Electrostatic drift kinetics and drift waves

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1. Introduction

In these notes, we study the simplest kinetic model for a magnetized plasma: electrostatic drift kinetics. In the electrostatic limit, the magnetic field is assumed to be determined by very strong external magnets. The plasma energy is small compared to the magnetic energy, and the plasma is unable to modify the magnetic field. Thus, in addition to assuming that the plasma is quasineutral, $\lambda_D/L \ll 1$ and $\omega/\omega_{pe} \ll 1$, and that the plasma is magnetized, $\rho_{s*} \ll 1$ and $\omega/\Omega_s \ll 1$, the electrostatic limit is characterized by

$$\beta = \frac{2\mu_0 p}{B^2} \ll 1. \tag{1.1}$$

Here $\lambda_D = \sqrt{\epsilon_0 T_e/e^2 n_e}$ is the Debye length, $\omega_p = \sqrt{e^2 n_e/\epsilon_0 m_e}$ is the plasma frequency, ϵ_0 and μ_0 are the vacuum permittivity and permeability, and $p = \sum_s p_s = \sum_s n_s m_s v_{ts}^2/2$ is the plasma pressure.

To show that $\beta \ll 1$ leads to electrostatic drift kinetics, we need to discuss the typical time and length scales of the problem. We will show that the electric field is well described by an electrostatic potential ϕ, $\mathbf{E} \simeq -\nabla\phi$. Since the electrostatic potential ϕ will be generated by the plasma, we expect it to be comparable to the characteristic plasma energy, that is, $e\phi/T \sim 1$, and hence

$$|\mathbf{E}| = |\nabla\phi| \sim \frac{T}{eL}. \tag{1.2}$$

Thus, the most natural ordering for electrostatic drift kinetics is the low flow regime, although there are situations in which an electrostatic electric field can be in the high flow regime. In the low flow regime, the particle motion is split into the fast parallel motion, with $v_\| \sim v_{ts}$, and the slow perpendicular drifts $\mathbf{v}_E + \mathbf{v}_\kappa + \mathbf{v}_{\nabla B} \sim \rho_{s*} v_{ts}$. We are interested in phenomena related to the drifts, that is, we will study phenomena with the characteristic frequency

$$\omega \sim \frac{|\mathbf{v}_E + \mathbf{v}_\kappa + \mathbf{v}_{\nabla B}|}{L} \sim \rho_{i*}\frac{v_{ti}}{L}, \tag{1.3}$$

where we are assuming that the main ions $s = i$ are the dominant species. Importantly, the difference between the perpendicular and parallel velocities leads to a separation between the characteristic parallel length $L_\| \sim |\hat{\mathbf{b}} \cdot \nabla \ln f_s|^{-1} \sim |\hat{\mathbf{b}} \cdot \nabla \ln \phi|^{-1}$ and the characteristic perpendicular length $L \sim |\nabla_\perp \ln f_s|^{-1} \sim |\nabla_\perp \ln \phi|^{-1}$ of the distribution functions and the potential. Imposing that $v_\| \hat{\mathbf{b}} \cdot \nabla f_s \sim (\mathbf{v}_E + \mathbf{v}_\kappa + \mathbf{v}_{\nabla B}) \cdot \nabla f_s$, we find

$$\frac{L}{L_\|} \sim \frac{|\mathbf{v}_E + \mathbf{v}_\kappa + \mathbf{v}_{\nabla B}|}{v_\|} \sim \rho_{i*} \ll 1. \tag{1.4}$$

Thus, $L_\|$ is typically much larger than L. Note that $L_\|$ is the characteristic parallel length of the potential and the distribution functions, and not of the magnetic field $\mathbf{B}$. The shape of $\mathbf{B}$ is determined by external magnets.

We have explained above that for low β, the magnetic field is not generated by the plasma, but by external sources. Indeed, the ratio between the magnetic field generated by currents in the plasma, $\mathbf{B}^p$, and the total magnetic field, $\mathbf{B}$, is of the order of the parameter β, $B^p/B \sim \beta \ll 1$, as we proceed to show. We start from Ampere's law,

$$\nabla \times \mathbf{B}^p = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{1.5}$$

We can estimate the size of the current density of the plasma, $\mathbf{J}$, from the size of the perpendicular current $\mathbf{J}_\perp$. Using that $f_s = \langle f_s \rangle_\varphi + \tilde{f}_s$, with $\tilde{f}_s \sim \rho_{s*} f_s$, we find

$$\begin{aligned} \mathbf{J} \sim \mathbf{J}_\perp &= \sum_s Z_s e \int f_s \mathbf{v}_\perp \,\mathrm{d}^3 v = \sum Z_s e \int f_s (\mathbf{w}_\perp + \mathbf{v}_E)\,\mathrm{d}^3 v \\ &= \sum_s Z_s e \int 2\pi B \langle \tilde{f}_s \mathbf{w}_\perp \rangle_\varphi \,\mathrm{d}v_\| \,\mathrm{d}\mu + \cancelto{0 \text{ due to quasineutrality}}{\sum_s Z_s e n_s \mathbf{v}_E} \sim e n_e \rho_{i*} v_{ti} \sim \frac{p}{BL}. \end{aligned} \tag{1.6}$$

The displacement current $c^{-2}(\partial \mathbf{E}/\partial t)$ is small compared to this current. Using that the speed of light is given by $c^2 = (\epsilon_0 \mu_0)^{-1}$, and employing $\omega \sim \rho_{i*} v_{ti}/L$ and $|\mathbf{E}| \sim T/eL$, we find

$$\frac{c^{-2}|\partial \mathbf{E}/\partial t|}{\mu_0 |\mathbf{J}|} \sim \frac{\lambda_D^2}{L^2} \ll 1 \tag{1.7}$$

Then, we can safely neglect the displacement current compared to the plasma current in Ampere's equation (1.5), leading to

$$\frac{B^p}{B} \sim \frac{\mu_0 |\mathbf{J}| L}{B} \sim \beta \ll 1, \tag{1.8}$$

as predicted above. The magnetic field is mostly generated by currents external to the plasma, giving

$$\nabla \times \mathbf{B} \simeq 0 \tag{1.9}$$

within the plasma. Since the plasma is unable to modify the background magnetic field, we consider $\mathbf{B}$ a given field that we do not need to determine.

We finish by showing that the electric field is indeed electrostatic for $\beta \ll 1$. In general, the electric field has an electrostatic piece and an inductive piece,

$$\mathbf{E} = -\nabla\phi + \mathbf{E}^A, \tag{1.10}$$

where $\mathbf{E}^A$ is the inductive electric field. For external magnetic fields with no time dependence, Faraday's law implies

$$\nabla \times \mathbf{E}^A = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial \mathbf{B}^p}{\partial t} \sim \beta \frac{\rho_{i*} v_{ti} B}{L} \sim \beta \frac{T}{eL^2}, \tag{1.11}$$

leading to an inductive piece of the electric field, $\mathbf{E}^A$, of order

$$|\mathbf{E}^A| \sim \beta \frac{T}{eL} \ll |\nabla\phi| \sim \frac{T}{eL}. \tag{1.12}$$

proving that the electric field is electrostatic.

2. Electrostatic drift kinetics

In the electrostatic limit, the low flow drift kinetic equation becomes

$$\frac{\partial \langle f_s \rangle_\varphi}{\partial t} + \dot{\mathbf{r}}^{DK} \cdot \nabla \langle f_s \rangle_\varphi + \dot{v}_\parallel^{DK} \frac{\partial \langle f_s \rangle_\varphi}{\partial v_\parallel} + \dot{\mu}^{DK} \frac{\partial \langle f_s \rangle_\varphi}{\partial \mu} = 0, \tag{2.1}$$

where

$$\dot{\mathbf{r}}^{DK} = (v_\parallel + v_B)\hat{\mathbf{b}} - \frac{1}{B}\nabla\phi \times \hat{\mathbf{b}} + \mathbf{v}_\kappa + \mathbf{v}_{\nabla B}, \tag{2.2}$$

$$\dot{v}_\parallel^{DK} = -\hat{\mathbf{b}} \cdot \left[\frac{Z_s e}{m_s}\nabla\phi + (\mu + \overline{\mu}_1)\nabla B \right] - \frac{v_\parallel}{\Omega_s}(\hat{\mathbf{b}} \times \boldsymbol{\kappa}) \cdot \left(\frac{Z_s e}{m_s}\nabla\phi + \mu\nabla B \right) - v_\parallel \hat{\mathbf{b}} \cdot \nabla v_B, \tag{2.3}$$

$$\dot{\mu}^{DK} = -v_\parallel \hat{\mathbf{b}} \cdot \nabla\overline{\mu}_1 + \hat{\mathbf{b}} \cdot \left(\frac{Z_s e}{m_s}\nabla\phi + \mu\nabla B \right) \frac{\partial \overline{\mu}_1}{\partial v_\parallel}. \tag{2.4}$$

Here $\mathbf{v}_\kappa = (v_\parallel^2/\Omega_s)\hat{\mathbf{b}} \times \boldsymbol{\kappa}$, $\mathbf{v}_{\nabla B} = (\mu/\Omega_s)\hat{\mathbf{b}} \times \nabla B$, $v_B = (m_s\mu/Z_s e)\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}$ and $\overline{\mu}_1 = -(v_\parallel \mu/\Omega_s)\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}$. According to (1.9), $\nabla \times \hat{\mathbf{b}} \simeq \hat{\mathbf{b}} \times \nabla \ln B$. Thus, $\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \simeq 0$, and v_B and $\overline{\mu}_1$ can be neglected, giving

$$\boxed{\frac{\partial \langle f_s \rangle_\varphi}{\partial t} + \left(v_\parallel \hat{\mathbf{b}} - \frac{1}{B}\nabla\phi \times \hat{\mathbf{b}} + \mathbf{v}_\kappa + \mathbf{v}_{\nabla B} \right) \cdot \nabla \langle f_s \rangle_\varphi - \left(\hat{\mathbf{b}} + \frac{v_\parallel}{\Omega_s}\hat{\mathbf{b}} \times \boldsymbol{\kappa} \right) \cdot \left(\frac{Z_s e}{m_s}\nabla\phi + \mu\nabla B \right) \frac{\partial \langle f_s \rangle_\varphi}{\partial v_\parallel} = 0.} \tag{2.5}$$

To close the problem, we need an equation for ϕ. We use quasineutrality,

$$\boxed{0 = \sum_s Z_s e n_s = \sum_s Z_s e \int 2\pi B \langle f_s \rangle_\varphi \, \mathrm{d}v_\parallel \, \mathrm{d}\mu.} \tag{2.6}$$

The potential appears implicitly in this equation through the gyroaveraged distribution functions $\langle f_s \rangle_\varphi$.

3. Drift waves

To understand the behavior of a low β, magnetized plasma, we consider a simple slab configuration. We consider a constant and uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ (see figure 1). In this magnetic field, the drift kinetic equation simplifies to

$$\frac{\partial \langle f_s \rangle_\varphi}{\partial t} + \left(v_\parallel \hat{\mathbf{z}} - \frac{1}{B}\nabla\phi \times \hat{\mathbf{z}} \right) \cdot \nabla \langle f_s \rangle_\varphi - \frac{Z_s e}{m_s}\hat{\mathbf{z}} \cdot \nabla\phi \frac{\partial \langle f_s \rangle_\varphi}{\partial v_\parallel} = 0. \tag{3.1}$$

We use this equation to study the response to small perturbations of a plasma composed of one ion species of charge Ze and mass m_i, and electrons of charge $-e$ and mass m_e. We consider a steady state without electric field ($\phi = 0$). Then, the drift kinetic equation becomes $v_\parallel \hat{\mathbf{z}} \cdot \nabla \langle f_s \rangle_\varphi = 0$. Thus, any distribution function constant along magnetic field lines will be a solution. We choose Maxwellian distribution functions for both electrons and ions,

$$\langle f_s \rangle_\varphi(x, v_\parallel, \mu) = f_{Ms}(x, v_\parallel, \mu) \equiv n_s(x) \left(\frac{m_s}{2\pi T_s(x)} \right)^{3/2} \exp\left(-\frac{m_s(v_\parallel^2/2 + \mu B)}{T_s(x)} \right). \tag{3.2}$$

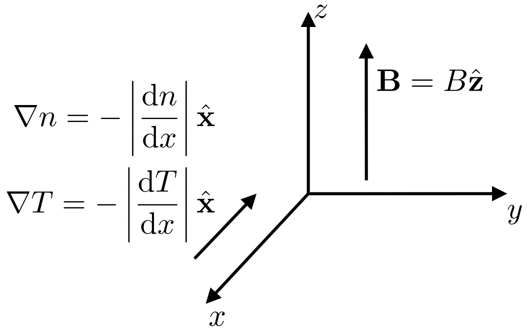

FIGURE 1. Magnetized plasma slab. It is conventional to assume that the gradients of density and temperature point in the negative x direction.

The densities n_s and temperatures T_s only depend on x (see figure 1). Electrons and ions must satisfy quasineutrality,

$$Zn_i(x) = n_e(x). \tag{3.3}$$

We also assume that the electron and ion temperature are of similar size, that is, $T_e \sim T_i$.

We perturb this equilibrium. We consider a perturbation to the potential $\delta\phi$, and a perturbation to the gyroaveraged distribution function due to this potential $\delta\langle f_s\rangle_\varphi$. Linearizing (3.1), we obtain

$$\frac{\partial \delta\langle f_s\rangle_\varphi}{\partial t} + v_\parallel \hat{\mathbf{z}}\cdot\nabla\delta\langle f_s\rangle_\varphi = \frac{1}{B}(\nabla\delta\phi\times\hat{\mathbf{z}})\cdot\nabla f_{Ms} + \frac{Z_s e}{m_s}\hat{\mathbf{z}}\cdot\nabla\delta\phi\frac{\partial f_{Ms}}{\partial v_\parallel}. \tag{3.4}$$

The perturbations also have to satisfy quasineutrality,

$$Z\int 2\pi B\,\delta\langle f_i\rangle_\varphi\,\mathrm{d}v_\parallel\,\mathrm{d}\mu = \int 2\pi B\,\delta\langle f_e\rangle_\varphi\,\mathrm{d}v_\parallel\,\mathrm{d}\mu. \tag{3.5}$$

Considering perturbations of the form $\delta\phi = \tilde{\phi}(x)\exp(-\mathrm{i}\omega t + \mathrm{i}k_y y + \mathrm{i}k_z z)$ and $\delta\langle f_s\rangle_\varphi = \tilde{g}_s(x, v_\parallel, \mu)\exp(-\mathrm{i}\omega t + \mathrm{i}k_y y + \mathrm{i}k_z z)$, and using

$$\nabla f_{Ms} = \hat{\mathbf{x}}\left[\frac{1}{n_s}\frac{\mathrm{d}n_s}{\mathrm{d}x} + \left(\frac{m_s(v_\parallel^2/2+\mu B)}{T_s} - \frac{3}{2}\right)\frac{1}{T_s}\frac{\mathrm{d}T_s}{\mathrm{d}x}\right] f_{Ms} \tag{3.6}$$

and

$$\frac{\partial f_{Ms}}{\partial v_\parallel} = -\frac{m_s v_\parallel}{T_s} f_{Ms}, \tag{3.7}$$

equation (3.4) becomes

$$(-\mathrm{i}\omega + \mathrm{i}k_z v_\parallel)\tilde{g}_s = \left\{\mathrm{i}\omega_{*s}\left[1 + \eta_s\left(\frac{m_s(v_\parallel^2/2+\mu B)}{T_s} - \frac{3}{2}\right)\right] - \mathrm{i}k_z v_\parallel\right\}\frac{Z_s e\tilde{\phi}}{T_s} f_{Ms}, \tag{3.8}$$

and the quasineutrality equation gives

$$Z\int 2\pi B\,\tilde{g}_i\,\mathrm{d}v_\parallel\,\mathrm{d}\mu = \int 2\pi B\,\tilde{g}_e\,\mathrm{d}v_\parallel\,\mathrm{d}\mu. \tag{3.9}$$

Here

$$\omega_{*s} = -\frac{k_y T_s}{Z_s e B L_{n_s}} \tag{3.10}$$

is the drift frequency of species s,

$$\eta_s = \frac{L_{n_s}}{L_{T_s}}, \tag{3.11}$$

and the density and temperature scale lengths L_{n_s} and L_{T_s} are defined by

$$L_Q = -\left(\frac{\mathrm{d}}{\mathrm{d}x}\ln Q\right)^{-1}. \tag{3.12}$$

Due to quasineutrality (see (3.3)), $L_{n_e} = L_{n_i} = L_n$.

Since $T_e \sim T_i$, the thermal speeds of ions and electrons are very different, $v_{te}/v_{ti} \sim \sqrt{m_i/m_e} \gg 1$. Thus, there exist modes that satisfy

$$k_z v_{ti} \ll \omega \sim \omega_{*e} \sim \omega_{*i} \ll k_z v_{te}. \tag{3.13}$$

Electrons and ions respond differently to waves in this frequency range. For electrons, equation (3.8) becomes

$$(\cancelto{\text{small}}{-\mathrm{i}\omega} + \mathrm{i}k_z v_\parallel)\tilde{g}_e = \left\{-\mathrm{i}\omega_{*e}\left[1 + \cancelto{\text{small}}{\eta_e\left(\frac{m_e(v_\parallel^2/2+\mu B)}{T_e} - \frac{3}{2}\right)}\right] + \mathrm{i}k_z v_\parallel\right\}\frac{e\tilde{\phi}}{T_e}f_{Me}. \tag{3.14}$$

Then, the electrons give a simple Maxwell-Boltzmann response,

$$\tilde{g}_e \simeq \frac{e\tilde{\phi}}{T_e}f_{Me}. \tag{3.15}$$

The ions satisfy the equation

$$(-\mathrm{i}\omega + \cancelto{\text{small}}{\mathrm{i}k_z v_\parallel})\tilde{g}_i = \left\{\cancelto{\text{small}}{-\mathrm{i}k_z v_\parallel} + \mathrm{i}\omega_{*i}\left[1 + \eta_i\left(\frac{m_i(v_\parallel^2/2+\mu B)}{T_i} - \frac{3}{2}\right)\right]\right\}\frac{Ze\tilde{\phi}}{T_i}f_{Mi}, \tag{3.16}$$

giving

$$\tilde{g}_i \simeq -\frac{\omega_{*i}}{\omega}\left[1 + \eta_i\left(\frac{m_i(v_\parallel^2/2+\mu B)}{T_i} - \frac{3}{2}\right)\right]\frac{Ze\tilde{\phi}}{T_i}f_{Mi}. \tag{3.17}$$

Using equations (3.15) and (3.17) in the quasineutrality equation (3.9), we obtain the dispersion relation for waves that satisfy (3.13),

$$-\frac{Z^2 n_i}{T_i}\frac{\omega_{*i}}{\omega} = \frac{n_e}{T_e}. \tag{3.18}$$

Using $Zn_i = n_e$ and $(ZT_e/T_i)\omega_{*i} = -\omega_{*e}$, we find that the wave frequency is

$$\omega = \omega_{*e} = \frac{k_y T_e}{eBL_n}. \tag{3.19}$$

The phase and group velocity of this wave are

$$\mathbf{v}_{*e} = -\frac{T_e}{en_eB}\hat{\mathbf{b}}\times\nabla n_e = \frac{T_e}{eBL_{n_e}}\hat{\mathbf{y}}. \tag{3.20}$$

In figure 2 we give a physical picture for the drift wave. Note that our derivation does not give the structure of the mode in the x direction. To calculate the structure in x, we would need higher order terms in ρ_{s*}.

Almost any low frequency ($\omega \ll \Omega_i \ll \Omega_e$) perturbation to a magnetized plasma with gradients will move with a velocity of the order of the drift velocity in (3.20). The drift

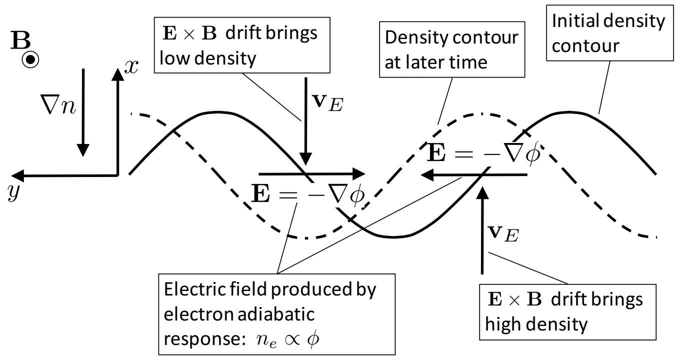

FIGURE 2. Sketch of a drift wave. An initial perturbed plasma density contour is sketched as a solid wavy line. The density below this line is higher than above it. Due to the electron Maxwell-Boltzmann response, caused by a parallel gradient not shown in the figure, there is a potential difference between the high and the low density region. This potential difference causes an electric field and hence an $\mathbf{E}\times\mathbf{B}$ drift, shown in the figure. The $\mathbf{E}\times\mathbf{B}$ drift moves ions in such a way that the density contour simply moves in the y direction. The density contour at a later time is sketched as a dashed wavy line.

velocity and the drift frequency are ubiquitous in plasma physics, and it is very common to find references to ω_* and $\mathbf{v}_*$ in the literature. The derivation that we have followed is based on $k_z v_{ti} \ll \omega$, but waves with $k_z v_{ti} \sim \omega$ will have frequencies of the order of ω_{*e} as well. In general, however, drift waves with $k_z v_{ti} \sim \omega$ will decay due to Landau damping. It is also possible to find unstable drift wave modes that grow by tapping the thermodynamic energy contained in plasma gradients.

4. Ion Temperature Gradient (ITG) instability

To demonstrate that drift waves can become unstable, we first consider perturbations that satisfy the assumptions in (3.13) and later we generalize the calculation to include perturbations with $k_z v_{ti} \sim \omega$. In both cases the electrons follow the Maxwell-Boltzmann response in (3.15).

4.1. *Fluid limit of the ITG instability*

In this subsection, we will look for perturbation that satisfy the assumptions in (3.13). The difference between the derivation that follows and the derivation of the drift wave in section 3 is that, for the derivation that follows, we consider cases with a large ion temperature gradient, $L_{T_i} \ll L_n$. We assume that $\eta_i = L_n/L_{T_i} \gg 1$, and we expand the equations assuming that

$$\frac{k_z v_{ti}}{\omega} \sim \frac{1}{\sqrt{\eta_i}} \ll 1. \tag{4.1}$$

Equation (3.8) for the ions can be written as

$$\tilde{g}_i = \frac{1}{\omega - k_z v_\parallel} \left\{ -\omega_{*i} \left[1 + \eta_i \left(\frac{m_i(v_\parallel^2/2 + \mu B)}{T_i} - \frac{3}{2} \right) \right] + k_z v_\parallel \right\} \frac{Ze\tilde{\phi}}{T_i} f_{Mi}. \tag{4.2}$$

Using (4.1), we can Taylor expand $(\omega - k_z v_\parallel)^{-1}$ for typical values of $v_\parallel$,

$$\frac{1}{\omega - k_z v_\parallel} = \frac{1}{\omega} \left[1 + \frac{k_z v_\parallel}{\omega} + \frac{k_z^2 v_\parallel^2}{\omega^2} + O(\eta_i^{-3/2}) \right]. \tag{4.3}$$

Note that by Taylor expanding, we have converted a resonant denominator into a smooth function. Resonant denominators have been treated in the notes for Kinetic Theory (Schekochihin 2015) with the help of Laplace transforms and Landau contours. We will come back to resonant denominators in the next subsection. For this subsection, it is sufficient to know that by employing equation (4.3), we will miss terms of order $\exp(-\omega^2/k_z^2 v_{ti}^2) \sim \exp(-\eta_i) \ll 1$. Using (4.3), the perturbation to the ion distribution function becomes

$$\tilde{g}_i = -\frac{\omega_{*i}}{\omega}\left[\left(\frac{m_i(v_\parallel^2/2+\mu B)}{T_i} - \frac{3}{2}\right)\left(\eta_i + \underbrace{\frac{\eta_i k_z v_\parallel}{\omega}}_{\sim \eta_i^{1/2}} + \underbrace{\frac{\eta_i k_z^2 v_\parallel^2}{\omega^2}}_{\sim 1}\right)\right.$$
$$\left. + 1 + O(\eta_i^{-1/2})\right]\frac{Ze\tilde{\phi}}{T_i} f_{Mi}. \tag{4.4}$$

Note that we are keeping terms of different size in the expansion in $\eta_i^{-1} \ll 1$. This may seem surprising, but it will become clear why we do so.

Substituting (3.15) and (4.4) into the quasineutrality equation (3.9), we obtain the dispersion relation

$$-\frac{Z^2 n_i}{T_i}\frac{\omega_{*i}}{\omega}\left(\frac{\eta_i k_z^2 T_i}{m_i \omega^2} + 1\right) = \frac{n_e}{T_e}. \tag{4.5}$$

Note that all the terms of order η_i and $\eta_i^{1/2}$ in (4.4) do not contribute to the density. We kept small terms in the expansion in $\eta_i^{-1} \ll 1$ because the small terms were the only ones that ended up contributing to the density. Using $Zn_i = n_e$ and $(ZT_e/T_i)\omega_{*i} = -\omega_{*e}$, we can rewrite the dispersion relation (4.5) as

$$\omega^3 - \omega_{*e}\omega^2 - \frac{k_z^2 T_i}{m_i}\omega_{*e}\eta_i = 0. \tag{4.6}$$

The general solution to (4.6) is

$$\omega = \frac{\omega_{*e}}{3}\left[1 + \left(A + \sqrt{A^2-1}\right)^{1/3}\exp\left(\frac{2\pi \mathrm{i} r}{3}\right)\right.$$
$$\left. + \left(A - \sqrt{A^2-1}\right)^{1/3}\exp\left(-\frac{2\pi \mathrm{i} r}{3}\right)\right], \tag{4.7}$$

where $r = 0, 1, 2$, and

$$A = 1 + \frac{27}{2}\frac{k_z^2 T_i \eta_i}{m_i \omega_{*e}^2}. \tag{4.8}$$

According to (4.7), the plasma is stable when the three solutions are purely real (when two solutions are complex, at least one of them has a positive imaginary part, leading to instability). All the solutions are purely real and hence stable when $A + \sqrt{A^2-1}$ is the complex conjugate of $A - \sqrt{A^2-1}$, that is, when $|A| \leqslant 1$, or equivalently

$$-\frac{4}{27} \leqslant \frac{k_z^2 T_i \eta_i}{m_i \omega_{*e}^2} \leqslant 0. \tag{4.9}$$

For η_i outside of this interval, the plasma is unstable (in fact, the plasma is always unstable in the limit $\eta_i \gg 1$, but for η_i in the interval (4.9), the growth rate is very small). We can find simple unstable solutions in some limits. For example, for $\eta_i > 0$ and

$k_z^2 T_i \eta_i / m_i \omega_{*e}^2 \ll 1$, the growing solution is

$$\omega \simeq \mathrm{i}\sqrt{\frac{k_z^2 T_i \eta_i}{m_i}}, \tag{4.10}$$

whereas for $k_z^2 T_i \eta_i / m_i \omega_{*e}^2 \gg 1$, the growing solution is

$$\omega \simeq \left(\frac{k_z^2 T_i}{m_i} |\omega_{*e} \eta_i| \right)^{1/3} \left(-\frac{1}{2} \frac{\omega_{*e}\eta_i}{|\omega_{*e}\eta_i|} + \frac{\sqrt{3}}{2}\mathrm{i} \right). \tag{4.11}$$

To give a physical picture for the fluid ITG instability, we consider first the lowest order term in equation (4.4),

$$\tilde{g}_i \simeq - \left(\frac{m_i (v_\parallel^2/2 + \mu B)}{T_i} - \frac{3}{2} \right) \frac{k_y \tilde{\phi}}{B\omega} \frac{\mathrm{d}\ln T_i}{\mathrm{d}x} f_{Mi}. \tag{4.12}$$

Note that to lowest order, the instability is mostly a fluctuation in the ion temperature. Indeed, an ion Maxwellian with a small electron temperature perturbation $\delta T_i = \tilde{T}_i(x) \exp(-\mathrm{i}\omega t + \mathrm{i}k_y y + \mathrm{i}k_z z) \ll T_i$ gives

$$n_i \left(\frac{m_i}{2\pi (T_i + \delta T_i)} \right)^{3/2} \exp\left(- \frac{m_i (v_\parallel^2/2 + \mu B)}{T_i + \delta T_i} \right) \simeq f_{Mi} + \left(\frac{m_i (v_\parallel^2/2 + \mu B)}{T_i} - \frac{3}{2} \right) \frac{\delta T_i}{T_i} f_{Mi} \tag{4.13}$$

Comparing this result with equation (4.12), we find the ion temperature perturbation

$$\tilde{T}_i = - \frac{k_y \tilde{\phi}}{B\omega} \frac{\mathrm{d}T_i}{\mathrm{d}x}. \tag{4.14}$$

This temperature perturbation is the equivalent to the density perturbation shown in figure 2: the fluctuating $\mathbf{E} \times \mathbf{B}$ drift brings into the magnetic field line of interest plasma with higher or lower temperature depending on the direction of the drift. Equation (4.14) can in fact be obtained from the intuitive energy equation $-\mathrm{i}\omega \tilde{T}_i + \tilde{\mathbf{v}}_E \cdot \hat{\mathbf{x}} (\mathrm{d}T_i/\mathrm{d}x) \simeq 0$, with $\tilde{\mathbf{v}}_E = -\mathrm{i}\hat{\mathbf{x}} k_y \tilde{\phi}/B$. In this simplified energy equation, we have been able to neglect the energy transport along magnetic field lines because the orderings in (4.1) imply the gradients along magnetic field lines are small. However, the parallel gradients become important for the ion flow. Multiplying the ion version of equation (3.4) by $m_i v_\parallel$ and integrating over velocity, we find an equation for the perturbed ion parallel velocity $\tilde{u}_{i\parallel} = n_i^{-1} \int \tilde{g}_i v_\parallel \, \mathrm{d}^3 v$,

$$-\mathrm{i}\omega n_i m_i \tilde{u}_{i\parallel} + \mathrm{i}k_z \int \tilde{g}_i m_i v_\parallel^2 \, \mathrm{d}^3 v = \cancel{-Z n_i \mathrm{i} k_z \tilde{\phi}}. \quad \text{small in } \eta_i^{-1} \ll 1 \tag{4.15}$$

Using the lowest order result (4.12) in the second term in the left side of (4.15), we obtain $-\mathrm{i}\omega n_i m_i \tilde{u}_{i\parallel} + \mathrm{i}k_z n_i \tilde{T}_i \simeq 0$, that is, the ion temperature perturbation drives flow because it exerts a pressure force along magnetic field lines. The resulting parallel velocity enters in the ion continuity equation

$$-\mathrm{i}\omega \tilde{n}_i + \mathrm{i}k_z n_i \tilde{u}_{i\parallel} + \tilde{\mathbf{v}}_E \cdot \hat{\mathbf{x}} \frac{\mathrm{d}n_i}{\mathrm{d}x} = 0, \tag{4.16}$$

that can be found by integrating over velocity the ion version of equation (3.4). Here,

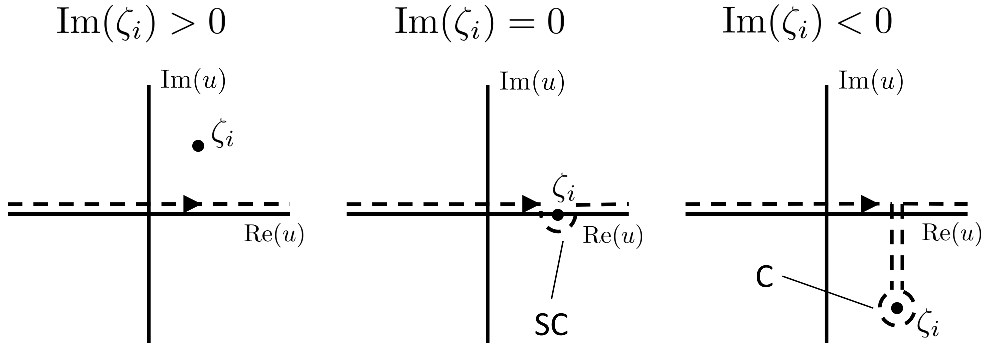

FIGURE 3. Landau contours for integration of resonant denominators such as the one in (4.19). For $\mathrm{Im}(\zeta_i) > 0$, the contour is the real line, as expected. For $\mathrm{Im}(\zeta_i) \leqslant 0$, the contour surrounds ζ_i to ensure that the integral is an analytic continuation of the integral with $\mathrm{Im}(\zeta_i) > 0$. For $\mathrm{Im}(\zeta_i) = 0$, the part of the contour that surrounds ζ_i is a semi-circumference, SC, whereas for $\mathrm{Im}(\zeta_i) < 0$, the piece of the contour that surrounds ζ_i is a a complete circumference, C.

$\tilde{n}_i = \int \tilde{g}_i \, \mathrm{d}^3 v$ is the perturbed ion density, and it is related to the perturbed electrostatic potential via quasineutrality and the electron Maxwell-Boltzmann response, $Z\tilde{n}_i = n_e(e\tilde{\phi}/T_e)$. Note that equation (4.16) gives the stable drift wave solution in (3.19) if we neglect the parallel flow term $\mathrm{i}k_z n_i \tilde{u}_{i\|}$. Since the ordering in (4.1) assumes that the parallel gradients are small, it is tempting to neglect this term, but for large η_i, the ion temperature gradient is large and it drives large pressure perturbations. These pressure perturbations push the plasma along magnetic field lines and give rise to a very large ion parallel flow that can destabilize the drift wave by significantly modifying the ion density perturbations.

4.2. *Kinetic ITG instability*

To obtain analytical results, we have assumed that $\eta_i \gg 1$ and that $k_z v_{ti}/\omega \ll 1$. These assumptions limit the validity of the results. It is possible to solve the dispersion relation for $\eta_i \sim 1$ numerically. For $\eta_i \sim 1$, we cannot assume $k_z v_{ti}/\omega \ll 1$. Thus, we have to integrate the ion distribution function in (4.2) without Taylor expanding the resonant denominator. The part of the integral over μ is straightforward. For the integral over $v_\|$, we change to the variable $u = (k_z/|k_z|)(v_\|/v_{ti})$, where $v_{ti} = \sqrt{2T_i/m_i}$. Then, the perturbed ion density is

$$\int \tilde{g}_i \, \mathrm{d}^3 v = \frac{Ze\tilde{\phi}}{T_i} \frac{n_i}{\sqrt{\pi}} \int_{C_L} \left\{ \frac{\omega_{*i}}{|k_z| v_{ti}} \left[1 + \eta_i \left(u^2 - \frac{1}{2} \right) \right] - u \right\} \frac{\exp(-u^2)}{u - \zeta_i} \, \mathrm{d}u. \tag{4.17}$$

Here

$$\zeta_i = \frac{\omega}{|k_z| v_{ti}} \tag{4.18}$$

and C_L is the Landau contour, explained in detail in (Schekochihin 2015). The resonant denominator in (4.2) indicates that for the time t, we need to use a Laplace transform instead of a Fourier transform. For the Laplace transform, we assume that $\mathrm{Im}(\omega) > 0$. Thus, for $\mathrm{Im}(\omega) > 0$, the Landau contour is simply the real axis. For $\mathrm{Im}(\omega) \leqslant 0$, to ensure that $\int \tilde{g}_i \, \mathrm{d}^3 v$ is analytic for all ω, we have to choose other contours that avoid ζ_i, as shown in figure 3. To reduce the size of the expressions, it is useful to define the

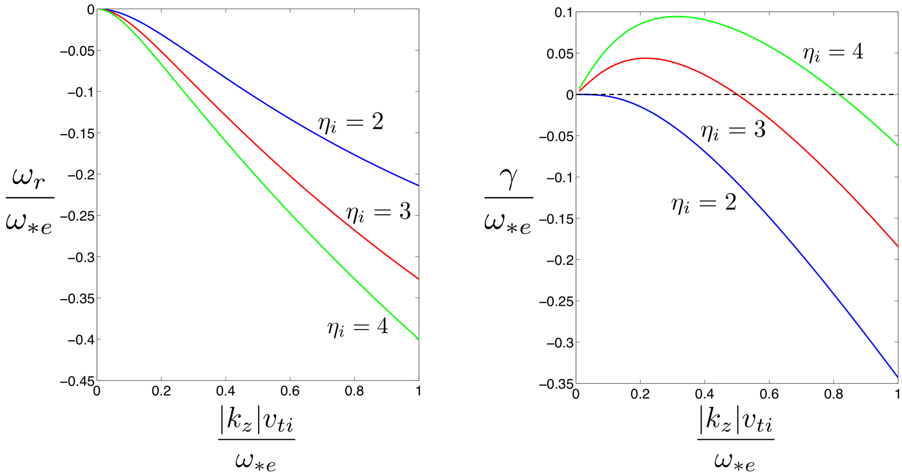

FIGURE 4. Real frequency ω_r and growth rate γ for the ITG instability as a function of $|k_z| v_{ti}/\omega_{*e}$ for different values of η_i in a plasma with $Z = 1$ and $T_e = T_i$.

plasma dispersion function

$$\mathcal{Z}(\zeta_i) = \frac{1}{\sqrt{\pi}} \int_{C_L} \frac{\exp(-u^2)}{u - \zeta_i} \mathrm{d}u. \tag{4.19}$$

Using this function, we obtain

$$\begin{aligned} \frac{1}{\sqrt{\pi}} \int_{C_L} \frac{u \exp(-u^2)}{u - \zeta_i} \mathrm{d}u &= \frac{1}{\sqrt{\pi}} \int_{C_L} \frac{(u - \zeta_i) \exp(-u^2)}{u - \zeta_i} \, \mathrm{d}u + \frac{\zeta_i}{\sqrt{\pi}} \int_{C_L} \frac{\exp(-u^2)}{u - \zeta_i} \mathrm{d}u \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-u^2) \, \mathrm{d}u + \frac{\zeta_i}{\sqrt{\pi}} \int_{C_L} \frac{\exp(-u^2)}{u - \zeta_i} \mathrm{d}u = 1 + \zeta_i \mathcal{Z}(\zeta_i) \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} \frac{1}{\sqrt{\pi}} \int_{C_L} \frac{u^2 \exp(-u^2)}{u - \zeta_i} \mathrm{d}u &= \frac{1}{\sqrt{\pi}} \int_{C_L} \frac{(u^2 - \zeta_i^2) \exp(-u^2)}{u - \zeta_i} \, \mathrm{d}u + \frac{\zeta_i^2}{\sqrt{\pi}} \int_{C_L} \frac{\exp(-u^2)}{u - \zeta_i} \mathrm{d}u \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (u + \zeta_i) \exp(-u^2) \, \mathrm{d}u + \frac{\zeta_i^2}{\sqrt{\pi}} \int_{C_L} \frac{\exp(-u^2)}{u - \zeta_i} \mathrm{d}u = \zeta_i + \zeta_i^2 \mathcal{Z}(\zeta_i) \end{aligned} \tag{4.21}$$

With these results, equation (4.17) becomes

$$\int \tilde{g}_i \, \mathrm{d}^3 v = \frac{Ze\tilde{\phi}}{T_i} n_i \left\{ \frac{\omega_{*i} \eta_i \zeta_i}{|k_z| v_{ti}} - 1 + \left[\frac{\omega_{*i}}{|k_z| v_{ti}} \left(1 + \eta_i \left(\zeta_i^2 - \frac{1}{2} \right) \right) - \zeta_i \right] \mathcal{Z}(\zeta_i) \right\}. \tag{4.22}$$

Substituting this equation and (3.15) into the quasineutrality equation (3.9), we obtain the dispersion relation

$$\frac{\omega_{*e} \eta_i \zeta_i}{|k_z| v_{ti}} + \frac{ZT_e}{T_i} + 1 + \left[\frac{\omega_{*e}}{|k_z| v_{ti}} \left(1 + \eta_i \left(\zeta_i^2 - \frac{1}{2} \right) \right) + \frac{ZT_e}{T_i} \zeta_i \right] \mathcal{Z}(\zeta_i) = 0. \tag{4.23}$$

where we have used $Zn_i = n_e$ and $(ZT_e/T_i)\omega_{*i} = -\omega_{*e}$.

We can solve the dispersion relation (4.23) numerically using a more convenient form of the plasma dispersion function,

$$\mathcal{Z}(\zeta_i) = \exp(-\zeta_i^2)\left[\mathrm{i}\sqrt{\pi} - 2\int_0^{\zeta_i} \exp(w^2)\,\mathrm{d}w\right] \tag{4.24}$$

(see Appendix A to see why the plasma dispersion function can be written in this way). In figure 4, for a plasma with $ZT_e/T_i = 1$, the complex frequency $\omega = \omega_r + \mathrm{i}\gamma$ of the ITG instability is plotted as a function of $|k_z|v_{ti}/\omega_{*e}$ for different values of η_i. The instability grows for $|k_z|v_{ti}/\omega_{*e}$ smaller than a certain value that depends on η_i and ZT_e/T_i. The approximate result in (4.7) suggests that the growth rate increases with increasing k_z without bound, but this trend, which we obtained by assuming $k_z v_{ti}/\omega \ll 1$, is only valid up to values of k_z such that $k_z v_{ti}/\omega \sim 1$.

We proceed to calculate the maximum value that $|k_z|v_{ti}/\omega_{*e}$ can take for instability to exist. At this value of $|k_z|v_{ti}/\omega_{*e}$, the growth rate vanishes. The complex frequency ω is such that both the real and the imaginary parts of the dispersion relation (4.23) vanish. For zero growth rate, ω and hence ζ_i are purely real. For a purely real ζ_i, the imaginary part of the plasma dispersion function is $\mathrm{Im}(\mathcal{Z}(\zeta_i)) = \sqrt{\pi}\exp(-\zeta_i^2)$ and cannot vanish (this result can be obtained from (4.24) or from the Landau contour for $\mathrm{Im}(\zeta_i) = 0$ sketched in figure 3). Thus, to set the imaginary part of the dispersion relation (4.23) to zero, the term that multiplies $\mathcal{Z}(\zeta_i)$ must vanish, giving

$$\frac{\omega_{*e}}{|k_z|v_{ti}}\left(1 + \eta_i\left(\zeta_i^2 - \frac{1}{2}\right)\right) + \frac{ZT_e}{T_i}\zeta_i = 0. \tag{4.25}$$

If this equation is satisfied, the dispersion relation (4.23) implies that

$$\frac{\omega_{*e}\eta_i\zeta_i}{|k_z|v_{ti}} + \frac{ZT_e}{T_i} + 1 = 0. \tag{4.26}$$

Solving equations (4.25) and (4.26) simultaneously, we find that the growth rate vanishes for

$$\left.\frac{|k_z|v_{ti}}{\omega_{*e}}\right|_{\max} = \sqrt{\frac{\eta_i(\eta_i - 2)}{1 + ZT_e/T_i}}. \tag{4.27}$$

This calculation gives the maximum value that $|k_z|v_{ti}/\omega_{*e}$ can take for instability to exist. Since we have used the fact that the growth rate is zero at this particular value of $|k_z|v_{ti}/\omega_{*e}$, it might be surprising that this calculation has not given the solution $k_z = 0$ since figure 4 clearly shows zero growth rate at that value of k_z. This apparent contradiction is resolved by the fact that for $k_z \to 0$ both the growth rate and the real frequency vanish in such a way that ζ_i is finite, that is, $\omega \propto k_z$ for $k_z \to 0$. In general, ζ_i for $k_z \to 0$ is not purely real, and we cannot use the procedure that led to (4.27).

From equation (4.27), we can deduce the values of η_i for which the plasma becomes stable. For some values of η_i, the maximum value that $|k_z|v_{ti}/\omega_{*e}$ can take for instability to exist, given in (4.27), does not exist. When this happens, the plasma is stable. Thus, the plasma is stable for $0 < \eta_i < 2$. For all the other values of η_i, the ITG instability exists.

Finally, our derivations above suggest that ITG modes with large k_y would grow faster since the growth rate is proportional to ω_{*e}. However, the drift kinetic model is only valid for $k_y\rho_i \ll 1$. For $k_y\rho_i \sim 1$ the ITG mode is stabilized by finite gyroradius effects.

REFERENCES

Schekochihin, A.A. 2015 Lecture notes on Kinetic Theory and Magnetohydrodynamics of Plasmas. http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/KT/.

Appendix A. Alternative expression for the plasma dispersion function

In this Appendix, we show that the plasma dispersion function, defined in (4.19), can be written as in equation (4.24). To do so, we first consider the case $\mathrm{Im}(\zeta_i) > 0$. In this case, we can write

$$\frac{1}{u - \zeta_i} = \mathrm{i} \int_0^\infty \exp(-\mathrm{i}\lambda(u - \zeta_i))\, \mathrm{d}\lambda. \tag{A 1}$$

Using this result, the plasma dispersion function in (4.19) becomes

$$\mathcal{Z}(\zeta_i) = \frac{\mathrm{i}}{\sqrt{\pi}} \int_{C_L} \mathrm{d}u \int_0^\infty \mathrm{d}\lambda\, \exp(-u^2 - \mathrm{i}\lambda(u - \zeta_i)). \tag{A 2}$$

Integrating in u by completing the square in the exponential, we find

$$\mathcal{Z}(\zeta_i) = \mathrm{i} \int_0^\infty \exp\left(-\frac{\lambda^2}{4} + \mathrm{i}\lambda\zeta_i\right) \mathrm{d}\lambda. \tag{A 3}$$

Using the integration variable $v = \lambda/2 - \mathrm{i}\zeta_i$, this integral becomes

$$\mathcal{Z}(\zeta_i) = 2\mathrm{i} \exp(-\zeta_i^2) \int_{-\mathrm{i}\zeta_i}^\infty \exp\left(-v^2\right)\, \mathrm{d}v. \tag{A 4}$$

This integral can be calculated by splitting it into two integrals

$$\mathcal{Z}(\zeta_i) = 2\mathrm{i} \exp(-\zeta_i^2) \left[\int_0^\infty \exp\left(-v^2\right)\, \mathrm{d}v - \int_0^{-\mathrm{i}\zeta_i} \exp\left(-v^2\right)\, \mathrm{d}v \right]. \tag{A 5}$$

Realizing that the first integral is $\int_0^\infty \exp\left(-v^2\right)\, \mathrm{d}v = \sqrt{\pi}/2$ and rewriting the second integral using the integration variable $w = \mathrm{i}v$, we finally obtain (4.24). The same result can be obtained for $\mathrm{Im}(\zeta_i) < 0$ using

$$\frac{1}{u - \zeta_i} = -\mathrm{i} \int_0^\infty \exp(\mathrm{i}\lambda(u - \zeta_i))\, \mathrm{d}\lambda \tag{A 6}$$

and recalling the Landau contour (shown in figure 3) has a part that surrounds ζ_i, giving

$$\begin{aligned} \mathcal{Z}(\zeta_i) &= -\frac{\mathrm{i}}{\sqrt{\pi}} \int_{C_L} \mathrm{d}u \int_0^\infty \mathrm{d}\lambda\, \exp(-u^2 + \mathrm{i}\lambda(u - \zeta_i)) \\ &= 2\mathrm{i}\sqrt{\pi} \exp(-\zeta_i^2) - \mathrm{i} \int_0^\infty \exp\left(-\frac{\lambda^2}{4} - \mathrm{i}\lambda\zeta_i\right) \mathrm{d}\lambda. \end{aligned} \tag{A 7}$$