Collisional Plasma Physics
Problem Set I
Due: Friday 14 May 2021

1.1 (10 points) The Fokker-Planck collision operator is often replaced by simpler model collision operators. For like particle collisions, one such model operator is the modified Brownian motion operator

$$C_{ss}[f_s] = \nu_{ss} \nabla_v \cdot \left[\frac{\Theta_s}{m_s} \nabla_v f_s + (\mathbf{v} - \mathbf{V}_s) f_s \right], \tag{1}$$

where ν_{ss} is a positive constant, and Θ_s and $\mathbf{V}_s$ are moments of f_s that need to be determined.

(a) Prove that this collision operator conserves particles.

(b) Determine $\mathbf{V}_s$ and Θ_s such that the collision operator conserves momentum and energy.

(c) For $\mathbf{V}_s$ and Θ_s as defined in (b), prove that the entropy production

$$\dot{\sigma}_{ss} = -\int \ln f_s \, C_{ss}[f_s] \, \mathrm{d}^3 v \tag{2}$$

is positive, $\dot{\sigma}_{ss} \geq 0$, and it is zero only for a Maxwellian distribution function. [Hint: show first that $\int \ln f_s \, C_{ss}[f_s] \, \mathrm{d}^3 v = \int (\ln f_s + m_s |\mathbf{v} - \mathbf{V}_s|^2 / 2\Theta_s) \, C_{ss}[f_s] \, \mathrm{d}^3 v$.]

1.2 (15 points) In this problem, we will use the Landau form of the Fokker-Planck collision operator to calculate the collisional energy exchange

$$W_{ss'} = \int \frac{1}{2} m_s v^2 \, C_{ss'}[f_{Ms}, f_{Ms'}] \, \mathrm{d}^3 v \tag{3}$$

between two species s and s' that have stationary Maxwellian distribution functions with different temperatures,

$$f_{Ms}(v) = n_s \left(\frac{m_s}{2\pi T_s} \right)^{3/2} \exp\left(-\frac{m_s v^2}{2T_s} \right), \tag{4}$$

$$f_{Ms'}(v) = n_{s'} \left(\frac{m_{s'}}{2\pi T_{s'}} \right)^{3/2} \exp\left(-\frac{m_{s'} v^2}{2T_{s'}} \right). \tag{5}$$

(a) Prove that

$$W_{ss'} = \gamma_{ss'} \int \mathrm{d}^3 v \int \mathrm{d}^3 v' f_{Ms}(v) f_{Ms'}(v') \mathbf{v} \cdot \nabla_g \nabla_g g \cdot \left(\frac{\mathbf{v}}{T_s} - \frac{\mathbf{v}'}{T_{s'}} \right). \tag{6}$$

(b) Change from the integration variables $\mathbf{v}$ and $\mathbf{v}'$ to the integration variables

$$\mathbf{U} = \frac{m_s \mathbf{v}/T_s + m_{s'} \mathbf{v}'/T_{s'}}{m_s/T_s + m_{s'}/T_{s'}}, \quad \mathbf{g} = \mathbf{v} - \mathbf{v}', \tag{7}$$

to find

$$W_{ss'} = K\left(\frac{1}{T_s} - \frac{1}{T_{s'}}\right)\int \mathrm{d}^3U \int \mathrm{d}^3g\, \mathbf{U}\cdot\nabla_g\nabla_g g\cdot\mathbf{U}\exp\left(-\frac{U^2}{2\sigma_U} - \frac{g^2}{2\sigma_g}\right), \tag{8}$$

where K, σ_U and σ_g are constants that you need to determine. [Hint: show that $\mathrm{d}^3v\,\mathrm{d}^3v' = \mathrm{d}^3U\,\mathrm{d}^3g$.]

(c) Integrate equation (8) to obtain $W_{ss'}$. Does your formula agree with the formula for the energy transfer between electrons and ions in the notes?

1.3 (30 points) In fusion devices, fusion-born alpha particles (Helium ions) are very energetic and satisfy $v_{ti} \ll v_\alpha \ll v_{te}$, where v_α is the typical alpha particle velocity, and v_{ti} and v_{te} are the ion and electron thermal speeds. Alpha particles slow down due to collisions with electrons and ions. We study this process by assuming that the ion and electron distribution function functions are stationary Maxwellians, that is,

$$f_s(v) = f_{Ms}(v) \equiv n_s\left(\frac{m_s}{2\pi T_s}\right)^{3/2}\exp\left(-\frac{m_s v^2}{2T_s}\right) \tag{9}$$

for $s = i, e$.

We assume that the alpha particles are produced always with velocity magnitude v_α at a volumetric rate S_α. We assume that the source is isotropic. Then, the slowing down process is described by the equation

$$C_{\alpha e}[f_\alpha, f_{Me}] + \sum_i C_{\alpha i}[f_\alpha, f_{Mi}] + \frac{S_\alpha}{4\pi v_\alpha^2}\delta(v - v_\alpha) = 0, \tag{10}$$

where $\delta(x)$ is the one-dimensional Dirac delta function. Due to the symmetry of this equation, the alpha particle distribution function is isotropic, that is, $f_\alpha(v)$ only depends on the magnitude of the velocity $v = |\mathbf{v}|$.

(a) Justify the ordering

$$\frac{v_{ti}}{v_\alpha} \ll \sqrt{\frac{m_\alpha}{m_i}} \sim 1, \tag{11}$$

and use it to show that

$$C_{\alpha i}[f_\alpha, f_{Mi}] \simeq \frac{3\sqrt{2\pi}}{2}\frac{Z_i^2 n_i T_e^{3/2}\ln\Lambda_{i\alpha}}{n_e m_i m_e^{1/2}\ln\Lambda_{e\alpha}}\nu_{\alpha e}\nabla_v\cdot\left(\frac{\mathbf{v}}{v^3}f_\alpha\right), \tag{12}$$

where

$$\nu_{\alpha e} = \frac{4\sqrt{2\pi}}{3}\frac{Z_\alpha^2 e^4 n_e\sqrt{m_e}\ln\Lambda_{e\alpha}}{(4\pi\epsilon_0)^2 m_\alpha T_e^{3/2}}. \tag{13}$$

(b) Justify the ordering

$$\sqrt{\frac{m_e}{m_\alpha}} \ll \frac{v_\alpha}{v_{te}} \ll 1, \tag{14}$$

and use it to show that

$$C_{\alpha e}[f_\alpha, f_{Me}] \simeq \nu_{\alpha e}\nabla_v\cdot(\mathbf{v}f_\alpha). \tag{15}$$

[Hint: expand first in the smallest parameter, $m_e/m_\alpha \lll 1$, assuming that $v_\alpha \sim v_{te}$. After that expansion is performed, proceed to do the subsidiary expansion $v_\alpha/v_{te} \ll 1$.]

(c) Using (15) and (12), show that equation (10) becomes

$$\frac{\nu_{\alpha e}}{v^2}\frac{\partial}{\partial v}\Big[(v^3+v_c^3)f_\alpha\Big] = -\frac{S_\alpha}{4\pi v_\alpha^2}\delta(v-v_\alpha), \tag{16}$$

where v_c is a constant that you need to determine.

(d) Show that the solution to equation (16) with boundary condition $f_\alpha(v\to\infty)\to 0$ is

$$f_\alpha(v) = \begin{cases} \frac{S_\alpha}{4\pi\nu_{\alpha e}}\frac{1}{v^3+v_c^3} & \text{for } v < v_\alpha \\ 0 & \text{for } v > v_\alpha \end{cases}. \tag{17}$$

This solution is known as the slowing down distribution function.

1.4 (30 points) The linearized collision operator $C_{ss'}^{(\ell)}[h_s;h_{s'}] = C_{ss'}[h_s, f_{Ms'}] + C_{ss'}[f_{Ms}, h_{s'}]$ is composed of a differential part, $C_{ss'}[h_s, f_{Ms'}]$, and an integral part, $C_{ss'}[f_{Ms}, h_{s'}]$. In this problem, we study the differential part,

$$C_{ss'}[h_s, f_{Ms'}] = \frac{\gamma_{ss'}}{m_s^2}\nabla_v\cdot\left[f_{Ms}(v)\int f_{Ms'}(v')\nabla_g\nabla_g g\cdot\nabla_v\left(\frac{h_s(\mathbf{v})}{f_{Ms}(v)}\right)\mathrm{d}^3v'\right]. \tag{18}$$

Here $f_{Ms'}(v)$ is a stationary Maxwellian,

$$f_{Ms'}(v) = n_{s'}\left(\frac{m_{s'}}{2\pi T}\right)^{3/2}\exp\left(-\frac{m_{s'}v^2}{2T}\right). \tag{19}$$

(a) Show that the differential piece of the linearized collision operator can be written as

$$C_{ss'}[h_s, f_{Ms'}] = \frac{\gamma_{ss'}}{m_s^2}\nabla_v\cdot\left[f_{Ms}\nabla_v\nabla_v H_{s'}\cdot\nabla_v\left(\frac{h_s}{f_{Ms}}\right)\right] \tag{20}$$

where

$$H_{s'}(\mathbf{v}) = \int g f_{Ms'}(v')\,\mathrm{d}^3v'. \tag{21}$$

In the plasma physics literature, the function $H_{s'}$ is one of the Rosenbluth potentials.

(b) Using spherical coordinates with the z-axis aligned with $\mathbf{v}$ (see Figure 1), take the integral over $\mathbf{v}'$ and show that

$$H_{s'}(\overline{v}) = n_{s'}v_{ts'}\left[\left(\overline{v}+\frac{1}{2\overline{v}}\right)\mathrm{erf}(\overline{v}) + \frac{1}{\sqrt{\pi}}\exp(-\overline{v}^2)\right], \tag{22}$$

where $v_{ts'} = \sqrt{2T/m_{s'}}$, $\mathrm{erf}(x) = (2/\sqrt{\pi})\int_0^x \exp(-s^2)\,\mathrm{d}s$ is the error function and $\overline{v} = v/v_{ts'}$. [Hint: integrate first over the angles α and β, and recall that $\sqrt{x^2} = |x|$ and not $= x$.]

(c) Show then that

$$C_{ss'}[h_s, f_{Ms'}] = \nabla_v\cdot\left[\frac{\nu_{ss',\perp}f_{Ms}}{4}(v^2\mathbf{I}-\mathbf{vv})\cdot\nabla_v\left(\frac{h_s}{f_{Ms}}\right) + \frac{\nu_{ss',\parallel}f_{Ms}}{2}\mathbf{vv}\cdot\nabla_v\left(\frac{h_s}{f_{Ms}}\right)\right], \tag{23}$$

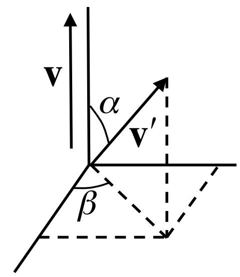

Figure 1: Spherical coordinates for the integral in $\mathbf{v}'$.

where

$$\nu_{ss',\perp} = \frac{8\pi Z_s^2 Z_{s'}^2 e^4 n_{s'} \ln \Lambda_{ss'}}{(4\pi\epsilon_0)^2 m_s^2 v^3} \Big(\mathrm{erf}(\overline{v}) - \Psi(\overline{v}) \Big) \tag{24}$$

and

$$\nu_{ss',\parallel} = \frac{8\pi Z_s^2 Z_{s'}^2 e^4 n_{s'} \ln \Lambda_{ss'}}{(4\pi\epsilon_0)^2 m_s^2 v^3} \Psi(\overline{v}). \tag{25}$$

Here

$$\Psi(x) = \frac{1}{2x^2} \left(\mathrm{erf}(x) - \frac{2x}{\sqrt{\pi}} \exp(-x^2) \right). \tag{26}$$

(d) Using spherical coordinates, explain why $\nu_{ss',\perp}$ is the pitch-angle scattering frequency and why $\nu_{ss',\parallel}$ is the energy diffusion frequency.

(e) Argue that for ion-electron collisions, we need to consider the limit $\overline{v} \ll 1$. Expand equation (23) in the limit $\overline{v} \ll 1$ and compare the result with the ion-electron collision operator calculated in class. [Hint: $\mathrm{erf}(x) = 2x/\sqrt{\pi} - 2x^3/3\sqrt{\pi} + \dots$ for $x \ll 1$.]

(f) Argue that for electron-ion collisions, we can assume $\overline{v} \gg 1$. Expand equation (23) in the limit $\overline{v} \gg 1$ and compare the result with the electron-ion collision operator calculated in class. [Hint: $\mathrm{erf}(x) \simeq 1$ for $x \gg 1$.]

1.5 (15 points) The conductivity σ_Z of an unmagnetized plasma formed by one ion species with charge Ze and mass m_i and electrons with charge $-e$ and mass m_e depends on the ion charge number Z. In this problem, we will evaluate this dependence.

(a) Using quasineutrality, argue that $\nu_{ee} = \nu_{ei}/Z$.

(b) Solve the Spitzer-Härm problem for a general Z using the truncated form

$$f_{e,\mathrm{SH}} = \Big[a_0 + a_1 L_1^{(3/2)}(x) \Big] \frac{e\mathbf{E}\cdot\mathbf{v}}{T_e} f_{Me}(v). \tag{27}$$

for the Spitzer-Härm distribution function. Here $\mathbf{E}$ is the electric field, T_e is the electron temperature, $L_p^{(\gamma)}(x)$ are modified Laguerre polynomials, $x = m_e v^2/2T_e$, and

$$f_{Me}(v) = n_e \left(\frac{m_e}{2\pi T_e} \right)^{3/2} \exp\left(-\frac{m_e v^2}{2T_e} \right) \tag{28}$$

is a stationary Maxwellian.

(c) Plot schematically σ_Z as a function of Z.

(d) Show that the Spitzer-Härm equation for $Z \gg 1$ is

$$\mathcal{L}_{ei}[f_{e,SH}] = \frac{e\mathbf{E} \cdot \mathbf{v}}{T_e} f_{Me}. \tag{29}$$

This equation can be solved exactly. Find the solution using spherical coordinates. [Hint: recall that $\cos\alpha$ is proportional to the spherical harmonic $Y_{1,0}$.]

(e) Using the solution in part (c), calculate the conductivity for $Z \gg 1$, and show that it is

$$\sigma_{Z\to\infty} = \frac{32}{3\pi} \frac{e^2 n_e}{m_e \nu_{ei}}. \tag{30}$$

Is this result consistent with the formula obtained in part (b)? Would you expect them to be exactly equal for $Z \gg 1$?