \frac{(2p+1)u^{2p}}{[\mathrm{Re}(\zeta_s)]^{2p}}\,\mathrm{d}u. \tag{B 13}$$

Taking the integrals in $u$, we obtain

$$\frac{1}{\sqrt{\pi}} \int_{|u|\lesssim 1} I(u)\,\mathrm{d}u \simeq \frac{\mathrm{Im}(\zeta_s)}{[\mathrm{Re}(\zeta_s)]^2\sqrt{\pi}} \sum_{p=0}^{P} \frac{(2p+1)\Gamma(p+1/2)}{[\mathrm{Re}(\zeta_s)]^{2p}}. \tag{B 14}$$

Around the peak $u_{E+}$, it is convenient to use the variable $v = (u - u_{E2+})/|\mathrm{Im}(\zeta_s)|$. In this variable, the function $I(u)$ becomes

$$I(u) \simeq \frac{\exp(-[\mathrm{Re}(\zeta_s)]^2)}{\mathrm{Im}(\zeta_s)} \frac{\exp(-2\mathrm{Re}(\zeta_s)|\mathrm{Im}(\zeta_s)|v)}{1+v^2}. \tag{B 15}$$

Since $|\mathrm{Re}(\zeta_s)\mathrm{Im}(\zeta_s)| \ll 1$, we can simplify this equation even further to

$$I(u) \simeq \frac{\exp(-[\mathrm{Re}(\zeta_s)]^2)}{\mathrm{Im}(\zeta_s)} \frac{1}{1+v^2}. \tag{B 16}$$

Then, the peak around $u_{E2+}$ scales as $\exp(-[\mathrm{Re}(\zeta_s)]^2)/|\mathrm{Im}(\zeta_s)| \sim 1/|\mathrm{Re}(\zeta_s)|^{\alpha}$, and it is much higher than the peak at $u_{E1}$, but it is confined to the region $|u - u_{E2+}| \lesssim |\mathrm{Im}(\zeta_s)|$ (equivalent to $|v| \lesssim 1$). Taking an integral over this peak using the integration variable $v = (u - u_{E2+})/|\mathrm{Im}(\zeta_s)|$, we find

$$\begin{aligned} \frac{1}{\sqrt{\pi}} \int_{|u-u_{E2+}|\lesssim|\mathrm{Im}(\zeta_s)|} I(u)\,\mathrm{d}u &\simeq \frac{\mathrm{Im}(\zeta_s)}{|\mathrm{Im}(\zeta_s)|} \frac{\exp(-[\mathrm{Re}(\zeta_s)]^2)}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\mathrm{d}v}{1+v^2} \\ &= \frac{\mathrm{Im}(\zeta_s)}{|\mathrm{Im}(\zeta_s)|} \sqrt{\pi} \exp(-[\mathrm{Re}(\zeta_s)]^2). \end{aligned} \tag{B 17}$$

Adding the two contributions (B 14) and (B 17), equation (B 7) finally becomes

$$\mathrm{Im}(\mathcal{Z}(\zeta_s)) \simeq \frac{\mathrm{Im}(\zeta_s)}{[\mathrm{Re}(\zeta_s)]^2\sqrt{\pi}} \sum_{p=0}^{P} \frac{(2p+1)\Gamma(p+1/2)}{[\mathrm{Re}(\zeta_s)]^{2p}} \,\mathrm{d}u + \sqrt{\pi}\exp(-[\mathrm{Re}(\zeta_s)]^2). \tag{B 18}$$

The dependence on the sign of $\mathrm{Im}(\zeta_s)$ has disappeared because $\sigma\sqrt{\pi}\mathrm{Re}(\exp(-\zeta_s^2)) + (\mathrm{Im}(\zeta_s)/|\mathrm{Im}(\zeta_s)|)\sqrt{\pi}\exp(-[\mathrm{Re}(\zeta_s)]^2) \simeq \sqrt{\pi}\exp(-[\mathrm{Re}(\zeta_s)]^2)$.

The first term in equation (B 18) is the imaginary part of the expansion given in equation (B 5) for $|\mathrm{Im}(\zeta_s)| \ll |\mathrm{Re}(\zeta_s)|$,

$$\mathrm{Im}\left(-\sum_{p=0}^{P} \frac{\Gamma(p+1/2)}{\zeta_s^{2p+1}\sqrt{\pi}}\right) \simeq \sum_{p=0}^{P} \frac{(2p+1)\mathrm{Im}(\zeta_s)\Gamma(p+1/2)}{[\mathrm{Re}(\zeta_s)]^{2p+2}\sqrt{\pi}}. \tag{B 19}$$

Thus, equation (B 18) is the correct lowest order approximation to the imaginary component of $\mathcal{Z}(\zeta_s)$ when assumption (B 6) is satisfied, justifying the use of equation (3.47). Note that, in deriving this result, we have used repeatedly the fact that under assumption (B 6), $|\mathrm{Re}(\zeta_s)\mathrm{Im}(\zeta_s)| \ll 1$. For this reason, the result in equation (3.47) is sometimes referred to as the $|\mathrm{Re}(\zeta_s)\mathrm{Im}(\zeta_s)| \ll 1$ limit even though it is only really valid in the more stringent limit (B 6).