Drift kinetics

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1. Introduction

In the first set of notes, we demonstrated that if a particle is magnetized, its motion can be split into a fast gyration and the motion of its guiding center. The fast gyration around the guiding center is ignorable if we choose the right coordinates. In particular, we choose to describe the particle motion using its position $\mathbf{r}$, its parallel velocity $v_{\|}$ and the magnetic moment of the particle μ.

In this set of notes, we will learn how to derive the kinetic theory for guiding centers, known as drift kinetics. We will follow the pioneering derivation of drift kinetics in (Hazeltine 1973).

We consider a system is of size L with a characteristic frequency

$$\omega \sim \frac{v_{ts}}{L}, \tag{1.1}$$

where v_{ts} is the thermal speed of species s. We assume that

$$\rho_{s*} = \frac{\rho_s}{L} \ll 1, \quad \frac{\omega}{\Omega_s} \sim \rho_{s*} \ll 1. \tag{1.2}$$

Here $\rho_s = v_{ts}/\Omega_s$ and $\Omega_s = Z_s eB/m_s$ are the characteristic gyroradius and gyrofrequency of species s, m_s and $Z_s e$ are the mass and charge of species s ($Z_s = -1$ for electrons), and e is the proton charge. We assume that the thermal energy of the particles is similar for all species, that is,

$$m_s v_{ts}^2 \sim T \tag{1.3}$$

for all s. Here T is the characteristic temperature of the plasma.

The electric field is ordered as in the first set of notes:

- In the high flow regime, the parallel electric field is

$$E_{\|} \sim \frac{T}{eL} \tag{1.4}$$

and the perpendicular electric field is

$$\mathbf{E}_{\perp} \sim v_{ts} B \sim \frac{1}{\rho_{s*}} \frac{T}{eL}. \tag{1.5}$$

- In the low flow regime or drift ordering, the electric field is of order

$$\mathbf{E} \sim \frac{T}{eL}. \tag{1.6}$$

We proceed to derive the drift kinetic equation in both the high flow and the low flow regimes.

2. High flow drift kinetics

We describe the plasma using the distribution functions $f_s(\mathbf{r}, \mathbf{v}, t)$. The probability of finding a particle of species s at time t in a differential phase space volume $\mathrm{d}^3r\,\mathrm{d}^3v$ centered around the phase space position $(\mathbf{r}, \mathbf{v})$ is $f_s(\mathbf{r}, \mathbf{v}, t)\,\mathrm{d}^3r\,\mathrm{d}^3v$. Ignoring collisions, the time evolution of the distribution functions $f_s(\mathbf{r}, \mathbf{v}, t)$ is well described by the Vlasov equation

$$\underbrace{\frac{\partial f_s}{\partial t}}_{\sim f_s v_{ts}/L} + \underbrace{\mathbf{v}\cdot\nabla f_s}_{\sim f_s v_{ts}/L} + \underbrace{\frac{Z_s e}{m_s}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_v f_s}_{\sim f_s \rho_{s*}^{-1} v_{ts}/L} = 0. \tag{2.1}$$

There are terms of very different size in the equation. The reason for these different sizes is the existence of the two very different time scales: the Larmor gyration time Ω_s^{-1}, and the longer time scale L/v_{ts}. In magnetized plasmas, the most interesting time scale is L/v_{ts}, and it corresponds to the motion of the guiding center. We will manipulate equation (2.1) to extract the effects of the guiding center motion on the distribution function.

To obtain the guiding center motion, we first change to a convenient set of phase space coordinates and then we expand in $\rho_{s*} \ll 1$. Once we have a kinetic equation for guiding centers, we will take moments of it to obtain fluid equations valid for a magnetized plasma.

2.1. *Change of phase space coordinates*

In the previous set of notes, we could separate the guiding center motion from the fast gyration by choosing an appropriate set of phase space coordinates. To develop the kinetic theory for guiding centers, we use the same convenient coordinates: the parallel velocity

$$v_\parallel = \mathbf{v}\cdot\hat{\mathbf{b}}(\mathbf{r}, t), \tag{2.2}$$

the magnetic moment

$$\mu = \frac{w_\perp^2}{2B(\mathbf{r}, t)} = \frac{|\mathbf{v} - \mathbf{v}_E|^2 - [\mathbf{v}\cdot\hat{\mathbf{b}}(\mathbf{r}, t)]^2}{2B(\mathbf{r}, t)} \tag{2.3}$$

and the gyrophase

$$\varphi = -\arctan\left(\frac{(\mathbf{v} - \mathbf{v}_E)\cdot\hat{\mathbf{e}}_2(\mathbf{r}, t)}{(\mathbf{v} - \mathbf{v}_E)\cdot\hat{\mathbf{e}}_1(\mathbf{r}, t)}\right), \tag{2.4}$$

where the unit vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ form an orthonormal basis with $\hat{\mathbf{b}}$ such that $\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$. Note that we use the $\mathbf{E}\times\mathbf{B}$ drift

$$\mathbf{v}_E(\mathbf{r}, t) = \frac{1}{B(\mathbf{r}, t)}\mathbf{E}(\mathbf{r}, t)\times\hat{\mathbf{b}}(\mathbf{r}, t). \tag{2.5}$$

To change phase space coordinates from $\{\mathbf{r}, \mathbf{v}\}$ to $\{\mathbf{r}, v_\parallel, \mu, \varphi\}$, we use the chain rule. The derivatives in (2.1) become

$$\left.\frac{\partial f_s}{\partial t}\right|_{\mathbf{r},\mathbf{v}} = \left.\frac{\partial f_s}{\partial t}\right|_{\mathbf{r},v_\parallel,\mu,\varphi} + \left.\frac{\partial v_\parallel}{\partial t}\right|_{\mathbf{r},\mathbf{v}} \left.\frac{\partial f_s}{\partial v_\parallel}\right|_{\mathbf{r},\mu,\varphi,t} + \left.\frac{\partial \mu}{\partial t}\right|_{\mathbf{r},\mathbf{v}} \left.\frac{\partial f_s}{\partial \mu}\right|_{\mathbf{r},v_\parallel,\varphi,t}$$
$$+ \left.\frac{\partial \varphi}{\partial t}\right|_{\mathbf{r},\mathbf{v}} \left.\frac{\partial f_s}{\partial \varphi}\right|_{\mathbf{r},v_\parallel,\mu,t}, \tag{2.6}$$

$$\nabla f_s|_{\mathbf{v},t} = \nabla f_s|_{v_\parallel,\mu,\varphi,t} + \nabla v_\parallel|_{\mathbf{v},t} \left.\frac{\partial f_s}{\partial v_\parallel}\right|_{\mathbf{r},\mu,\varphi,t} + \nabla \mu|_{\mathbf{v},t} \left.\frac{\partial f_s}{\partial \mu}\right|_{\mathbf{r},v_\parallel,\varphi,t} + \nabla \varphi|_{\mathbf{v},t} \left.\frac{\partial f_s}{\partial \varphi}\right|_{\mathbf{r},v_\parallel,\mu,t} \tag{2.7}$$

and

$$\nabla_v f_s|_{\mathbf{r},t} = \nabla_v v_\parallel|_{\mathbf{r},t} \left.\frac{\partial f_s}{\partial v_\parallel}\right|_{\mathbf{r},\mu,\varphi,t} + \nabla_v \mu|_{\mathbf{r},t} \left.\frac{\partial f_s}{\partial \mu}\right|_{\mathbf{r},v_\parallel,\varphi,t} + \nabla_v \varphi|_{\mathbf{r},t} \left.\frac{\partial f_s}{\partial \varphi}\right|_{\mathbf{r},v_\parallel,\mu,t}. \tag{2.8}$$

With these results, equation (2.1) becomes an equation for $f_s(\mathbf{r}, v_\parallel, \mu, \varphi, t)$,

$$\frac{\partial f_s}{\partial t} + \dot{\mathbf{r}} \cdot \nabla f_s + \dot{v}_\parallel \frac{\partial f_s}{\partial v_\parallel} + \dot{\mu} \frac{\partial f_s}{\partial \mu} + \dot{\varphi} \frac{\partial f_s}{\partial \varphi} = 0, \tag{2.9}$$

where we have used the operator

$$\dot{Q} = \left.\frac{\partial Q}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + \mathbf{v} \cdot \nabla Q|_{\mathbf{v},t} + \frac{Z_s e}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v Q|_{\mathbf{r},t}. \tag{2.10}$$

We had to calculate the coefficients $\dot{\mathbf{r}}$, $\dot{v}_\parallel$ and $\dot{\varphi}$ in the first set of notes about particle motion in magnetized plasmas. The coefficient $\dot{\mu}$ is obtained in a very similar way. The final result is

$$\dot{\mathbf{r}} = v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{w}_\perp, \tag{2.11}$$

$$\dot{v}_\parallel = \left[\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \right] \cdot \mathbf{w}_\perp + \mathbf{w}_\perp \cdot \nabla \hat{\mathbf{b}} \cdot \mathbf{w}_\perp + \mathbf{w}_\perp \cdot \nabla \hat{\mathbf{b}} \cdot \mathbf{v}_E + \frac{Z_s e}{m_s} \left[\hat{\mathbf{b}} + \frac{1}{\Omega_s} \hat{\mathbf{b}} \times \left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \right) \right] \cdot \mathbf{E}, \tag{2.12}$$

$$\dot{\mu} = -\frac{\mu}{B} \left[\frac{\partial B}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla B \right] - \frac{\mu}{B} \mathbf{w}_\perp \cdot \nabla B - \frac{1}{B} \mathbf{w}_\perp \cdot (v_\parallel \nabla \hat{\mathbf{b}} + \nabla \mathbf{v}_E) \cdot \mathbf{w}_\perp - \frac{v_\parallel}{B} \left[\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \right] \cdot \mathbf{w}_\perp - \frac{1}{B} \left[\frac{\partial \mathbf{v}_E}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \mathbf{v}_E \right] \cdot \mathbf{w}_\perp, \tag{2.13}$$

$$\dot{\varphi} = \Omega_s + O(v_{ts}/L), \tag{2.14}$$

where

$$\mathbf{w}_\perp = \sqrt{2\mu B(\mathbf{r}, t)} [\cos\varphi \, \hat{\mathbf{e}}_1(\mathbf{r}, t) - \sin\varphi \, \hat{\mathbf{e}}_2(\mathbf{r}, t)]. \tag{2.15}$$

Importantly, the volume in velocity space is not a trivial function of $\{t, \mathbf{r}, v_\parallel, \mu, \varphi\}$. The infinitesimal element of volume in velocity space is given by

$$\mathrm{d}^3 v = \left| \det \left(\frac{\partial \mathbf{v}}{\partial (v_\parallel, \mu, \varphi)} \right) \right| \mathrm{d}v_\parallel \, \mathrm{d}\mu \, \mathrm{d}\varphi, \tag{2.16}$$

where the determinant of the Jacobian of the transformation $(v_\parallel, \mu, \varphi) \to \mathbf{v}$ is

$$\det \left(\frac{\partial \mathbf{v}}{\partial (v_\parallel, \mu, \varphi)} \right) = \frac{1}{\nabla_v v_\parallel \cdot (\nabla_v \mu \times \nabla_v \varphi)} = -B. \tag{2.17}$$

Then,

$$\mathrm{d}^3 v = B(\mathbf{r}, t) \, \mathrm{d}v_\parallel \, \mathrm{d}\mu \, \mathrm{d}\varphi, \tag{2.18}$$

and the probability of finding a particle of species s at a time t, within a volume

d^3r around the point $\mathbf{r}$, within the range $dv_\parallel$ of the parallel velocity $v_\parallel$, within the range $d\mu$ of the magnetic moment μ, and within the range $d\varphi$ of the gyrophase φ is $B(\mathbf{r},t)f_s(\mathbf{r},v_\parallel,\mu,\varphi)\,d^3r\,dv_\parallel\,d\mu\,d\varphi$.

The determinant of the Jacobian is also needed to obtain a useful relation: the conservation of phase space volume. In the usual coordinates $\mathbf{X} = (X_0, X_1, X_2, X_3, X_4, X_5, X_6) = (t, \mathbf{r}, \mathbf{v})$ (note that we have added the time to the coordinates), the conservation of phase space volume is

$$\nabla \cdot \dot{\mathbf{r}} + \nabla_v \cdot \dot{\mathbf{v}} = 0. \tag{2.19}$$

This is a divergence in the 7-dimensional space $\mathbf{X}$, and it can be written in Einstein's index notation as

$$\frac{\partial V_i}{\partial X_i} = 0, \tag{2.20}$$

where $\mathbf{V} = (V_0, V_1, V_2, V_3, V_4, V_5, V_6) = (1, \dot{\mathbf{r}}, \dot{\mathbf{v}})$. To change to other coordinates $\mathbf{Q} = (Q_0, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6) = (t, \mathbf{r}, v_\parallel, \mu, \varphi)$, we use the formula for the coordinate transformation of a divergence (see Appendix A),

$$\frac{\partial}{\partial Q_i}\left[\det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{Q}}\right) V_j \frac{\partial Q_i}{\partial X_j}\right] = 0, \tag{2.21}$$

where $\partial\mathbf{X}/\partial\mathbf{Q}$ is the Jacobian of the transformation $\mathbf{Q} \to \mathbf{X}$. In this case,

$$\det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{Q}}\right) = \det\left(\frac{\partial(t,\mathbf{r},\mathbf{v})}{\partial(t,\mathbf{r},v_\parallel,\mu,\varphi)}\right) = \det\left(\frac{\partial \mathbf{v}}{\partial(v_\parallel,\mu,\varphi)}\right) = -B. \tag{2.22}$$

Note also that

$$V_j \frac{\partial Q_i}{\partial X_j} = \frac{\partial Q_i}{\partial t} + \dot{\mathbf{r}} \cdot \nabla Q_i + \dot{\mathbf{v}} \cdot \nabla_v Q_i = \dot{Q}_i. \tag{2.23}$$

Thus, equation (2.21) gives

$$\frac{\partial B}{\partial t} + \nabla \cdot (B\dot{\mathbf{r}}) + \frac{\partial}{\partial v_\parallel}(B\dot{v}_\parallel) + \frac{\partial}{\partial \mu}(B\dot{\mu}) + \frac{\partial}{\partial \varphi}(B\dot{\varphi}) = 0. \tag{2.24}$$

This expression is the conservation of phase space volume in the coordinates $\{\mathbf{r}, v_\parallel, \mu, \varphi\}$. Using this expression, we can rewrite equation (2.9) in conservative form,

$$\frac{\partial}{\partial t}(Bf_s) + \nabla \cdot (B\dot{\mathbf{r}}f_s) + \frac{\partial}{\partial v_\parallel}(B\dot{v}_\parallel f_s) + \frac{\partial}{\partial \mu}(B\dot{\mu}f_s) + \frac{\partial}{\partial \varphi}(B\dot{\varphi}f_s) = 0. \tag{2.25}$$

This form is useful when we want to take moments of the Vlasov equation to obtain fluid equations, as we will see in subsection 2.3.

2.2. *Expansion in $\rho_{s*} \ll 1$*

Equation (2.9) can be written as

$$\underbrace{\Omega_s \frac{\partial f_s}{\partial \varphi}}_{\sim f_s \rho_{s*}^{-1} v_{ts}/L} + \underbrace{\mathcal{L}[f_s]}_{\sim f_s v_{ts}/L} = 0, \tag{2.26}$$

where the linear operator $\mathcal{L}$ is

$$\mathcal{L}[f] = \frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \nabla f + \dot{v}_\parallel \frac{\partial f}{\partial v_\parallel} + \dot{\mu}\frac{\partial f}{\partial \mu} + (\dot{\varphi} - \Omega_s)\frac{\partial f}{\partial \varphi}. \tag{2.27}$$

Here it will be useful to split the distribution function into its gyrophase independent piece, $\langle f_s \rangle_\varphi$, where

$$\langle g \rangle_\varphi = \frac{1}{2\pi} \int_0^{2\pi} g(\mathbf{r}, v_\|, \mu, \varphi, t)\, \mathrm{d}\varphi \tag{2.28}$$

is the gyroaverage, and its gyrophase dependent piece,

$$\tilde{f}_s = f_s - \langle f_s \rangle_\varphi. \tag{2.29}$$

Using this separation, equation (2.26) becomes

$$\Omega_s \frac{\partial \tilde{f}_s}{\partial \varphi} + \mathcal{L}[\langle f_s \rangle_\varphi] + \mathcal{L}[\tilde{f}_s] = 0. \tag{2.30}$$

This equation, in turn, can be split into its gyrophase independent and dependent pieces,

$$\underbrace{\langle \mathcal{L}[\langle f_s \rangle_\varphi] \rangle_\varphi}_{\sim \langle f_s \rangle_\varphi v_{ts}/L} + \underbrace{\langle \mathcal{L}[\tilde{f}_s] \rangle_\varphi}_{\sim \tilde{f}_s v_{ts}/L} = 0 \tag{2.31}$$

and

$$\underbrace{\Omega_s \frac{\partial \tilde{f}_s}{\partial \varphi}}_{\sim \tilde{f}_s \rho_{s*}^{-1} v_{ts}/L} + \underbrace{\widetilde{\mathcal{L}[\tilde{f}_s]}}_{\sim \tilde{f}_s v_{ts}/L} = - \underbrace{\widetilde{\mathcal{L}[\langle f_s \rangle_\varphi]}}_{\sim \langle f_s \rangle_\varphi v_{ts}/L}, \tag{2.32}$$

where we have used the fact that $\tilde{f}_s$ is periodic in φ to find $\langle \partial \tilde{f}_s / \partial \varphi \rangle_\varphi = 0$.

We use equation (2.32) to obtain the gyrophase dependent piece $\tilde{f}_s$ as a functional of $\langle f_s \rangle_\varphi$. Equation (2.32) can be solved by expanding $\tilde{f}_s$ as a power series in ρ_{s*}, that is,

$$\tilde{f}_s = \tilde{f}_{s,1} + \tilde{f}_{s,2} + \ldots, \tag{2.33}$$

where $\tilde{f}_{s,n} \sim \rho_{s*}^n \langle f_s \rangle_\varphi$. Note that $\tilde{f}_s \ll \langle f_s \rangle_\varphi$. Using the expansion (2.33), equation (2.32) gives

$$\Omega_s \frac{\partial \tilde{f}_{s,1}}{\partial \varphi} = -\widetilde{\mathcal{L}[\langle f_s \rangle_\varphi]} = -\tilde{\dot{\mathbf{r}}} \cdot \nabla \langle f_s \rangle_\varphi - \tilde{\dot{v}}_\| \frac{\partial \langle f_s \rangle_\varphi}{\partial v_\|} - \tilde{\dot{\mu}} \frac{\partial \langle f_s \rangle_\varphi}{\partial \mu} \tag{2.34}$$

to lowest order, and

$$\Omega_s \frac{\partial \tilde{f}_{s,2}}{\partial \varphi} = -\widetilde{\mathcal{L}[\tilde{f}_{s,1}]} \tag{2.35}$$

to next order. Continuing the expansion, we can calculate $\tilde{f}_{s,n+1}$ from $\tilde{f}_{s,n}$,

$$\Omega_s \frac{\partial \tilde{f}_{s,n+1}}{\partial \varphi} = -\widetilde{\mathcal{L}[\tilde{f}_{s,n}]}. \tag{2.36}$$

We calculate $\tilde{f}_{s,1}$ as an example. Integrating (2.34) only requires writing $\widetilde{\mathcal{L}[\langle f_s \rangle_\varphi]}$ as a Fourier series of sines and cosines of φ, and then integrating. This is what we show in Appendix B, although done in a more elegant manner. The final answer is

$$\boxed{\tilde{f}_{s,1} = -\tilde{\mathbf{r}}_1 \cdot \nabla \langle f_s \rangle_\varphi - \tilde{v}_{\|,1} \frac{\partial \langle f_s \rangle_\varphi}{\partial v_\|} - \tilde{\mu}_1 \frac{\partial \langle f_s \rangle_\varphi}{\partial \mu},} \tag{2.37}$$

where

$$\tilde{\mathbf{r}}_1 = \frac{1}{\Omega_s} \int^{\varphi} \widetilde{\dot{\mathbf{r}}}(\varphi')\, \mathrm{d}\varphi' = \frac{1}{\Omega_s} \hat{\mathbf{b}} \times \mathbf{w}_\perp, \tag{2.38}$$

$$\begin{aligned} \tilde{v}_{\|,1} &= \frac{1}{\Omega_s} \int^{\varphi} \widetilde{\dot{v}}_{\|}(\varphi')\, \mathrm{d}\varphi' \\ &= \frac{1}{\Omega_s} \left[\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\| \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} + \nabla \hat{\mathbf{b}} \cdot \mathbf{v}_E \right] \cdot (\hat{\mathbf{b}} \times \mathbf{w}_\perp) \\ &+ \frac{1}{4\Omega_s} [\mathbf{w}_\perp (\hat{\mathbf{b}} \times \mathbf{w}_\perp) + (\hat{\mathbf{b}} \times \mathbf{w}_\perp) \mathbf{w}_\perp] : \nabla \hat{\mathbf{b}}, \end{aligned} \tag{2.39}$$

$$\begin{aligned} \tilde{\mu}_1 &= \frac{1}{\Omega_s} \int^{\varphi} \widetilde{\dot{\mu}}(\varphi')\, \mathrm{d}\varphi' \\ &= -\frac{1}{B\Omega_s} \left[\mu \nabla B + v_\| \left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\| \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \right) \right. \\ &+ \left. \frac{\partial \mathbf{v}_E}{\partial t} + (v_\| \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \mathbf{v}_E \right] \cdot (\hat{\mathbf{b}} \times \mathbf{w}_\perp) \\ &- \frac{1}{4B\Omega_s} [\mathbf{w}_\perp (\hat{\mathbf{b}} \times \mathbf{w}_\perp) + (\hat{\mathbf{b}} \times \mathbf{w}_\perp) \mathbf{w}_\perp] : (v_\| \nabla \hat{\mathbf{b}} + \nabla \mathbf{v}_E). \end{aligned} \tag{2.40}$$

In these expressions, the indefinite integrals $\int^{\varphi} \widetilde{\dot{Q}}_i(\varphi')\, \mathrm{d}\varphi'$ are chosen such that $\langle \widetilde{Q}_{i,1} \rangle_\varphi = 0$. Note that we are using the double contraction of two matrices, $\mathbf{M} : \mathbf{N}$. This operation gives a scalar, and in Einstein's index notation, it corresponds to

$$\mathbf{M} : \mathbf{N} = M_{ij} N_{ji}. \tag{2.41}$$

The tensor $\mathbf{w}_\perp (\hat{\mathbf{b}} \times \mathbf{w}_\perp) + (\hat{\mathbf{b}} \times \mathbf{w}_\perp) \mathbf{w}_\perp$ is just a convenient way to write the second order harmonics in φ. Using (2.15), we obtain

$$\mathbf{w}_\perp (\hat{\mathbf{b}} \times \mathbf{w}_\perp) + (\hat{\mathbf{b}} \times \mathbf{w}_\perp) \mathbf{w}_\perp = 2\mu B [\sin 2\varphi\, \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \cos 2\varphi (\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_1) - \sin 2\varphi\, \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2]. \tag{2.42}$$

In the orthonormal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{b}}\}$, the tensor in (2.42) is the matrix

$$\mathbf{w}_\perp (\hat{\mathbf{b}} \times \mathbf{w}_\perp) + (\hat{\mathbf{b}} \times \mathbf{w}_\perp) \mathbf{w}_\perp = 2\mu B \begin{pmatrix} \sin 2\varphi & \cos 2\varphi & 0 \\ \cos 2\varphi & -\sin 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.43}$$

Since we have calculated $\tilde{f}_s$ as a functional of $\langle f_s \rangle_\varphi$ using (2.32), equation (2.31) becomes an equation for the gyrophase independent piece $\langle f_s \rangle_\varphi$. This is the drift kinetic equation. We can choose the order of accuracy of the drift kinetic equation by deciding to what order we calculate $\tilde{f}_s$. If we choose to neglect $\tilde{f}_s$, the drift kinetic equation is missing terms of order $\langle \mathcal{L}[\tilde{f}_{s,1}] \rangle_\varphi \sim \langle f_s \rangle_\varphi \rho_{s*} v_{ts}/L$. The drift kinetic equation to this order is

$$\langle \mathcal{L}[\langle f_s \rangle_\varphi] \rangle_\varphi = \frac{\partial \langle f_s \rangle_\varphi}{\partial t} + \langle \dot{\mathbf{r}} \rangle_\varphi \cdot \nabla \langle f_s \rangle_\varphi + \langle \dot{v}_\| \rangle_\varphi \frac{\partial \langle f_s \rangle_\varphi}{\partial v_\|} + \langle \dot{\mu} \rangle_\varphi \frac{\partial \langle f_s \rangle_\varphi}{\partial \mu} = 0. \tag{2.44}$$

We calculated the quantities $\langle \dot{\mathbf{r}} \rangle_\varphi$ and $\langle \dot{v}_\| \rangle_\varphi$ when we derived the guiding center motion. We can deduce that $\langle \dot{\mu} \rangle_\varphi \simeq 0$ from the quantities $\langle \dot{\mathbf{r}} \rangle_\varphi$, $\langle \dot{v}_\| \rangle_\varphi$ and $\langle \dot{w}_\perp \rangle_\varphi$ calculated for the guiding center motion equations, or it can be directly shown by gyroaveraging (2.13) and by using $\nabla \cdot \mathbf{B} = 0$ and Faraday's induction law $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ (see Appendix C).

The final result is that equation (2.44) becomes

$$\boxed{\frac{\partial\langle f_s\rangle_\varphi}{\partial t} + (v_\parallel\hat{\mathbf{b}} + \mathbf{v}_E)\cdot\nabla\langle f_s\rangle_\varphi + \left[\frac{Z_s e}{m_s}\left(\hat{\mathbf{b}} + \frac{1}{\Omega_s}\hat{\mathbf{b}}\times\frac{\mathrm{D}\hat{\mathbf{b}}}{\mathrm{D}t}\right)\cdot\mathbf{E} - \mu\hat{\mathbf{b}}\cdot\nabla B\right]\frac{\partial\langle f_s\rangle_\varphi}{\partial v_\parallel} = 0,} \tag{2.45}$$

where we have defined the operator

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (v_\parallel\hat{\mathbf{b}} + \mathbf{v}_E)\cdot\nabla. \tag{2.46}$$

Equation (2.45) is the lowest order high flow drift kinetic equation. The coefficients in the equation are clearly related to the guiding center motion equations.

In this course we do not consider the high flow equation to higher order than this. To see how to go to next order, we will use the low flow ordering because the next order terms tend to be more important in the low flow regime.

2.3. *Moments of the drift kinetic distribution function*

According to (2.37), $\tilde{f}_s \sim \rho_{s*}\langle f_s\rangle_\varphi$ and hence

$$f_s = \langle f_s\rangle_\varphi + \tilde{f}_s \simeq \langle f_s\rangle_\varphi. \tag{2.47}$$

This is important for the lowest order moments of the distribution function. For example, the density and the average flow are

$$n_s = \int f_s\,\mathrm{d}^3v = \int_{-\infty}^{\infty}\mathrm{d}v_\parallel\int_0^\infty\mathrm{d}\mu\int_0^{2\pi}\mathrm{d}\varphi\,Bf_s = 2\pi\int_{-\infty}^{\infty}\mathrm{d}v_\parallel\int_0^\infty\mathrm{d}\mu\,B\langle f_s\rangle_\varphi \tag{2.48}$$

and

$$\begin{aligned} n_s\mathbf{u}_s &= \int f_s\mathbf{v}\,\mathrm{d}^3v = \int Bf_s(v_\parallel\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{w}_\perp)\,\mathrm{d}v_\parallel\,\mathrm{d}\mu\,\mathrm{d}\varphi \\ &\simeq \int B\langle f_s\rangle_\varphi(v_\parallel\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{w}_\perp)\,\mathrm{d}v_\parallel\,\mathrm{d}\mu\,\mathrm{d}\varphi = 2\pi\int B\langle f_s\rangle_\varphi v_\parallel\hat{\mathbf{b}}\,\mathrm{d}v_\parallel\,\mathrm{d}\mu + n_s\mathbf{v}_E. \end{aligned} \tag{2.49}$$

Thus, the average velocity is, to lowest order in $\rho_{s*} \ll 1$,

$$\mathbf{u}_s \simeq u_{s\parallel}\hat{\mathbf{b}} + \mathbf{v}_E, \tag{2.50}$$

where

$$u_{s\parallel} = \frac{1}{n_s}\int f_s v_\parallel\,\mathrm{d}^3v = \frac{2\pi}{n_s}\int B\langle f_s\rangle_\varphi v_\parallel\,\mathrm{d}v_\parallel\,\mathrm{d}\mu. \tag{2.51}$$

As we will see, there are other interesting quantities, such as the parallel and perpendicular pressures,

$$p_{s\parallel} = \int f_s m_s(v_\parallel - u_{s\parallel})^2\,\mathrm{d}^3v = 2\pi\int B\langle f_s\rangle_\varphi m_s(v_\parallel - u_{s\parallel})^2\,\mathrm{d}v_\parallel\,\mathrm{d}\mu \tag{2.52}$$

and

$$p_{s\perp} = \int f_s\frac{m_s|\mathbf{v}_\perp - \mathbf{u}_{s\perp}|^2}{2}\,\mathrm{d}^3v \simeq \int f_s\frac{m_s w_\perp^2}{2}\,\mathrm{d}^3v = 2\pi\int B^2\langle f_s\rangle_\varphi m_s\mu\,\mathrm{d}v_\parallel\,\mathrm{d}\mu. \tag{2.53}$$

The factor of 2 dividing $m_s|\mathbf{v}_\perp - \mathbf{u}_{s\perp}|^2$ in the perpendicular pressure is due to the fact that there are two spatial dimension perpendicular to the magnetic field.

Taking moments of (2.45), we can obtain fluid equations that relate the different moments of f_s. To take moments, we need to write equation (2.45) in what is known as "conservative form". Gyroaveraging equation (2.24), we obtain

$$\frac{\partial B}{\partial t} + \nabla \cdot (B\langle \dot{\mathbf{r}} \rangle_\varphi) + \frac{\partial}{\partial v_\|}(B\langle \dot{v}_\| \rangle_\varphi) = 0, \tag{2.54}$$

where we have used that $\langle \dot{\mu} \rangle_\varphi \simeq 0$ (see Appendix C). With (2.44) and (2.54), we obtain

$$\frac{\partial}{\partial t}(B\langle f_s \rangle_\varphi) + \nabla \cdot (B\langle \dot{\mathbf{r}} \rangle_\varphi \langle f_s \rangle_\varphi) + \frac{\partial}{\partial v_\|}(B\langle \dot{v}_\| \rangle_\varphi \langle f_s \rangle_\varphi) = 0. \tag{2.55}$$

A more explicit version of the equation is

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}(B\langle f_s \rangle_\varphi) + \nabla \cdot [B(v_\| \hat{\mathbf{b}} + \mathbf{v}_E)\langle f_s \rangle_\varphi] + \frac{\partial}{\partial v_\|}\Bigg\{ B\Bigg[\frac{Z_s e}{m_s}\left(\hat{\mathbf{b}} + \frac{1}{\Omega_s}\hat{\mathbf{b}} \times \frac{\mathrm{D}\hat{\mathbf{b}}}{\mathrm{D}t}\right) \cdot \mathbf{E} \\ - \mu \hat{\mathbf{b}} \cdot \nabla B\Bigg] \langle f_s \rangle_\varphi \Bigg\} = 0.\end{aligned}} \tag{2.56}$$

We can integrate equation (2.56) over $v_\|$ and μ (the integral over φ is just a factor of 2π) to find the drift kinetic continuity equation. For the term under the partial derivative with respect to $v_\|$, we assume that $\langle f_s \rangle_\varphi$ vanishes for sufficiently large $v_\|$ so that

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial v_\|}\left(B\langle \dot{v}_\| \rangle_\varphi \langle f_s \rangle_\varphi\right) \mathrm{d}v_\| = \Big[B\langle \dot{v}_\| \rangle_\varphi \langle f_s \rangle_\varphi\Big]_{v_\| = -\infty}^{v_\| = \infty} = 0. \tag{2.57}$$

Then, the drift kinetic continuity equation is

$$\boxed{\frac{\partial n_s}{\partial t} + \nabla \cdot [n_s(u_{s\|}\hat{\mathbf{b}} + \mathbf{v}_E)] = 0.} \tag{2.58}$$

We can also multiply equation (2.56) by $m_s v_\|$ and integrate over $v_\|$ and μ to find the parallel momentum conservation equation. For the term under the partial derivative with respect to $v_\|$, we integrate by parts and we assume that $v_\| \langle f_s \rangle_\varphi$ vanishes for sufficiently large $v_\|$ so that

$$\int_{-\infty}^{\infty} v_\| \frac{\partial}{\partial v_\|}(B\langle \dot{v}_\| \rangle_\varphi \langle f_s \rangle_\varphi)\, \mathrm{d}v_\| = -\int_{-\infty}^{\infty} B\langle \dot{v}_\| \rangle_\varphi \langle f_s \rangle_\varphi \, \mathrm{d}v_\|. \tag{2.59}$$

Then, the parallel momentum conservation equation becomes

$$\begin{aligned}\frac{\partial}{\partial t}(n_s m_s u_{s\|}) + \nabla \cdot \left[\hat{\mathbf{b}} \int \langle f_s \rangle_\varphi m_s v_\|^2 \, \mathrm{d}^3 v + n_s m_s u_{s\|} \mathbf{v}_E\right] + p_{s\perp} \hat{\mathbf{b}} \cdot \nabla \ln B \\ - Z_s e n_s \left[\hat{\mathbf{b}} + \frac{1}{\Omega_s}\hat{\mathbf{b}} \times \left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + (u_{s\|}\hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}}\right)\right] \cdot \mathbf{E} = 0.\end{aligned} \tag{2.60}$$

This equation can be rewritten using (2.52) to obtain $\int \langle f_s \rangle_\varphi m_s v_\|^2 \, \mathrm{d}^3 v = p_{s\|} + n_s m_s u_{s\|}^2$. Moreover, employing (2.58), $\hat{\mathbf{b}} \cdot \nabla \ln B = -\nabla \cdot \hat{\mathbf{b}}$ (deduced from $\nabla \cdot \mathbf{B} = 0$) and

$$\frac{1}{B}\left[\hat{\mathbf{b}} \times \left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + (u_{s\|}\hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}}\right)\right] \cdot \mathbf{E} = -\left(\frac{\partial \mathbf{v}_E}{\partial t} + (u_{s\|}\hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \mathbf{v}_E\right) \cdot \hat{\mathbf{b}}, \tag{2.61}$$

equation (2.60) can be manipulated to become the final drift kinetic parallel momentum equation,

$$\boxed{\begin{aligned} n_s m_s \left[\frac{\partial}{\partial t}(u_{s\|}\hat{\mathbf{b}} + \mathbf{v}_E) + (u_{s\|}\hat{\mathbf{b}} + \mathbf{v}_E)\cdot\nabla(u_{s\|}\hat{\mathbf{b}} + \mathbf{v}_E)\right]\cdot\hat{\mathbf{b}} \\ = -\hat{\mathbf{b}}\cdot\nabla p_{s\|} + (p_{s\perp} - p_{s\|})\nabla\cdot\hat{\mathbf{b}} + Z_s e n_s E_\|. \end{aligned}} \tag{2.62}$$

Note that the magnetic bottling force appears as a term proportional to the perpendicular pressure $p_{s\perp}$, and that part of $\langle \dot{v}_\| \rangle_\varphi$ becomes part of the inertial term. We will discuss the parallel momentum equation further when we formulate kinetic MHD.

3. Low flow drift kinetics

In the low flow regime, the electric field is ordered as (1.6), and the $\mathbf{E}\times\mathbf{B}$ drift is of order

$$\mathbf{v}_E \sim \rho_{s*} v_{ts} \ll v_{ts}. \tag{3.1}$$

Using Faraday's induction law, we find

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \sim \rho_{s*}\frac{v_{ts}}{L}B \ll \frac{v_{ts}}{L}B, \tag{3.2}$$

that is, the time derivative of the magnetic field is much smaller than the other time derivatives in the problem. As a result, the time derivatives of quantities related to $\mathbf{B}$, such as B, $\hat{\mathbf{b}}$, $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, are small,

$$\frac{\partial}{\partial t}\ln B \sim \frac{\partial \hat{\mathbf{b}}}{\partial t} \sim \frac{\partial \hat{\mathbf{e}}_1}{\partial t} \sim \frac{\partial \hat{\mathbf{e}}_2}{\partial t} \sim \rho_{s*}\frac{v_{ts}}{L} \ll \frac{v_{ts}}{L}. \tag{3.3}$$

Using (1.6), (3.1) and (3.3) in (2.45), and neglecting the terms of order $\langle f_s\rangle_\varphi \rho_{s*} v_{ts}/L$, we obtain

$$\frac{\partial \langle f_s\rangle_\varphi}{\partial t} + v_\|\hat{\mathbf{b}}\cdot\nabla\langle f_s\rangle_\varphi + \left(\frac{Z_s e}{m_s}\hat{\mathbf{b}}\cdot\mathbf{E} - \mu\hat{\mathbf{b}}\cdot\nabla B\right)\frac{\partial\langle f_s\rangle_\varphi}{\partial v_\|} = 0. \tag{3.4}$$

This equation does not include the perpendicular component of the gradient $\nabla\langle f_s\rangle_\varphi$ because particles only move along magnetic field lines to lowest order. In reality, particles drift slowly across magnetic field lines. This small perpendicular drift is important because it is the only drift in this direction, and it is the only mechanism by which different magnetic field lines communicate. Thus, we need to obtain the drift kinetic equation to next order in ρ_{s*}.

To obtain the next order drift kinetic equation, we use the lowest order approximation to $\tilde{f}_s$, $\tilde{f}_s \simeq \tilde{f}_{s,1}$, in (2.31) to find

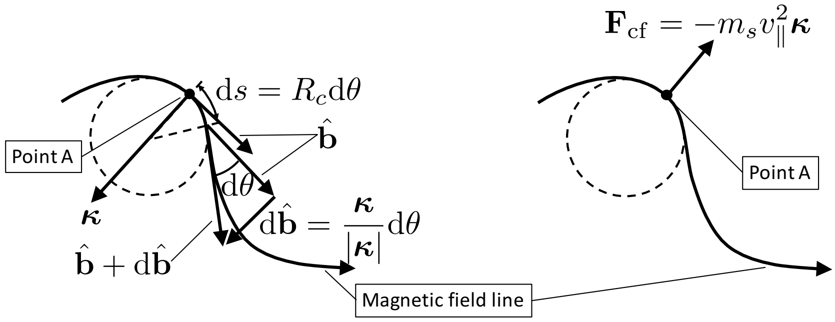
$$\langle \mathcal{L}[\langle f_s\rangle_\varphi]\rangle_\varphi + \langle \mathcal{L}[\tilde{f}_{s,1}]\rangle_\varphi = 0. \tag{3.5}$$

In this equation, we are missing terms of order $\langle \mathcal{L}[\tilde{f}_{s,2}]\rangle_\varphi \sim \langle f_s\rangle_\varphi \rho_{s*}^2 v_{ts}/L$. We proceed to evaluate the two terms in (3.5).

3.1. *Evaluation of* $\langle \mathcal{L}[\langle f_s\rangle_\varphi]\rangle_\varphi$

Using the formulas for $\langle\dot{\mathbf{r}}\rangle_\varphi$ and $\langle \dot{v}_\|\rangle_\varphi$ that we obtained in the notes for particle motion, Appendix C (equation (C 10) in particular), and equations (1.6), (3.1) and (3.3), we find

$$\langle \mathcal{L}[\langle f_s\rangle_\varphi]\rangle_\varphi = \frac{\partial\langle f_s\rangle_\varphi}{\partial t} + \langle\dot{\mathbf{r}}\rangle_\varphi\cdot\nabla\langle f_s\rangle_\varphi + \langle\dot{v}_\|\rangle_\varphi\frac{\partial\langle f_s\rangle_\varphi}{\partial v_\|} + \langle\dot{\mu}\rangle_\varphi\frac{\partial\langle f_s\rangle_\varphi}{\partial\mu}, \tag{3.6}$$



$$\boldsymbol{\kappa} = \frac{1}{R_c}\frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|} = \frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}\theta} = \frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}s} = \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}$$

FIGURE 1. Definition of the curvature of a magnetic field line, and its relation to the local centrifugal force. The curvature $\boldsymbol{\kappa}$ of a magnetic field line at a point A is a vector whose magnitude is the inverse of the radius of the circle that best fits the line at point A (radius of curvature R_c), and whose direction is the direction that points from point A to the center of said circle. The direction and magnitude of the curvature is given by the infinitesimal change in direction of the unit vector $\hat{\mathbf{b}}$ along the curve. In the sketch, the angle $\mathrm{d}\theta$ between $\hat{\mathbf{b}}$ at point A and $\hat{\mathbf{b}}+\mathrm{d}\hat{\mathbf{b}}$ at an infinitesimal distance $\mathrm{d}s$ away from A is related to the angular separation in the circle that best fits the curve at point A. Using this relation, we find that the curvature vector is $\boldsymbol{\kappa} = \mathrm{d}\hat{\mathbf{b}}/\mathrm{d}s = \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}$. Note that according to the definition of curvature, the centrifugal force felt in a frame moving with the particle along the line is $\mathbf{F}_{\rm cf} = -(m_s v_\parallel^2/R_c)(\boldsymbol{\kappa}/|\boldsymbol{\kappa}|) = -m_s v_\parallel^2\boldsymbol{\kappa}$.

where

$$\langle\dot{\mathbf{r}}\rangle_\varphi = v_\parallel\hat{\mathbf{b}} + \mathbf{v}_E, \tag{3.7}$$

$$\langle\dot{v}_\parallel\rangle_\varphi = \frac{Z_s e}{m_s}\left(\hat{\mathbf{b}} + \frac{v_\parallel}{\Omega_s}\hat{\mathbf{b}}\times\boldsymbol{\kappa}\right)\cdot\mathbf{E} - \mu\hat{\mathbf{b}}\cdot\nabla B + O(\rho_{s*}^2 v_{ts}^2/L), \tag{3.8}$$

$$\langle\dot{\mu}\rangle_\varphi = \frac{E_\parallel\mu}{B}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}. \tag{3.9}$$

Here

$$\boldsymbol{\kappa} = \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}} \tag{3.10}$$

is the curvature of the magnetic field line (see figure 1).

3.2. *Evaluation of* $\langle\mathcal{L}[\tilde{f}_{s,1}]\rangle_\varphi$

The function $\tilde{f}_{s,1}$ is given in equation (2.37) for the high flow ordering. Using (1.6), (3.1) and (3.3), and neglecting terms of order $\rho_{s*}^2\langle f_s\rangle_\varphi$, the function $\tilde{f}_{s,1}$ is of the form (2.37), but $\tilde{v}_{\parallel,1}$ and $\tilde{\mu}_1$ become

$$\tilde{v}_{\parallel,1} = \frac{v_\parallel}{\Omega_s}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\hat{\mathbf{b}}\times\mathbf{w}_\perp) + \frac{1}{4\Omega_s}[\mathbf{w}_\perp(\hat{\mathbf{b}}\times\mathbf{w}_\perp) + (\hat{\mathbf{b}}\times\mathbf{w}_\perp)\mathbf{w}_\perp] : \nabla\hat{\mathbf{b}}, \tag{3.11}$$

$$\tilde{\mu}_1 = -\frac{1}{B\Omega_s}(\mu\nabla B + v_\parallel^2\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot(\hat{\mathbf{b}}\times\mathbf{w}_\perp) - \frac{v_\parallel}{4B\Omega_s}[\mathbf{w}_\perp(\hat{\mathbf{b}}\times\mathbf{w}_\perp) + (\hat{\mathbf{b}}\times\mathbf{w}_\perp)\mathbf{w}_\perp] : \nabla\hat{\mathbf{b}}. \tag{3.12}$$

Using the low flow $\tilde{f}_{s,1}$, we can directly calculate $\langle\mathcal{L}[\tilde{f}_{s,1}]\rangle_\varphi$. It is a tedious operation, but there are some useful tricks that make it bearable. We start by realizing that in

Einstein's index notation,

$$\langle \mathcal{L}[\tilde{f}_{s,1}] \rangle_\varphi = \left\langle \tilde{\dot{\mathbf{r}}} \cdot \nabla \tilde{f}_{s,1} \right\rangle_\varphi + \left\langle \widetilde{\dot{v}}_\| \frac{\partial \tilde{f}_{s,1}}{\partial v_\|} \right\rangle_\varphi + \left\langle \widetilde{\dot{\mu}} \frac{\partial \tilde{f}_{s,1}}{\partial \mu} \right\rangle_\varphi + \left\langle \widetilde{\dot{\varphi}} \frac{\partial \tilde{f}_{s,1}}{\partial \varphi} \right\rangle_\varphi \equiv \left\langle \widetilde{\dot{Q}}_i \frac{\partial \tilde{f}_{s,1}}{\partial Q_i} \right\rangle_\varphi , \tag{3.13}$$

where $\mathbf{Q} = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6) = (\mathbf{r}, v_\|, \mu, \varphi)$. Before continuing, we use the gyrophase dependent piece of (2.24) to write

$$\frac{\partial}{\partial Q_i} \left(B \widetilde{\dot{Q}}_i \right) \equiv \nabla \cdot (B \tilde{\dot{\mathbf{r}}}) + \frac{\partial}{\partial v_\|} (B \widetilde{\dot{v}}_\|) + \frac{\partial}{\partial \mu} (B \widetilde{\dot{\mu}}) + \frac{\partial}{\partial \varphi} (B \widetilde{\dot{\varphi}}) = 0. \tag{3.14}$$

Using this result, equation (3.13) becomes

$$\langle \mathcal{L}[\tilde{f}_{s,1}] \rangle_\varphi = \frac{1}{B} \frac{\partial}{\partial Q_i} \left(B \left\langle \widetilde{\dot{Q}}_i \tilde{f}_{s,1} \right\rangle_\varphi \right). \tag{3.15}$$

Note that equation (2.37) can be written as

$$\tilde{f}_{s,1} = - \widetilde{Q}_{i,1} \frac{\partial \langle f_s \rangle_\varphi}{\partial Q_i}, \tag{3.16}$$

where $\widetilde{\mathbf{Q}}_1 = (\widetilde{Q}_{1,1}, \widetilde{Q}_{2,1}, \widetilde{Q}_{3,1}, \widetilde{Q}_{4,1}, \widetilde{Q}_{5,1}, \widetilde{Q}_{6,1}) = (\tilde{\mathbf{r}}_1, \tilde{v}_{\|,1}, \tilde{\mu}_1, \tilde{\varphi}_1)$, and

$$\Omega_s \frac{\partial \widetilde{Q}_{i,1}}{\partial \varphi} = \widetilde{\dot{Q}}_i. \tag{3.17}$$

Using (3.16) and (3.17), equation (3.15) becomes

$$\langle \mathcal{L}[\tilde{f}_{s,1}] \rangle_\varphi = - \frac{1}{B} \frac{\partial}{\partial Q_i} \left(B \Omega_s \left\langle \widetilde{Q}_{j,1} \frac{\partial \widetilde{Q}_{i,1}}{\partial \varphi} \right\rangle_\varphi \frac{\partial \langle f_s \rangle_\varphi}{\partial Q_j} \right). \tag{3.18}$$

Finally, integrating by parts in φ, we find that

$$\left\langle \widetilde{Q}_{j,1} \frac{\partial \widetilde{Q}_{i,1}}{\partial \varphi} \right\rangle_\varphi = - \left\langle \widetilde{Q}_{i,1} \frac{\partial \widetilde{Q}_{j,1}}{\partial \varphi} \right\rangle_\varphi , \tag{3.19}$$

leading to the expression

$$\langle \mathcal{L}[\tilde{f}_{s,1}] \rangle_\varphi = \frac{1}{B} \frac{\partial}{\partial Q_i} \left(\frac{B \Omega_s}{2} \left\langle \widetilde{Q}_{i,1} \frac{\partial \widetilde{Q}_{j,1}}{\partial \varphi} - \widetilde{Q}_{j,1} \frac{\partial \widetilde{Q}_{i,1}}{\partial \varphi} \right\rangle_\varphi \frac{\partial \langle f_s \rangle_\varphi}{\partial Q_j} \right)$$

0 because $\frac{\partial^2}{\partial Q_i \partial Q_j} = \frac{\partial^2}{\partial Q_j \partial Q_i}$

$$= \frac{\Omega_s}{2} \left\langle \widetilde{Q}_{i,1} \frac{\partial \widetilde{Q}_{j,1}}{\partial \varphi} - \widetilde{Q}_{j,1} \frac{\partial \widetilde{Q}_{i,1}}{\partial \varphi} \right\rangle_\varphi \frac{\partial^2 \langle f_s \rangle_\varphi}{\partial Q_i \partial Q_j} + \dot{Q}_{j,1}^{DK} \frac{\partial \langle f_s \rangle_\varphi}{\partial Q_j}, \tag{3.20}$$

where we have defined the coefficients

$$\dot{Q}_{j,1}^{DK} = \frac{1}{B} \frac{\partial}{\partial Q_i} \left(\frac{B \Omega_s}{2} \left\langle \widetilde{Q}_{i,1} \frac{\partial \widetilde{Q}_{j,1}}{\partial \varphi} - \widetilde{Q}_{j,1} \frac{\partial \widetilde{Q}_{i,1}}{\partial \varphi} \right\rangle_\varphi \right). \tag{3.21}$$

Using expression (3.20), and after many tedious vector manipulations (see Appendix D),

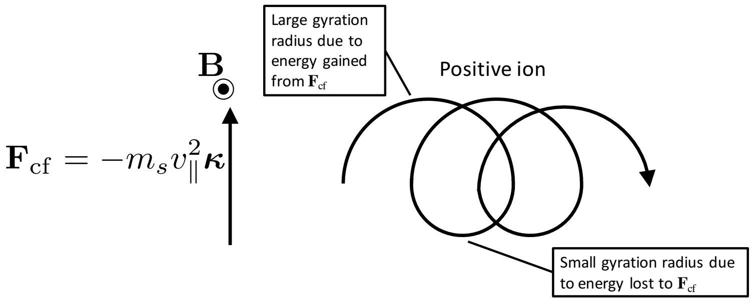


FIGURE 2. The curvature drift is the result of the centrifugal force (see figure 1) accelerating and decelerating the particle in its gyration, and consequently changing the radius of gyration.

we find

$$\langle \mathcal{L}[\tilde{f}_{s,1}] \rangle_\varphi = \dot{\mathbf{r}}_1^{DK} \cdot \nabla \langle f_s \rangle_\varphi + \dot{v}_{\parallel,1}^{DK} \frac{\partial \langle f_s \rangle_\varphi}{\partial v_\parallel} + \dot{\mu}_1^{DK} \frac{\partial \langle f_s \rangle_\varphi}{\partial \mu}, \tag{3.22}$$

where the coefficients are

$$\dot{\mathbf{r}}_1^{DK} = v_B \hat{\mathbf{b}} + \mathbf{v}_\kappa + \mathbf{v}_{\nabla B}, \tag{3.23}$$

$$\dot{v}_{\parallel,1}^{DK} = -\frac{v_\parallel \mu}{\Omega_s} (\hat{\mathbf{b}} \times \boldsymbol{\kappa}) \cdot \nabla B - \overline{\mu}_1 \hat{\mathbf{b}} \cdot \nabla B - v_\parallel \hat{\mathbf{b}} \cdot \nabla v_B, \tag{3.24}$$

$$\dot{\mu}_1^{DK} = -v_\parallel \hat{\mathbf{b}} \cdot \nabla \overline{\mu}_1 + \mu \hat{\mathbf{b}} \cdot \nabla B \frac{\partial \overline{\mu}_1}{\partial v_\parallel}. \tag{3.25}$$

Here

$$\mathbf{v}_\kappa = \frac{v_\parallel^2}{\Omega_s} \hat{\mathbf{b}} \times \boldsymbol{\kappa}, \quad \mathbf{v}_{\nabla B} = \frac{\mu}{\Omega_s} \hat{\mathbf{b}} \times \nabla B \tag{3.26}$$

are the curvature and ∇B drifts. They will be very important for low flow plasmas. The Baños parallel drift, $v_B = (m_s \mu / Z_s e) \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}$, and the correction to the magnetic moment $\overline{\mu}_1 = -(v_\parallel \mu / \Omega_s) \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}$ hardly ever matter. We give them here for completeness.

3.3. *Low flow drift kinetic equation*

Summing (3.6) and (3.22), we find the first order, low flow, drift kinetic equation

$$\boxed{\frac{\partial \langle f_s \rangle_\varphi}{\partial t} + \dot{\mathbf{r}}^{DK} \cdot \nabla \langle f_s \rangle_\varphi + \dot{v}_\parallel^{DK} \frac{\partial \langle f_s \rangle_\varphi}{\partial v_\parallel} + \dot{\mu}^{DK} \frac{\partial \langle f_s \rangle_\varphi}{\partial \mu} = 0,} \tag{3.27}$$

where

$$\dot{\mathbf{r}}^{DK} = \langle \dot{\mathbf{r}} \rangle_\varphi + \dot{\mathbf{r}}_1^{DK} = (v_\parallel + v_B) \hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_\kappa + \mathbf{v}_{\nabla B}, \tag{3.28}$$

$$\dot{v}_\parallel^{DK} = \langle \dot{v}_\parallel \rangle_\varphi + \dot{v}_{\parallel,1}^{DK} = \hat{\mathbf{b}} \cdot \left[\frac{Z_s e}{m_s} \mathbf{E} - (\mu + \overline{\mu}_1) \nabla B \right] + \frac{v_\parallel}{\Omega_s} (\hat{\mathbf{b}} \times \boldsymbol{\kappa}) \cdot \left(\frac{Z_s e}{m_s} \mathbf{E} - \mu \nabla B \right) - v_\parallel \hat{\mathbf{b}} \cdot \nabla v_B, \tag{3.29}$$

$$\dot{\mu}^{DK} = \langle \dot{\mu} \rangle_\varphi + \dot{\mu}_1^{DK} = -v_\parallel \hat{\mathbf{b}} \cdot \nabla \overline{\mu}_1 - \left(\frac{Z_s e}{m_s} \hat{\mathbf{b}} \cdot \mathbf{E} - \mu \hat{\mathbf{b}} \cdot \nabla B \right) \frac{\partial \overline{\mu}_1}{\partial v_\parallel}. \tag{3.30}$$

As explained at the beginning of this section, equation (3.27) is missing terms of order $\langle f_s \rangle_\varphi \rho_{s*}^2 v_{ts} / L$.

Equation (3.27) contains two new perpendicular drifts, the curvature and ∇B drifts,

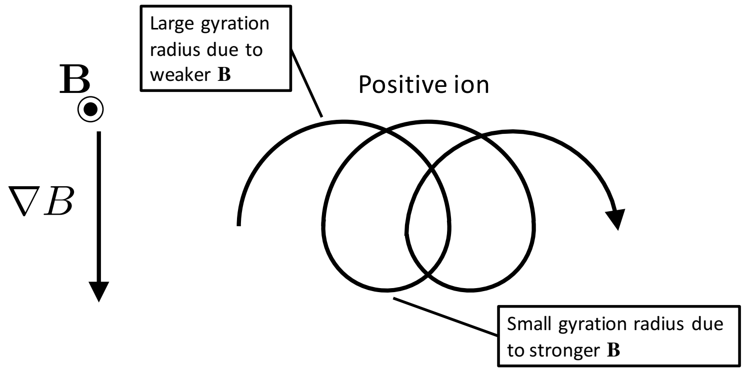


FIGURE 3. The ∇B drift is the result of the radius of gyration changing because of the magnetic field magnitude variations along the path of the particle.

the Baños parallel drift v_B, and the correction to the magnetic moment $\overline{\mu}_1$. The corrections v_B and $\overline{\mu}_1$ are rarely useful, and we discuss them in Appendix E. The curvature and ∇B drifts, $\mathbf{v}_\kappa$ and $\mathbf{v}_{\nabla B}$, are important in the low flow regime because they are comparable to the small $\mathbf{E} \times \mathbf{B}$ drift.

The curvature drift is the result of the parallel motion of the particle. The particle follows the magnetic field line, even if it is curved. When the magnetic field line curves, the particle's trajectory does as well, and that implies that a force is being applied, in this case by the magnetic field. The magnetic field requires a perpendicular velocity to exert a force. The curvature drift is the perpendicular velocity that gives the necessary magnetic force, that is, $Z_s e \mathbf{v}_\kappa \times \mathbf{B}$ is the force that turns the particle when the magnetic field turns. A different way to understand the force is to move with the particle along the magnetic field line. In this frame, the particle is feeling the centrifugal force shown in figure 1. This force accelerates and decelerates the particle in its gyration, and consequently, it changes the radius of gyration. The net result of these changes in the radius of gyration is the curvature drift, as shown in figure 2.

The ∇B drift is the result of the radius of gyration changing along the path of the particle. Due to the gradient in the magnitude of the magnetic field, the magnetic force is smaller in one half of the orbit than in the other half. The region with smaller magnetic field will have a larger radius of gyration, whereas the region with larger magnetic field will have a smaller radius of gyration. The net result of these changes in the radius of gyration is the ∇B drift, as shown in figure 3.

3.4. *Conservative low flow drift kinetic equation*

Equation (3.27) can be written in conservative form. According to (3.6) and (3.20), the coefficients $\dot{\mathbf{r}}^{DK}$, $\dot{v}_\parallel^{DK}$ and $\dot{\mu}^{DK}$ are

$$\dot{Q}_i^{DK} = \langle \dot{Q}_i \rangle_\varphi + \dot{Q}_{i,1}^{DK} = \langle \dot{Q}_i \rangle_\varphi + \frac{1}{B} \frac{\partial}{\partial Q_j} \left(\frac{B\Omega_s}{2} \left\langle \widetilde{Q}_{j,1} \frac{\partial \widetilde{Q}_{i,1}}{\partial \varphi} - \widetilde{Q}_{i,1} \frac{\partial \widetilde{Q}_{j,1}}{\partial \varphi} \right\rangle_\varphi \right). \quad (3.31)$$

Since $\partial^2/\partial Q_i \partial Q_j$ is symmetric, we find

$$\frac{\partial}{\partial Q_i} (B \dot{Q}_{i,1}^{DK}) = \frac{\partial^2}{\partial Q_i \partial Q_j} \left(\frac{B\Omega_s}{2} \left\langle \widetilde{Q}_{j,1} \frac{\partial \widetilde{Q}_{i,1}}{\partial \varphi} - \widetilde{Q}_{i,1} \frac{\partial \widetilde{Q}_{j,1}}{\partial \varphi} \right\rangle_\varphi \right) = 0, \quad (3.32)$$

and hence

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial Q_i}(B\dot{Q}_i^{DK}) = \frac{\partial B}{\partial t} + \frac{\partial}{\partial Q_i}(B\langle\dot{Q}_i\rangle_\varphi). \tag{3.33}$$

Equation (2.54) implies $\partial B/\partial t + \partial(B\langle\dot{Q}_i\rangle_\varphi)/\partial Q_i = 0$, leading to

$$\frac{\partial B}{\partial t} + \nabla\cdot(B\dot{\mathbf{r}}^{KD}) + \frac{\partial}{\partial v_\|}(B\dot{v}_\|^{DK}) + \frac{\partial}{\partial \mu}(B\dot{\mu}^{DK}) = 0. \tag{3.34}$$

With equations (3.34) and (3.27), we obtain

$$\boxed{\frac{\partial}{\partial t}(B\langle f_s\rangle_\varphi) + \nabla\cdot(B\dot{\mathbf{r}}^{DK}\langle f_s\rangle_\varphi) + \frac{\partial}{\partial v_\|}(B\dot{v}_\|^{DK}\langle f_s\rangle_\varphi) + \frac{\partial}{\partial \mu}(B\dot{\mu}^{DK}\langle f_s\rangle_\varphi) = 0.} \tag{3.35}$$

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Appendix A. Change of coordinates of a divergence

In this appendix, we prove that a change from coordinates $\mathbf{X}$ to coordinates $\mathbf{Q}$ changes expression (2.20) to (2.21). We consider a general n-dimensional space (in the main text, the space has seven dimensions: time, three spatial dimensions and three velocity dimensions).

Using the chain rule in (2.20), we find

$$\frac{\partial V_i}{\partial X_i} = \frac{\partial Q_j}{\partial X_i}\frac{\partial V_i}{\partial Q_j} = 0. \tag{A 1}$$

To show that this equation gives (2.21), we need to prove

$$\frac{\partial}{\partial Q_j}\left[\det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{Q}}\right)\frac{\partial Q_j}{\partial X_i}\right] = 0. \tag{A 2}$$

The left side of this expression can be written as

$$\frac{\partial}{\partial Q_j}\left[\det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{Q}}\right)\frac{\partial Q_j}{\partial X_i}\right] = \frac{\partial}{\partial Q_j}\left[\det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{Q}}\right)\right]\frac{\partial Q_j}{\partial X_i} + \det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{Q}}\right)\frac{\partial}{\partial Q_j}\left(\frac{\partial Q_j}{\partial X_i}\right). \tag{A 3}$$

We proceed to evaluate these two terms.

To evaluate the derivative of the determinant of the Jacobian, we make it explicit that the Jacobian is composed of n row vectors,

$$\det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{Q}}\right) = \det\left(\frac{\partial \mathbf{X}}{\partial Q_1}, \frac{\partial \mathbf{X}}{\partial Q_2}, \ldots, \frac{\partial \mathbf{X}}{\partial Q_{n-1}}, \frac{\partial \mathbf{X}}{\partial Q_n}\right). \tag{A 4}$$

Considering the determinant as a linear function of each of the rows of the matrix within the determinant, the derivative of the determinant can be written as a sum of n terms,

$$\begin{aligned}\frac{\partial}{\partial Q_j}\left[\det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{Q}}\right)\right] &= \det\left(\frac{\partial^2 \mathbf{X}}{\partial Q_j \partial Q_1}, \frac{\partial \mathbf{X}}{\partial Q_2}, \ldots, \frac{\partial \mathbf{X}}{\partial Q_{n-1}}, \frac{\partial \mathbf{X}}{\partial Q_n}\right) \\ &+ \det\left(\frac{\partial \mathbf{X}}{\partial Q_1}, \frac{\partial^2 \mathbf{X}}{\partial Q_j \partial Q_2}, \ldots, \frac{\partial \mathbf{X}}{\partial Q_{n-1}}, \frac{\partial \mathbf{X}}{\partial Q_n}\right) \\ &+ \ldots + \det\left(\frac{\partial \mathbf{X}}{\partial Q_1}, \frac{\partial \mathbf{X}}{\partial Q_2}, \ldots, \frac{\partial \mathbf{X}}{\partial Q_{n-1}}, \frac{\partial^2 \mathbf{X}}{\partial Q_j \partial Q_n}\right).\end{aligned} \tag{A 5}$$

We define the family of linear operators

$$L_k(\mathbf{a}) \equiv l_{km} a_m = \det\left(\frac{\partial \mathbf{X}}{\partial Q_1}, \frac{\partial \mathbf{X}}{\partial Q_2}, \ldots, \frac{\partial \mathbf{X}}{\partial Q_{k-1}}, \underbrace{\mathbf{a}}_{k-\text{th row}}, \frac{\partial \mathbf{X}}{\partial Q_{k+1}}, \ldots, \frac{\partial \mathbf{X}}{\partial Q_{n-1}}, \frac{\partial \mathbf{X}}{\partial Q_n}\right) \tag{A 6}$$

to rewrite (A 5) as

$$\begin{aligned}\frac{\partial}{\partial Q_j}\left[\det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{Q}}\right)\right] &= L_1\left(\frac{\partial^2 \mathbf{X}}{\partial Q_j \partial Q_1},\right) + L_2\left(\frac{\partial^2 \mathbf{X}}{\partial Q_j \partial Q_2}\right) + \ldots + L_n\left(\frac{\partial^2 \mathbf{X}}{\partial Q_j \partial Q_n}\right) \\ &= L_k\left(\frac{\partial^2 \mathbf{X}}{\partial Q_j \partial Q_k}\right) \equiv l_{km}\frac{\partial^2 X_m}{\partial Q_j \partial Q_k}.\end{aligned} \tag{A 7}$$

The operator $L_k(\mathbf{a})$ satisfies

$$L_k\left(\frac{\partial \mathbf{X}}{\partial Q_p}\right) \equiv l_{km}\frac{\partial X_m}{\partial Q_p} = \det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{Q}}\right)\delta_{kp} \tag{A 8}$$

because for $k = p$, we obtain the determinant of the Jacobian, and for $k \neq p$, we repeat one of the rows of the matrix within the determinant, and the result is zero. Since the vectors $\{\partial\mathbf{X}/\partial Q_1, \partial\mathbf{X}/\partial Q_2, \ldots, \partial\mathbf{X}/\partial Q_n\}$ form a basis of the n-dimensional space, condition (A 8) is sufficient to determine the operators $L_k(\mathbf{a})$. Based on (A 8), we find that

$$L_k(\mathbf{a}) \equiv l_{km} a_m = \det\left(\frac{\partial\mathbf{X}}{\partial\mathbf{Q}}\right) \frac{\partial Q_k}{\partial X_m} a_m \tag{A 9}$$

because

$$\frac{\partial Q_k}{\partial X_m} \frac{\partial X_m}{\partial Q_p} = \delta_{kp}. \tag{A 10}$$

Then,

$$l_{km} = \det\left(\frac{\partial\mathbf{X}}{\partial\mathbf{Q}}\right) \frac{\partial Q_k}{\partial X_m}. \tag{A 11}$$

Substituting (A 11) into (A 7), we find

$$\frac{\partial}{\partial Q_j}\left[\det\left(\frac{\partial\mathbf{X}}{\partial\mathbf{Q}}\right)\right] = \det\left(\frac{\partial\mathbf{X}}{\partial\mathbf{Q}}\right) \frac{\partial Q_k}{\partial X_m} \frac{\partial^2 X_m}{\partial Q_j \partial Q_k}. \tag{A 12}$$

We proceed to calculate the term

$$\frac{\partial}{\partial Q_j}\left(\frac{\partial Q_j}{\partial X_i}\right). \tag{A 13}$$

Differentiating (A 10) with respect to Q_k, we find

$$\frac{\partial}{\partial Q_k}\left(\frac{\partial Q_k}{\partial X_m}\right) \frac{\partial X_m}{\partial Q_p} + \frac{\partial Q_k}{\partial X_m} \frac{\partial^2 X_m}{\partial Q_p \partial Q_k} = 0. \tag{A 14}$$

Multiplying by $\partial Q_p/\partial X_j$, and using (A 10) again, we find

$$\frac{\partial}{\partial Q_k}\left(\frac{\partial Q_k}{\partial X_j}\right) = -\frac{\partial Q_k}{\partial X_m} \frac{\partial Q_p}{\partial X_j} \frac{\partial^2 X_m}{\partial Q_p \partial Q_k}. \tag{A 15}$$

Using equations (A 12) and (A 15), equation (A 3) becomes

$$\frac{\partial}{\partial Q_j}\left[\det\left(\frac{\partial\mathbf{X}}{\partial\mathbf{Q}}\right) \frac{\partial Q_j}{\partial X_i}\right] = \det\left(\frac{\partial\mathbf{X}}{\partial\mathbf{Q}}\right)\left(\frac{\partial Q_k}{\partial X_m} \frac{\partial^2 X_m}{\partial Q_j \partial Q_k} \frac{\partial Q_j}{\partial X_i} - \frac{\partial Q_k}{\partial X_m} \frac{\partial Q_p}{\partial X_i} \frac{\partial^2 X_m}{\partial Q_p \partial Q_k}\right) = 0, \tag{A 16}$$

proving (A 2).

Using (A 1) and (A 2), we recover (2.21).

Appendix B. Derivation of the gyrophase dependent piece $\tilde{f}_{s,1}$

Integrating equation (2.34), we obtain (2.37) with

$$\tilde{\mathbf{r}}_1 = \frac{1}{\Omega_s} \int^{\varphi} \widetilde{\mathbf{r}}(\varphi')\, \mathrm{d}\varphi', \quad \tilde{v}_{\|,1} = \frac{1}{\Omega_s} \int^{\varphi} \widetilde{v}_{\|}(\varphi')\, \mathrm{d}\varphi', \quad \tilde{\mu}_1 = \frac{1}{\Omega_s} \int^{\varphi} \widetilde{\mu}(\varphi')\, \mathrm{d}\varphi', \tag{B 1}$$

where the indefinite integrals are taken such that $\langle \tilde{\mathbf{r}}_1 \rangle_\varphi = 0$, $\langle \tilde{v}_{\|,1} \rangle_\varphi = 0$ and $\langle \tilde{\mu}_1 \rangle_\varphi = 0$. We proceed to find the indefinite integrals.

To obtain the functions $\tilde{\mathbf{r}}_1$, $\tilde{v}_{\|,1}$ and $\tilde{\mu}_1$, we need the functions $\widetilde{\dot{\mathbf{r}}}$, $\widetilde{\dot{v}}_\|$ and $\widetilde{\dot{\mu}}$, given by

$$\widetilde{\dot{\mathbf{r}}} = \mathbf{w}_\perp, \tag{B 2}$$

$$\widetilde{\dot{w}}_\| = \left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\| \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} + \nabla \hat{\mathbf{b}} \cdot \mathbf{v}_E \right) \cdot \mathbf{w}_\perp + (\mathbf{w}_\perp \mathbf{w}_\perp - \langle \mathbf{w}_\perp \mathbf{w}_\perp \rangle_\varphi) : \nabla \hat{\mathbf{b}}, \tag{B 3}$$

$$\widetilde{\dot{\mu}} = -\frac{1}{B} \left[\mu \nabla B + v_\| \left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\| \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \right) + \frac{\partial \mathbf{v}_E}{\partial t} + (v_\| \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \mathbf{v}_E \right] \cdot \mathbf{w}_\perp$$
$$- \frac{1}{B} (\mathbf{w}_\perp \mathbf{w}_\perp - \langle \mathbf{w}_\perp \mathbf{w}_\perp \rangle_\varphi) : \nabla (v_\| \hat{\mathbf{b}} + \mathbf{v}_E). \tag{B 4}$$

Before integrating over φ, we rewrite $\mathbf{w}_\perp \mathbf{w}_\perp - \langle \mathbf{w}_\perp \mathbf{w}_\perp \rangle_\varphi$ in a convenient form. We first note that using (2.15), we can write

$$\mathbf{w}_\perp \mathbf{w}_\perp = 2\mu B [\cos^2 \varphi \, \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 - \sin \varphi \cos \varphi (\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_1) + \sin^2 \varphi \, \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2] \tag{B 5}$$

and

$$(\hat{\mathbf{b}} \times \mathbf{w}_\perp)(\hat{\mathbf{b}} \times \mathbf{w}_\perp) = 2\mu B [\sin^2 \varphi \, \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \sin \varphi \cos \varphi (\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_1) + \cos^2 \varphi \, \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2]. \tag{B 6}$$

In the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{b}}\}$, the tensors $\mathbf{w}_\perp \mathbf{w}_\perp$ and $(\hat{\mathbf{b}} \times \mathbf{w}_\perp)(\hat{\mathbf{b}} \times \mathbf{w}_\perp)$ are the matrices

$$\mathbf{w}_\perp \mathbf{w}_\perp = 2\mu B \begin{pmatrix} \cos^2 \varphi & -\sin \varphi \cos \varphi & 0 \\ -\sin \varphi \cos \varphi & \sin^2 \varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{B 7}$$

and

$$(\hat{\mathbf{b}} \times \mathbf{w}_\perp)(\hat{\mathbf{b}} \times \mathbf{w}_\perp) = 2\mu B \begin{pmatrix} \sin^2 \varphi & \sin \varphi \cos \varphi & 0 \\ \sin \varphi \cos \varphi & \cos^2 \varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B 8}$$

Summing (B 5) and (B 6), we obtain

$$\mathbf{w}_\perp \mathbf{w}_\perp + (\hat{\mathbf{b}} \times \mathbf{w}_\perp)(\hat{\mathbf{b}} \times \mathbf{w}_\perp) = 2\mu B (\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2) = 2\mu B \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= 2\mu B (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}). \tag{B 9}$$

Using this result and the fact that

$$\langle \mathbf{w}_\perp \mathbf{w}_\perp \rangle_\varphi = \mu B (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}), \tag{B 10}$$

we find

$$\langle \mathbf{w}_\perp \mathbf{w}_\perp \rangle_\varphi = \frac{1}{2} [\mathbf{w}_\perp \mathbf{w}_\perp + (\hat{\mathbf{b}} \times \mathbf{w}_\perp)(\hat{\mathbf{b}} \times \mathbf{w}_\perp)] \tag{B 11}$$

and

$$\mathbf{w}_\perp \mathbf{w}_\perp - \langle \mathbf{w}_\perp \mathbf{w}_\perp \rangle_\varphi = \frac{1}{2} [\mathbf{w}_\perp \mathbf{w}_\perp - (\hat{\mathbf{b}} \times \mathbf{w}_\perp)(\hat{\mathbf{b}} \times \mathbf{w}_\perp)]$$
$$= \mu B [\cos 2\varphi \, \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 - \sin 2\varphi (\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_1) - \cos 2\varphi \, \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2]$$
$$= \mu B \begin{pmatrix} \cos 2\varphi & -\sin 2\varphi & 0 \\ -\sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B 12}$$

The form (B 12) is useful because it is easy to integrate. Using (2.15), we find

$$\frac{\partial}{\partial \varphi}(\hat{\mathbf{b}} \times \mathbf{w}_\perp) = \mathbf{w}_\perp \tag{B 13}$$

and

$$\frac{\partial \mathbf{w}_\perp}{\partial \varphi} = -\hat{\mathbf{b}} \times \mathbf{w}_\perp. \tag{B 14}$$

Using these two expressions, we find

$$\frac{\partial}{\partial \varphi}[\mathbf{w}_\perp(\hat{\mathbf{b}} \times \mathbf{w}_\perp) + (\hat{\mathbf{b}} \times \mathbf{w}_\perp)\mathbf{w}_\perp] = 2[\mathbf{w}_\perp\mathbf{w}_\perp - (\hat{\mathbf{b}} \times \mathbf{w}_\perp)(\hat{\mathbf{b}} \times \mathbf{w}_\perp)]. \tag{B 15}$$

Employing (B 12), (B 13) and (B 15), we find

$$\int \mathbf{w}_\perp \, \mathrm{d}\varphi = \hat{\mathbf{b}} \times \mathbf{w}_\perp \tag{B 16}$$

and

$$\int (\mathbf{w}_\perp\mathbf{w}_\perp - \langle \mathbf{w}_\perp\mathbf{w}_\perp \rangle_\varphi) \, \mathrm{d}\varphi = \frac{1}{4}[\mathbf{w}_\perp(\hat{\mathbf{b}} \times \mathbf{w}_\perp) + (\hat{\mathbf{b}} \times \mathbf{w}_\perp)\mathbf{w}_\perp], \tag{B 17}$$

and we can integrate (B 2), (B 3) and (B 4) in φ to obtain (2.38), (2.39) and (2.40).

Appendix C. Derivation of $\langle \dot{\mu} \rangle_\varphi$

In this appendix, we derive $\langle \dot{\mu} \rangle_\varphi$ from (2.13). From (2.13), we obtain

$$\langle \dot{\mu} \rangle_\varphi = -\frac{\mu}{B}\left[\frac{\partial B}{\partial t} + (v_\| \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla B\right] - \frac{v_\|}{B}\langle \mathbf{w}_\perp\mathbf{w}_\perp \rangle_\varphi : \nabla \hat{\mathbf{b}} - \frac{1}{B}\langle \mathbf{w}_\perp\mathbf{w}_\perp \rangle_\varphi : \nabla \mathbf{v}_E. \tag{C 1}$$

Using (B 10), $\mathbf{I} : \nabla \hat{\mathbf{b}} = \nabla \cdot \hat{\mathbf{b}}$ and $\mathbf{I} : \nabla \mathbf{v}_E = \nabla \cdot \mathbf{v}_E$, equation (C 1) becomes

$$\langle \dot{\mu} \rangle_\varphi = -\frac{\mu}{B}\left[\frac{\partial B}{\partial t} + (v_\| \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla B\right] - v_\| \mu \nabla \cdot \hat{\mathbf{b}} - \mu(\nabla \cdot \mathbf{v}_E - \hat{\mathbf{b}} \cdot \nabla \mathbf{v}_E \cdot \hat{\mathbf{b}}). \tag{C 2}$$

Using $\nabla \cdot \mathbf{B} = 0$ to write

$$B \nabla \cdot \hat{\mathbf{b}} = -\hat{\mathbf{b}} \cdot \nabla B, \tag{C 3}$$

we rewrite equation (C 2) as

$$\langle \dot{\mu} \rangle_\varphi = -\frac{\mu}{B}\left(\frac{\partial B}{\partial t} + \mathbf{v}_E \cdot \nabla B\right) - \mu(\nabla \cdot \mathbf{v}_E - \hat{\mathbf{b}} \cdot \nabla \mathbf{v}_E \cdot \hat{\mathbf{b}}). \tag{C 4}$$

To simplify equation (C 4) further, we use that

$$\mathbf{E} + \mathbf{v}_E \times \mathbf{B} = E_\| \hat{\mathbf{b}}. \tag{C 5}$$

Taking the curl of this equation, we obtain

$$\nabla \times \mathbf{E} + \nabla \times (\mathbf{v}_E \times \mathbf{B}) = \nabla \times (E_\| \hat{\mathbf{b}}). \tag{C 6}$$

Using Faraday's induction law $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, and

$$\nabla \times (\mathbf{v}_E \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{v}_E - (\nabla \cdot \mathbf{v}_E)\mathbf{B} - \mathbf{v}_E \cdot \nabla \mathbf{B}, \tag{C 7}$$

equation (C 6) becomes

$$\mathbf{B} \cdot \nabla \mathbf{v}_E - (\nabla \cdot \mathbf{v}_E)\mathbf{B} = \frac{\partial \mathbf{B}}{\partial t} + \mathbf{v}_E \cdot \nabla \mathbf{B} + \nabla \times (E_\| \hat{\mathbf{b}}). \tag{C 8}$$

Projecting this equation on $\hat{\mathbf{b}}$, and using that $(\partial \mathbf{B}/\partial t)\cdot\hat{\mathbf{b}} = \partial B/\partial t$ and that $\nabla\mathbf{B}\cdot\hat{\mathbf{b}} = \nabla B$, we finally get

$$\hat{\mathbf{b}}\cdot\nabla\mathbf{v}_E\cdot\hat{\mathbf{b}} - \nabla\cdot\mathbf{v}_E = \frac{1}{B}\left(\frac{\partial B}{\partial t} + \mathbf{v}_E\cdot\nabla B\right) + \frac{E_\parallel}{B}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}. \tag{C 9}$$

Using this expression, equation (C 4) becomes

$$\langle\dot{\mu}\rangle_\varphi = \frac{E_\parallel \mu}{B}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}. \tag{C 10}$$

Since the parallel electric field is ordered as in (1.4), $\langle\dot{\mu}\rangle_\varphi \sim \rho_{s*}\mu v_{ts}/L$ and the contribution of $\langle\dot{\mu}\rangle_\varphi$ to equation (2.45) is as small as other terms that we have neglected. Then, for equation (2.45), we can use

$$\langle\dot{\mu}\rangle_\varphi \simeq 0. \tag{C 11}$$

Appendix D. Derivation of $\langle\mathcal{L}[\tilde{f}_{s,1}]\rangle_\varphi$ in the low flow regime

Equation (3.20) gives

$$\langle\mathcal{L}[\tilde{f}_{s,1}]\rangle_\varphi = \dot{\mathbf{r}}_1^{DK}\cdot\nabla\langle f_s\rangle_\varphi + \dot{v}_{\parallel,1}^{DK}\frac{\partial\langle f_s\rangle_\varphi}{\partial v_\parallel} + \dot{\mu}_1^{DK}\frac{\partial\langle f_s\rangle_\varphi}{\partial\mu}, \tag{D 1}$$

where we have defined the coefficients

$$\dot{\mathbf{r}}_1^{DK} = \frac{1}{B}\left[\nabla\times\left(\frac{B\Omega_s}{2}\left\langle\frac{\partial\tilde{\mathbf{r}}_1}{\partial\varphi}\times\tilde{\mathbf{r}}_1\right\rangle_\varphi\right) + \frac{\partial}{\partial v_\parallel}\left(\frac{B\Omega_s}{2}\left\langle\tilde{v}_{\parallel,1}\frac{\partial\tilde{\mathbf{r}}_1}{\partial\varphi} - \tilde{\mathbf{r}}_1\frac{\partial\tilde{v}_{\parallel,1}}{\partial\varphi}\right\rangle_\varphi\right)\right.$$
$$\left. + \frac{\partial}{\partial\mu}\left(\frac{B\Omega_s}{2}\left\langle\tilde{\mu}_1\frac{\partial\tilde{\mathbf{r}}_1}{\partial\varphi} - \tilde{\mathbf{r}}_1\frac{\partial\tilde{\mu}_1}{\partial\varphi}\right\rangle_\varphi\right)\right], \tag{D 2}$$

$$\dot{v}_{\parallel,1}^{DK} = \frac{1}{B}\left[\nabla\cdot\left(\frac{B\Omega_s}{2}\left\langle\tilde{\mathbf{r}}_1\frac{\partial\tilde{v}_{\parallel,1}}{\partial\varphi} - \tilde{v}_{\parallel,1}\frac{\partial\tilde{\mathbf{r}}_1}{\partial\varphi}\right\rangle_\varphi\right) + \frac{\partial}{\partial\mu}\left(\frac{B\Omega_s}{2}\left\langle\tilde{\mu}_1\frac{\partial\tilde{v}_{\parallel,1}}{\partial\varphi} - \tilde{v}_{\parallel,1}\frac{\partial\tilde{\mu}_1}{\partial\varphi}\right\rangle_\varphi\right)\right], \tag{D 3}$$

$$\dot{\mu}_1^{DK} = \frac{1}{B}\left[\nabla\cdot\left(\frac{B\Omega_s}{2}\left\langle\tilde{\mathbf{r}}_1\frac{\partial\tilde{\mu}_1}{\partial\varphi} - \tilde{\mu}_1\frac{\partial\tilde{\mathbf{r}}_1}{\partial\varphi}\right\rangle_\varphi\right) + \frac{\partial}{\partial v_\parallel}\left(\frac{B\Omega_s}{2}\left\langle\tilde{v}_{\parallel,1}\frac{\partial\tilde{\mu}_1}{\partial\varphi} - \tilde{\mu}_1\frac{\partial\tilde{v}_{\parallel,1}}{\partial\varphi}\right\rangle_\varphi\right)\right]. \tag{D 4}$$

We use $\tilde{\mathbf{r}}_1$ in (2.38), $\tilde{v}_{\parallel,1}$ in (3.11) and $\tilde{\mu}_1$ in (3.12) to calculate $\dot{\mathbf{r}}_1^{DK}$, $\dot{v}_{\parallel,1}^{DK}$ and $\dot{\mu}_1^{DK}$. Employing

$$\left\langle[\mathbf{w}_\perp(\hat{\mathbf{b}}\times\mathbf{w}_\perp) + (\hat{\mathbf{b}}\times\mathbf{w}_\perp)\mathbf{w}_\perp]:\nabla\hat{\mathbf{b}}\,\frac{\partial}{\partial\varphi}\left\{[\mathbf{w}_\perp(\hat{\mathbf{b}}\times\mathbf{w}_\perp) + (\hat{\mathbf{b}}\times\mathbf{w}_\perp)\mathbf{w}_\perp]:\nabla\hat{\mathbf{b}}\right\}\right\rangle_\varphi$$
$$= \frac{1}{2}\left\langle\frac{\partial}{\partial\varphi}\left\{[\mathbf{w}_\perp(\hat{\mathbf{b}}\times\mathbf{w}_\perp) + (\hat{\mathbf{b}}\times\mathbf{w}_\perp)\mathbf{w}_\perp]:\nabla\hat{\mathbf{b}}\right\}^2\right\rangle_\varphi = 0, \tag{D 5}$$

we obtain

$$\frac{B\Omega_s}{2}\left(\frac{\partial\tilde{\mathbf{r}}_1}{\partial\varphi}\times\tilde{\mathbf{r}}_1\right) = \frac{m_s}{2Z_s e}\mathbf{w}_\perp\times(\hat{\mathbf{b}}\times\mathbf{w}_\perp) = \frac{m_s|\mathbf{w}_\perp|^2}{2Z_s e}\hat{\mathbf{b}} = \frac{m_s\mu B}{Z_s e}\hat{\mathbf{b}}, \tag{D 6}$$

$$B\Omega_s\left\langle \tilde{v}_{\parallel,1}\frac{\partial\tilde{\mathbf{r}}_1}{\partial\varphi}\right\rangle_\varphi = -B\Omega_s\left\langle \tilde{\mathbf{r}}_1\frac{\partial\tilde{v}_{\parallel,1}}{\partial\varphi}\right\rangle_\varphi = -\frac{m_s v_\parallel \mu B}{Z_s e}\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}), \tag{D 7}$$

$$B\Omega_s\left\langle \tilde{\mu}_1\frac{\partial\tilde{\mathbf{r}}_1}{\partial\varphi}\right\rangle_\varphi = -B\Omega_s\left\langle \tilde{\mathbf{r}}_1\frac{\partial\tilde{\mu}_1}{\partial\varphi}\right\rangle_\varphi = \frac{m_s \mu}{Z_s e}\hat{\mathbf{b}}\times(v_\parallel^2\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}+\mu\nabla B) \tag{D 8}$$

and

$$B\Omega_s\left\langle \tilde{\mu}_1\frac{\partial\tilde{v}_{\parallel,1}}{\partial\varphi}\right\rangle_\varphi = -B\Omega_s\left\langle \tilde{v}_{\parallel,1}\frac{\partial\tilde{\mu}_1}{\partial\varphi}\right\rangle_\varphi = -\frac{m_s v_\parallel \mu^2}{Z_s e}[\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})]\cdot\nabla B. \tag{D 9}$$

Substituting these results into equations (D 2), (D 3) and (D 4), we get

$$\begin{aligned}
\dot{\mathbf{r}}_1^{DK} &= \frac{v_\parallel^2}{\Omega_s}\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}) + \frac{\mu}{\Omega_s}\hat{\mathbf{b}}\times\nabla B + \frac{m_s\mu}{Z_s e}[\nabla\times\hat{\mathbf{b}} - \hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})], && \text{(D 10)}\\
\dot{v}_{\parallel,1}^{DK} &= -\frac{v_\parallel\mu}{\Omega_s}[\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})]\cdot\nabla B + \frac{m_s v_\parallel\mu}{Z_s e}\nabla\cdot[\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})], && \text{(D 11)}\\
\dot{\mu}_1^{DK} &= -\frac{v_\parallel^2\mu}{\Omega_s}\nabla\cdot[\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})] - \frac{\mu^2}{\Omega_s}[\nabla\times\hat{\mathbf{b}} - \hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})]\cdot\nabla B. && \text{(D 12)}
\end{aligned}$$

Finally, using

$$\nabla\times\hat{\mathbf{b}} = \hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} + \hat{\mathbf{b}}\times[(\nabla\times\hat{\mathbf{b}})\times\hat{\mathbf{b}}] = \hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} + \hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}) \tag{D 13}$$

and

$$\begin{aligned}
\nabla\cdot[\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})] &= \nabla\cdot(\nabla\times\hat{\mathbf{b}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}) = -\nabla\cdot(\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}})\\
&= -\mathbf{B}\cdot\nabla\left(\frac{1}{B}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}\right) = \frac{1}{B}(\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}})(\hat{\mathbf{b}}\cdot\nabla B) - \hat{\mathbf{b}}\cdot\nabla(\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}), && \text{(D 14)}
\end{aligned}$$

we find

$$\begin{aligned}
\dot{\mathbf{r}}_1^{DK} &= \frac{v_\parallel^2}{\Omega_s}\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}) + \frac{\mu}{\Omega_s}\hat{\mathbf{b}}\times\nabla B + \frac{m_s\mu}{Z_s e}\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}, && \text{(D 15)}\\
\dot{v}_{\parallel,1}^{DK} &= -\frac{v_\parallel\mu}{\Omega_s}[\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}) - \hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}]\cdot\nabla B - \frac{m_s v_\parallel\mu}{Z_s e}\hat{\mathbf{b}}\cdot\nabla(\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}), && \text{(D 16)}\\
\dot{\mu}_1^{DK} &= \frac{m_s v_\parallel^2\mu}{Z_s e}\hat{\mathbf{b}}\cdot\nabla\left(\frac{1}{B}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}\right) - \frac{\mu^2}{\Omega_s}(\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}})(\hat{\mathbf{b}}\cdot\nabla B). && \text{(D 17)}
\end{aligned}$$

Substituting these values into (D 1), we obtain (3.22).

Appendix E. The Baños parallel drift and the correction to the magnetic moment

The Baños parallel drift

$$v_B = \frac{m_s\mu}{Z_s e}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} \tag{E 1}$$

and the correction to the magnetic moment

$$\overline{\mu}_1 = -\frac{v_\parallel\mu}{\Omega_s}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} \tag{E 2}$$

originate from $\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}\neq 0$. The quantity $\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}$ is proportional to the current density parallel to the magnetic field,

$$J_\parallel = \frac{1}{\mu_0}(\nabla\times\mathbf{B})\cdot\hat{\mathbf{b}} = \frac{B}{\mu_0}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}, \tag{E 3}$$

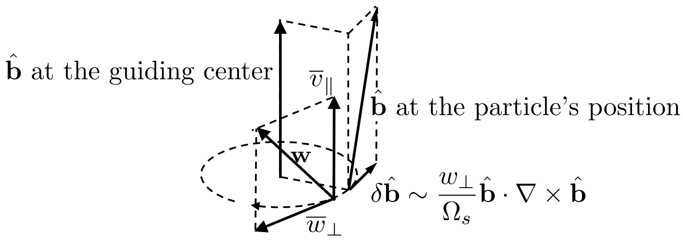

FIGURE 4. Sketch of the magnetic field direction at the position of the particle and at the position of the guiding center.

where μ_0 is the vacuum permeability. For non-zero parallel current, the circulation of $\mathbf{B}$ over any curve around the magnetic field line will be non-zero, and in particular, this implies that there is a small magnetic field component along the gyromotion that changes the direction $\hat{\mathbf{b}}$ by an amount

$$\delta\hat{\mathbf{b}} \sim \frac{\mu_0 \rho J_\parallel}{B} \sim \frac{w_\perp}{\Omega_s}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} \tag{E 4}$$

(see figure 4). This magnetic field along the gyromotion means that the direction of the magnetic field at the real position of the particle is different from the direction of the magnetic field at the guiding center. This difference is important because the parallel and perpendicular velocities should be defined with respect to the direction of the magnetic field at the guiding center. According to figure 4, the component of the velocity parallel to the magnetic field at the guiding center is

$$\overline{v}_\parallel \sim \mathbf{w}\cdot[\hat{\mathbf{b}} \text{ at particle's position} - \delta\hat{\mathbf{b}}] = v_\parallel + O\left(\frac{w_\perp^2}{\Omega_s}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}\right) \tag{E 5}$$

and the component of the velocity perpendicular to the magnetic field at the guiding center is

$$\overline{w}_\perp \sim |\mathbf{w} - \overline{v}_\parallel\hat{\mathbf{b}}| = w_\perp - O\left(\frac{v_\parallel w_\perp}{\Omega_s}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}\right). \tag{E 6}$$

From $\overline{w}_\perp$, we obtain $\overline{\mu} = \overline{w}_\perp^2/2B$. The exact definitions of $\overline{v}_\parallel$ and $\overline{\mu}$ are

$$\overline{v}_\parallel = v_\parallel + v_B = v_\parallel + \frac{m_s\mu}{Z_s e}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} \tag{E 7}$$

and

$$\overline{\mu} = \mu + \overline{\mu}_1 = \mu - \frac{v_\parallel \mu}{\Omega_s}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}. \tag{E 8}$$

Equation (3.27) is more intuitive in the new coordinates $\overline{v}_\parallel$ and $\overline{\mu}$. We transform (3.27) to these new coordinates to demonstrate it. We use the chain rule and (1.6), (3.1) and (3.3) to write

$$\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial t}\right|_{\mathbf{r},v_\parallel,\mu} = \left.\frac{\partial\langle f_s\rangle_\varphi}{\partial t}\right|_{\mathbf{r},\overline{v}_\parallel,\overline{\mu}} + \left.\frac{\partial\overline{v}_\parallel}{\partial t}\right|_{\mathbf{r},v_\parallel,\mu}\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{v}_\parallel}\right|_{\mathbf{r},\overline{\mu},t} + \left.\frac{\partial\overline{\mu}}{\partial t}\right|_{\mathbf{r},v_\parallel,\mu}\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{\mu}}\right|_{\mathbf{r},\overline{v}_\parallel,t}$$
$$= \left.\frac{\partial\langle f_s\rangle_\varphi}{\partial t}\right|_{\mathbf{r},\overline{v}_\parallel,\overline{\mu}} + O\left(\rho_{s*}^2\frac{v_{ts}}{L}\langle f_s\rangle_\varphi\right), \tag{E 9}$$

$$\nabla\langle f_s\rangle_\varphi|_{v_\parallel,\mu,t} = \nabla\langle f_s\rangle_\varphi|_{\overline{v}_\parallel,\overline{\mu},t} + \nabla\overline{v}_\parallel|_{v_\parallel,\mu,t}\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{v}_\parallel}\right|_{\mathbf{r},\overline{\mu},t} + \nabla\overline{\mu}|_{v_\parallel,\mu,t}\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{\mu}}\right|_{\mathbf{r},\overline{v}_\parallel,t}$$
$$= \nabla\langle f_s\rangle_\varphi|_{\overline{v}_\parallel,\overline{\mu},t} + \nabla v_B|_{v_\parallel,\mu,t}\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{v}_\parallel}\right|_{\mathbf{r},\overline{\mu},t} + \nabla\overline{\mu}_1|_{v_\parallel,\mu,t}\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{\mu}}\right|_{\mathbf{r},\overline{v}_\parallel,t}, \quad (E\,10)$$

$$\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial v_\parallel}\right|_{\mathbf{r},\mu,t} = \left.\frac{\partial\overline{v}_\parallel}{\partial v_\parallel}\right|_{\mathbf{r},\mu,t}\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{v}_\parallel}\right|_{\mathbf{r},\overline{\mu},t} + \left.\frac{\partial\overline{\mu}}{\partial v_\parallel}\right|_{\mathbf{r},\mu,t}\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{\mu}}\right|_{\mathbf{r},\overline{v}_\parallel,t}$$
$$= \left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{v}_\parallel}\right|_{\mathbf{r},\overline{\mu},t} + \left.\frac{\partial\overline{\mu}_1}{\partial v_\parallel}\right|_{\mathbf{r},\mu,t}\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{\mu}}\right|_{\mathbf{r},\overline{v}_\parallel,t} \quad (E\,11)$$

and

$$\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\mu}\right|_{\mathbf{r},v_\parallel,t} = \left.\frac{\partial\overline{v}_\parallel}{\partial\mu}\right|_{\mathbf{r},v_\parallel,t}\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{v}_\parallel}\right|_{\mathbf{r},\overline{\mu},t} + \left.\frac{\partial\overline{\mu}}{\partial\mu}\right|_{\mathbf{r},v_\parallel,t}\left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{\mu}}\right|_{\mathbf{r},\overline{v}_\parallel,t}$$
$$= \left.\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{\mu}}\right|_{\mathbf{r},\overline{v}_\parallel,t} + O\left(\rho_{s*}\frac{v_{ts}}{L}\langle f_s\rangle_\varphi\right). \quad (E\,12)$$

Substituting these expressions into (3.27), we find the equation

$$\boxed{\frac{\partial\langle f_s\rangle_\varphi}{\partial t} + \dot{\mathbf{r}}^{DK}\cdot\nabla\langle f_s\rangle_\varphi + \dot{\overline{v}}_\parallel^{DK}\frac{\partial\langle f_s\rangle_\varphi}{\partial\overline{v}_\parallel} = 0} \quad (E\,13)$$

for $\langle f_s\rangle_\varphi(\mathbf{r},\overline{v}_\parallel,\overline{\mu},t)$. Here,

$$\dot{\mathbf{r}}^{DK} \simeq \overline{v}_\parallel\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_\kappa + \mathbf{v}_{\nabla B}, \quad (E\,14)$$
$$\dot{\overline{v}}_\parallel^{DK} \simeq \dot{v}_\parallel^{DK} + v_\parallel\hat{\mathbf{b}}\cdot\nabla v_B \simeq \left(\hat{\mathbf{b}} + \frac{\overline{v}_\parallel}{\Omega_s}\hat{\mathbf{b}}\times\boldsymbol{\kappa}\right)\cdot\left(\frac{Z_s e}{m_s}\mathbf{E} - \overline{\mu}\nabla B\right). \quad (E\,15)$$

To prove that the coefficient in front of $\partial\langle f_s\rangle_\varphi/\partial\overline{\mu}$ vanishes to the relevant order, we have used

$$\dot{\mu}^{DK} + v_\parallel\hat{\mathbf{b}}\cdot\nabla\overline{\mu}_1 + \left(\frac{Z_s e}{m_s}\hat{\mathbf{b}}\cdot\mathbf{E} - \mu\hat{\mathbf{b}}\cdot\nabla B\right)\frac{\partial\overline{\mu}_1}{\partial v_\parallel} = 0. \quad (E\,16)$$