Resistive MHD, reconnection and resistive tearing modes

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1. Introduction

In these notes we show how MagnetoHydroDynamics (MHD) can be deduced from Braginskii equations. We then study how keeping some of the small Braginskii terms, such as resistivity, can be important. Resistivity and other Braginskii (or two-fluid) effects break the frozen-in law that ties plasma and magnetic field lines together, allowing magnetic reconnection (breaks in magnetic field lines) to happen. This new effect can lead to instabilities that cannot be described using MHD.

2. MagnetoHydroDynamics (MHD)

In MHD (Schekochihin 2015) the plasma is modeled as a single fluid characterized by its density $\rho(\mathbf{r}, t)$, its average velocity $\mathbf{u}(\mathbf{r}, t)$, its pressure $p(\mathbf{r}, t)$ and its current density $\mathbf{J}(\mathbf{r}, t)$. For a Braginskii plasma composed of one ion species of charge e and mass m_i and electrons with charge $-e$ and mass m_e, the plasma density is

$$\rho = n_i m_i + n_e m_e = n_e(m_i + m_e) \simeq n_e m_i, \tag{2.1}$$

the plasma average velocity is

$$\mathbf{u} = \frac{n_i m_i \mathbf{u}_i + n_e m_e \mathbf{u}_e}{n_i m_i + n_e m_e} \simeq \mathbf{u}_i, \tag{2.2}$$

the plasma total pressure is

$$p = n_i T_i + n_e T_e = n_e(T_i + T_e), \tag{2.3}$$

and the plasma current density is

$$\mathbf{J} = e n_e(\mathbf{u}_i - \mathbf{u}_e). \tag{2.4}$$

To obtain the MHD equations for ρ, $\mathbf{u}$, p and $\mathbf{J}$, we start from the Braginskii equations, derived using the orderings

$$\frac{\rho_i}{L} \ll \frac{\lambda_{ee}}{L} \sim \frac{\lambda_{ei}}{L} \sim \frac{\lambda_{ii}}{L} \sim \sqrt{\frac{m_e}{m_i}} \ll 1, \tag{2.5}$$

$$|\mathbf{u}_i| \sim |\mathbf{u}_e| \sim |u_{i\|} - u_{e\|}| \sim v_{ti} \gg |\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}| \sim \frac{\rho_i}{L} v_{ti}, \tag{2.6}$$

and

$$T_i \sim T_e \sim |T_i - T_e|, \tag{2.7}$$

among other assumptions. To find the MHD equations, we will modify these assumptions and perform a subsidiary expansion in

$$\frac{\rho_i}{L} \ll \boxed{\frac{\lambda_{ee}}{L} \sim \frac{\lambda_{ei}}{L} \sim \frac{\lambda_{ii}}{L} \ll \sqrt{\frac{m_e}{m_i}}} \ll 1. \tag{2.8}$$

In this limit, the parallel collisional friction force and the collisional temperature equilibration term, of order

$$F_{ei,\|} \sim n_e m_e \nu_{ei} |u_{i\|} - u_{e\|}| \sim \sqrt{\frac{m_e}{m_i}} \frac{L}{\lambda_{ei}} \frac{|u_{i\|} - u_{e\|}|}{v_{ti}} \frac{p_e}{L} \tag{2.9}$$

and

$$\tilde{W}_{ie} = \frac{3 n_e m_e \nu_{ei}}{m_i}(T_e - T_i) \sim \sqrt{\frac{m_e}{m_i}} \frac{L}{\lambda_{ei}} \frac{|T_e - T_i|}{T_e} \frac{p_e v_{ti}}{L}, \tag{2.10}$$

must be of the same order as the other terms in their respective equations, the parallel momentum equations and the energy equations of both ions and electrons. Thus, we assume that $F_{ei,\|} \sim p_e/L$, giving

$$|\mathbf{u}_i| \sim |\mathbf{u}_e| \sim v_{ti} \gg |u_{i\|} - u_{e\|}| \sim \sqrt{\frac{m_i}{m_e}} \frac{\lambda_{ei}}{L} v_{ti} \gg |\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}| \sim \frac{\rho_i}{L} v_{ti}, \tag{2.11}$$

and that $\tilde{W}_{ie} \sim p_e v_{ti}/L$, leading to

$$T_i \sim T_e \gg |T_i - T_e| \sim \sqrt{\frac{m_i}{m_e}} \frac{\lambda_{ei}}{L} T_e. \tag{2.12}$$

These assumptions are different from Braginskii's more general ordering in (2.6) and (2.7). Equations (2.11) and (2.12) imply that

$$\mathbf{u}_e \simeq \mathbf{u}_i = \mathbf{u} \tag{2.13}$$

and

$$T_i \simeq T_e = T. \tag{2.14}$$

The plasma is sufficiently collisional that both electrons and ions move at the same average velocity and have the same temperature, and can then be treated as a single fluid.

We proceed to derive the MHD equations from Braginskii equations.

- Multiplying Braginskii's ion continuity equation by m_i, we obtain the MHD continuity equation,

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.} \tag{2.15}$$

- The equation for $\mathbf{u}$ is Braginskii's total momentum equation. Note that under the assumptions in (2.5), we showed that the ion viscosity was small in $\lambda_{ii}/L \ll 1$, giving

$$\boxed{\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \mathbf{J} \times \mathbf{B},} \tag{2.16}$$

- We also use the electron momentum equation. We have shown that under the assumptions in (2.5), the electromagnetic forces dominate, giving $-e n_e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = 0$. Using (2.13), this equation finally becomes

$$\boxed{\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0.} \tag{2.17}$$

• Summing the electron and ion thermal energy equations and using (2.13) and (2.14), we find

$$3n_e\left(\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T\right)=-2n_eT\nabla\cdot\mathbf{u}, \tag{2.18}$$

where we have neglected the terms $\nabla\cdot\mathbf{q}_i$ and $\mathbf{\Pi}_i:\nabla\mathbf{u}_i$ because they are small in $\lambda_{ii}/L\ll 1$, and the terms $\nabla\cdot\mathbf{q}_e$ and $\mathbf{F}_{ei}\cdot(\mathbf{u}_i-\mathbf{u}_e)$ because they are small in $\sqrt{m_i/m_e}(\lambda_{ee}/L)\ll 1$. Using (2.15) and $p=n_e(T_i+T_e)\simeq 2n_eT$, equation (2.18) can be rewritten as the usual MHD energy equation

$$\boxed{\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)\left(\frac{p}{\rho^{5/3}}\right)=0.} \tag{2.19}$$

In addition to the plasma model in (2.15), (2.16), (2.17) and (2.19), we need Maxwell's equations for the evolution of $\mathbf{E}$ and $\mathbf{B}$: the induction equation

$$\boxed{\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},} \tag{2.20}$$

and Ampere's law without the displacement current,

$$\boxed{\nabla\times\mathbf{B}=\mu_0\mathbf{J}.} \tag{2.21}$$

3. Resistive MHD and reconnection

We have neglected several terms to derive the MHD equations. Some of them are in fact important even though they are small. We will focus on the electron momentum equation. The electron momentum equation with next order terms is

$$0\simeq -en_e(\mathbf{E}+\mathbf{u}_e\times\mathbf{B})-\nabla p_e+0.51n_em_e\nu_{ei}(u_{i\|}-u_{e\|})\hat{\mathbf{b}}-0.71n_e\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla T_e, \tag{3.1}$$

where we have neglected $n_em_e(\partial\mathbf{u}_e+\mathbf{u}_e\cdot\nabla\mathbf{u}_e)$, $\nabla\cdot\mathbf{\Pi}_e$ and $\mathbf{F}_{ei,\perp}$ because they are much smaller than the electron pressure gradient. We rewrite this equation performing the following manipulations.

• According to (2.11), $|u_{i\|}-u_{e\|}|\gg|\mathbf{u}_{i\perp}-\mathbf{u}_{e\perp}|$, leading to $(u_{i\|}-u_{e\|})\hat{\mathbf{b}}\simeq\mathbf{u}_i-\mathbf{u}_e$. Hence, we can write

$$0.51n_em_e\nu_{ei}(u_{i\|}-u_{e\|})\hat{\mathbf{b}}\simeq 0.51n_em_e\nu_{ei}(\mathbf{u}_i-\mathbf{u}_e)=en_e\eta\mathbf{J}, \tag{3.2}$$

where

$$\eta=\frac{0.51m_e\nu_{ei}}{e^2n_e} \tag{3.3}$$

is the plasma resistivity.

• We can rewrite $\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla T_e$ as

$$\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla T_e=\nabla T_e-(\hat{\mathbf{b}}\times\nabla T_e)\times\hat{\mathbf{b}}. \tag{3.4}$$

Using these results, equation (3.1) becomes

$$\mathbf{E}+\mathbf{u}_B\times\mathbf{B}=-\frac{\nabla p_e}{en_e}+\eta\mathbf{J}-\frac{0.71}{e}\nabla T_e, \tag{3.5}$$

where the velocity $\mathbf{u}_B$ is

$$\mathbf{u}_B=\underbrace{\mathbf{u}_e}_{\sim v_{ti}}-\underbrace{\frac{0.71}{eB}\hat{\mathbf{b}}\times\nabla T_e}_{\sim\frac{\rho_i}{L}v_{ti}}\simeq\mathbf{u}. \tag{3.6}$$

There are two differences between the usual MHD electron momentum equation, $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$, and equation (3.5): the difference between the plasma velocity $\mathbf{u}$ and the new velocity $\mathbf{u}_B$, and the terms in the right side of (3.5). Both of these differences are small, but some of the terms in the right side of (3.5) are important because they change a fundamental property of MHD: the frozen-in law.

To see what happens to the frozen-in law, we obtain $\mathbf{E}$ from (3.5) and substitute it into the induction equation (2.20) to find

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u}_B \times \mathbf{B}) - \frac{\nabla n_e \times \nabla p_e}{e n_e^2} - \nabla \times (\eta \mathbf{J}). \tag{3.7}$$

The usual MHD frozen-in law is $\partial \mathbf{B}/\partial t = \nabla \times (\mathbf{u} \times \mathbf{B})$, and it implies that the magnetic field lines move with the plasma velocity $\mathbf{u}$. Since plasma infinitesimal volume elements do not split into two ($\mathbf{u}$ would have to be discontinuous), a magnetic field line that moves with the plasma cannot be broken. Conversely, equation (3.7) implies that the magnetic field lines move with the velocity $\mathbf{u}_B \simeq \mathbf{u}$, and in addition to this motion, the magnetic field changes in time due to two new terms: $-\nabla n_e \times \nabla p_e/e n_e^2$ and $-\nabla \times (\eta \mathbf{J})$. In these notes, we focus on the term $-\nabla \times (\eta \mathbf{J})$ due to resistivity. The term $-\nabla n_e \times \nabla p_e/e n_e^2$ is in general as important as the resistivity term, but it can be made zero if we take the solution $p/\rho^{5/3} = \text{constant}$ to equation (2.19), implying that $p_e \propto n_e^{5/3}$ and hence $\nabla n_e \times \nabla p_e = 0$. Then, in the particular case $p/\rho^{5/3} = \text{constant}$, and using (2.21) to obtain $\mathbf{J}$, equation (3.7) becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u}_B \times \mathbf{B}) - \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right). \tag{3.8}$$

Using $\nabla \times [(\eta/\mu_0) \nabla \times \mathbf{B}] = (\eta/\mu_0) \nabla (\nabla \cdot \mathbf{B}) + \mu_0^{-1} \nabla \mathbf{B} \cdot \nabla \eta - \nabla \cdot [(\eta/\mu_0) \nabla \mathbf{B}]$ and $\nabla \cdot \mathbf{B} = 0$, we rewrite (3.8) as

ignore

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u}_B \times \mathbf{B}) + \nabla \cdot \left(\frac{\eta}{\mu_0} \nabla \mathbf{B} \right) - \frac{1}{\mu_0} \nabla \mathbf{B} \cdot \cancel{\nabla \eta}. \tag{3.9}$$

Thus, if we ignore $\nabla \eta$ to avoid unnecessary complications, the magnetic field moves with velocity $\mathbf{u}_B$, which is uninteresting because $\mathbf{u}_B$ is almost the plasma velocity $\mathbf{u}$, and it diffuses with a diffusion coefficient $D_\eta = \eta/\mu_0$. The fact that the magnetic field diffuses implies that magnetic field lines can be broken. The resistivity is not the only term that can break magnetic field lines, and there is a large area of research that studies the different physical effects that can lead to magnetic reconnection.

From here on, we only consider resistivity, and instead of (3.5), we consider the simpler equation

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J} \tag{3.10}$$

that contains the necessary terms to obtain (3.9). To simplify the problem even further, we consider η a constant, i.e. we consider a plasma in which the temperature variation is small.

We characterize the strength of the resistive term using the characteristic time of the magnetic field diffusion $\tau_\eta = L^2/D_\eta = \mu_0 L^2/\eta$, where L is the length of the system. We measure the importance of the resistivity in the plasma by comparing this characteristic time with the Alfven time, $\tau_A = L/v_A$, where $v_A = B/\sqrt{\rho \mu_0}$ is the Alfven speed. This comparison gives the Lundquist number,

$$S = \frac{\tau_\eta}{\tau_A} = \frac{\mu_0 v_A L}{\eta} = \frac{BL}{\eta} \sqrt{\frac{\mu_0}{\rho}}. \tag{3.11}$$

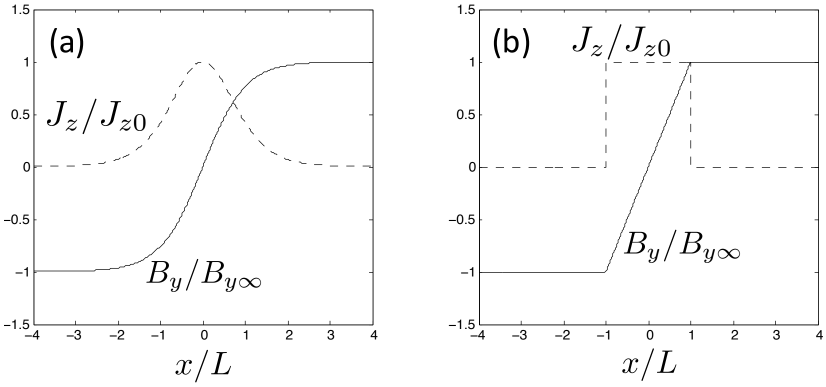

FIGURE 1. Sketches of $B_y(x)$ (solid) and $J_z(x)$ (dashed) normalized by their values at $x \to \infty$, $B_{y\infty} = B_y(x \to \infty)$, and at $x = 0$, $J_{z0} = J_z(x = 0)$, respectively. Figure (a) represents a general current sheet, and figure (b) is the specific example in equations (4.2) and (4.5).

Since we are assuming that resistivity is a small effect, we are considering the case $S \gg 1$.

4. Resistive tearing modes

To study the effect of the magnetic diffusion, we consider a current sheet, which is a magnetic field configuration that is stable to ideal MHD modes, but unstable to small but finite resistivity. The unstable mode "reconnects" the magnetic field, converting the magnetic energy into plasma energy.

In this section, we first describe the equilibrium that we will consider, and we then study its stability by linearizing the equations.

4.1. *Current sheet equilibrium*

We consider a simple sheared magnetic field

$$\mathbf{B} = B_z\hat{\mathbf{z}} + B_y(x)\hat{\mathbf{y}}. \tag{4.1}$$

The function $B_y(x)$ is of the form shown in figure 1. To simplify the calculation, we assume in these notes that $B_y(x)$ is antisymmetric with respect to $x = 0$, i.e. $B_y(x) = -B_y(-x)$, and hence $B_y = 0$ at $x = 0$. We also assume that B_y tends to a constant for $x \to \pm\infty$. A possible example, shown in figure 1(b), is

$$B_y(x) = \left\{ \begin{array}{ll} B_{y0}x/L & \text{for } |x| \leqslant L \\ B_{y0} & \text{for } |x| > L \end{array} \right. . \tag{4.2}$$

Assuming that $B_y(x)$ is antisymmetric with respect to $x = 0$ is particularly restrictive and not necessary for the derivation, but this assumption simplifies the calculation around $x = 0$. The sheared magnetic field in (4.1) is similar to the magnetic field used in the seminal calculation by Furth, Killeen and Rosenbluth (FKR) that we follow in these notes (Furth *et al.* 1963).

Using Ampere's law, given in (2.21), we find the current density corresponding to the magnetic field in (4.1),

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B} = J_z(x)\hat{\mathbf{z}}, \tag{4.3}$$

where

$$J_z(x) = \frac{1}{\mu_0}B_y'(x). \tag{4.4}$$

From here on, the prime $'$ indicates differentiation with respect to x. The dependence of $\mathbf{J}$ on x is sketched in figure 1. The localization of the current is the reason why this type of equilibria is called a current sheet. For the magnetic field in (4.2), we find

$$J_z(x) = \begin{cases} B_{y0}/\mu_0 L & \text{for } |x| \leqslant L \\ 0 & \text{for } |x| > L \end{cases}. \tag{4.5}$$

This current profile is sketched in figure 1(b).

The current must be carried by the plasma. For the plasma equilibrium, we assume that $\mathbf{u} = 0$. Then, equation (2.16) gives $\nabla p = \mathbf{J} \times \mathbf{B} = -(B_y B_y'/\mu_0)\hat{\mathbf{x}}$, and we can integrate in x to obtain the equilibrium pressure $p(x)$. Equation (3.10) gives $\mathbf{E} = E_z(x)\hat{\mathbf{z}}$, with $E_z(x) = \eta J_z(x)$. This electric field is externally imposed to drive the current in the current sheet. Note that the plasma density $\rho(x)$ is undetermined. We assume that it only depends on x.

The equilibrium that we have just described is stable in the MHD model because the sheared magnetic field makes moving the magnetic field lines impossible (they cannot break and cross each other). Including resistivity into the analysis will change the stability properties.

4.2. *Perturbation equations*

Linearizing equations (2.16), (3.10), (2.20) and (2.21), we find

$$\rho \frac{\partial \mathbf{u}_1}{\partial t} = -\nabla p_1 + \mathbf{J}_1 \times \mathbf{B} + \mathbf{J} \times \mathbf{B}_1, \tag{4.6}$$

$$\mathbf{E}_1 + \mathbf{u}_1 \times \mathbf{B} = \eta \mathbf{J}_1, \tag{4.7}$$

$$\nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t} \tag{4.8}$$

and

$$\nabla \times \mathbf{B}_1 = \mu_0 \mathbf{J}_1. \tag{4.9}$$

We will not need the continuity equation or the energy equation.

Due to the symmetry of the equilibrium (it only depends on x and B_z is a constant), we can choose a form of the perturbation that simplifies the derivation. We will assume that $\mathbf{B}_1(x, y, t)$, $\mathbf{J}_1(x, y, t)$, $\mathbf{E}_1(x, y, t)$, $\mathbf{u}_1(x, y, t)$ and $p_1(x, y, t)$ depend on x and y, but not on z. Moreover, we will assume that $\mathbf{B}_1 \cdot \hat{\mathbf{z}} = 0$. Since the perturbed magnetic field must satisfy $\nabla \cdot \mathbf{B}_1 = 0$, we choose

$$\mathbf{B}_1 = \hat{\mathbf{z}} \times \nabla \psi, \tag{4.10}$$

where $\psi(x, y, t)$ is the flux function. With this form of $\mathbf{B}_1$ and (4.9), we obtain the perturbed current density

$$\mathbf{J}_1 = \frac{1}{\mu_0} \nabla \times \mathbf{B}_1 = \frac{\nabla^2 \psi}{\mu_0} \hat{\mathbf{z}}. \tag{4.11}$$

The form of $\mathbf{B}_1$ in (4.10) gives the form of the electric field $\mathbf{E}_1$ and the velocity $\mathbf{u}_1$. Equation (4.8) gives

$$\mathbf{E}_1 = \frac{\partial \psi}{\partial t} \hat{\mathbf{z}} - \nabla \phi, \tag{4.12}$$

where $\phi(x, y, t)$ is the electrostatic potential. Using (4.12), the components of the vector equation (4.7) perpendicular to $\hat{\mathbf{z}}$ give

$$\mathbf{u}_1 = \frac{1}{B_z} \hat{\mathbf{z}} \times \nabla \phi, \tag{4.13}$$

where we have assumed $\mathbf{u}_1 \cdot \hat{\mathbf{z}} = 0$ to simplify the derivation.

We need equations for ψ and ϕ. The $\hat{\mathbf{z}}$-component of (4.7) gives

$$\frac{\partial \psi}{\partial t} - \frac{B_y}{B_z}\frac{\partial \phi}{\partial y} = \frac{\eta}{\mu_0}\nabla^2\psi. \tag{4.14}$$

To find an equation for ϕ, we use (4.6). To eliminate the term ∇p_1, we take a curl of this equation, obtaining the vorticity equation,

$$\frac{\partial}{\partial t}[\nabla \times (\rho \mathbf{u}_1)] = \mathbf{B} \cdot \nabla \mathbf{J}_1 - \cancelto{0}{\mathbf{J}_1 \cdot \nabla \mathbf{B}} + \mathbf{B}_1 \cdot \nabla \mathbf{J} - \cancelto{0}{\mathbf{J} \cdot \nabla \mathbf{B}_1}. \tag{4.15}$$

Using (4.10), (4.11) and (4.13), equation (4.15) gives

$$\frac{\mu_0}{B_z}\frac{\partial}{\partial t}\left[\nabla \cdot (\rho \nabla \phi)\right] = B_y \nabla^2 \left(\frac{\partial \psi}{\partial y}\right) - B_y'' \frac{\partial \psi}{\partial y}. \tag{4.16}$$

To simplify the equations even further, we take the forms

$$\begin{aligned} \psi(x,y,t) &= \tilde{\psi}(x)\exp(\gamma t + \mathrm{i}k_y y) + \text{c.c.}, \\ \phi(x,y,t) &= \tilde{\phi}(x)\exp(\gamma t + \mathrm{i}k_y y) + \text{c.c.}, \end{aligned} \tag{4.17}$$

where γ is the growth rate of the instability and k_y is the mode number in the direction of symmetry y. With this form of the solution, equations (4.14) and (4.16) become

$$\underbrace{\gamma\tilde{\psi}}_{\sim\gamma\tau_A\frac{\psi}{\tau_A}} - \underbrace{\frac{\mathrm{i}k_y B_y}{B_z}\tilde{\phi}}_{\sim\frac{\phi}{v_A\psi}\frac{\psi}{\tau_A}} = \underbrace{\frac{\eta}{\mu_0}\left(\tilde{\psi}'' - k_y^2\tilde{\psi}\right)}_{\sim S^{-1}\frac{\psi}{\tau_A}} \tag{4.18}$$

and

$$\underbrace{\frac{\mu_0\gamma}{\mathrm{i}k_y B_z B_y}\left[(\rho\tilde{\phi}')' - \rho k_y^2\tilde{\phi}\right]}_{\sim\gamma\tau_A\frac{\phi}{v_A\psi}\frac{\psi}{L^2}} = \underbrace{\tilde{\psi}'' - k_y^2\tilde{\psi} - \frac{B_y''}{B_y}\tilde{\psi}}_{\sim\frac{\psi}{L^2}}. \tag{4.19}$$

The order of magnitude estimates in these equations are obtained by assuming that $k_y L \sim 1$ and $\mathrm{d}/\mathrm{d}x \sim 1/L$.

The boundary conditions for equations (4.18) and (4.19) are that $\tilde{\psi}, \tilde{\phi} \to 0$ for $x \to \pm\infty$. For Lundquist number $S \gg 1$, the mode that satisfies these boundary conditions and equations (4.18) and (4.19) is composed of two very distinct regions: an outer region in which the resistivity is negligible, and an inner region in which the resistivity is comparable to other terms. We solve the equations in these two regions and then match them to find the shape of the mode and the growth rate γ.

4.2.1. *Outer region*

We will obtain that

$$S^{-1} \ll \gamma\tau_A \ll 1. \tag{4.20}$$

Then, the right side of equation (4.18) is small and can be neglected, leading to

$$\tilde{\phi}_o = -\frac{\mathrm{i}\gamma B_z}{k_y B_y}\tilde{\psi}_o, \tag{4.21}$$

where the subindex $_o$ indicates that this is only valid in the outer region. According to this result, the left side of (4.19) is of order $(\gamma\tau_A)^2\psi/L^2 \ll \psi/L^2$, and hence, equation

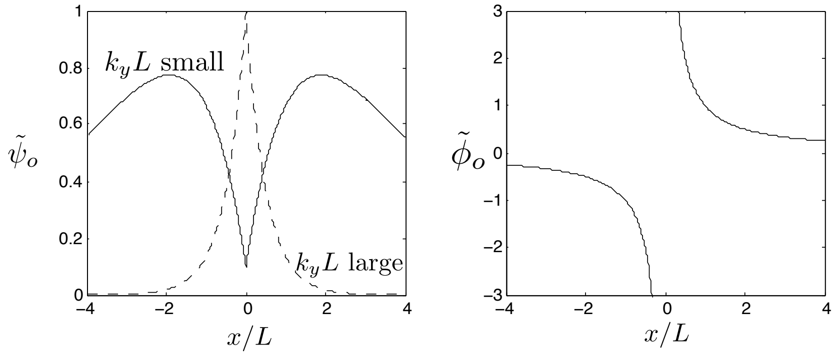

FIGURE 2. The functions $\tilde{\psi}_o(x)$ and $\tilde{\phi}_o(x)$ in the outer region. For $\tilde{\psi}_o(x)$, we give typical solutions for $k_y L$ small (solid) and $k_y L$ large (dashed).

(4.19) becomes

$$\tilde{\psi}_o'' - k_y^2 \tilde{\psi}_o - \frac{B_y''}{B_y} \tilde{\psi}_o = 0. \tag{4.22}$$

Note however, that the left side of (4.19) is small only away from $x = 0$ because at $x = 0$, $B_y = 0$. Thus, equation (4.22) is only valid for $]-\infty, 0[$ and $]0, \infty[$. We solve the equation with the boundary conditions $\tilde{\psi}_o \to 0$ for $x \to \pm\infty$, and

$$\tilde{\psi}_o(0^-) = \tilde{\psi}_o(0^+) \tag{4.23}$$

at $x = 0$. It is not obvious that $\tilde{\psi}_o$ should be continuous at $x = 0$, but we will see that this condition is consistent with the solution in the inner region.

For $x \to \pm\infty$, $B_y''/B_y \to 0$, and equation (4.22) becomes

$$\tilde{\psi}_o'' - k_y^2 \tilde{\psi}_o = 0. \tag{4.24}$$

Then, the solutions are $\tilde{\psi}_o \sim \exp(\pm k_y x)$ for $x \to \pm\infty$, and we must choose the solution that goes like $\exp(k_y x)$ for $x < 0$, and the solution that goes like $\exp(-k_y x)$ for $x > 0$ (here we have assumed without loss of generality that $k_y > 0$). This choice fixes the derivative at $x = 0^-$ and $x = 0^+$. Thus, in general the solution is of the form shown in figure 2, with a discontinuous derivate at $x = 0$. Because we have assumed that $B_y(x)$ is antisymmetric, the solutions are symmetric in x, giving

$$\tilde{\psi}_o(x) \simeq \tilde{\psi}_o(0) \left(1 + \frac{\Delta'}{2} |x| \right). \tag{4.25}$$

The parameter

$$\Delta' = \frac{\tilde{\psi}_o'(0^+) - \tilde{\psi}_o'(0^-)}{\tilde{\psi}_o(0)}. \tag{4.26}$$

is the standard measure of the size of the discontinuity in $\tilde{\psi}_o'$.

The function $\tilde{\phi}_o(x)$, given by (4.21), diverges at $x = 0$ as shown in figure 2 due to $B_y(x) \simeq B_y'(0) x$ for x close to zero.

To find an example of outer region solution, we use the magnetic field in (4.2). For this magnetic field, equation (4.22) becomes

$$\tilde{\psi}_o'' - k_y^2 \tilde{\psi}_o - \frac{1}{L} [\delta(x + L) - \delta(x - L)] \tilde{\psi}_o = 0. \tag{4.27}$$

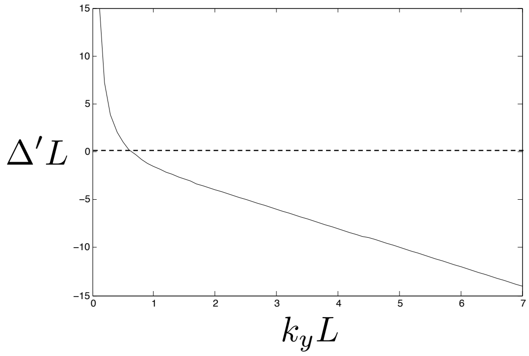

FIGURE 3. The quantity Δ' as a function of $k_y L$. We have used the particular case in (4.29).

The solution to this equation is

$$\frac{\tilde{\psi}_o(x)}{\tilde{\psi}_o(0)} = \begin{cases} \cosh(k_y x) + \dfrac{1 - k_y L - k_y L \tanh(k_y L)}{k_y L - (1 - k_y L)\tanh(k_y L)} \sinh(k_y |x|) & \text{for } |x| \leqslant L \\ \dfrac{k_y L}{k_y L \cosh(k_y L) - (1 - k_y L)\sinh(k_y L)} \exp(-k_y(|x| - L)) & \text{for } |x| > L \end{cases}. \tag{4.28}$$

As expected, this solution has a discontinuous derivative at $x = 0$. We find

$$\Delta' = \frac{2k_y[1 - k_y L - k_y L \tanh(k_y L)]}{k_y L - (1 - k_y L)\tanh(k_y L)}. \tag{4.29}$$

The quantity Δ' ranges from $+\infty$ at $k_y L \ll 1$ to $-\infty$ for $k_y L \gg 1$, as can be seen in figure 3. In fact, the quantity Δ' ranges from $-\infty$ to $+\infty$ when $k_y L$ decreases from ∞ to 0 for any profile of $B_y(x)$ that is of the form shown in figure 1 (see the different profiles of $\tilde{\psi}_o(x)$ for different values of $k_y L$ in figure 2). We show that this is the case in Appendix A.

The magnetic field is unknown in a region around $x = 0$. The discontinuity in $\tilde{\psi}_o'$ indicates that near $x = 0$, the assumptions that we have used to obtain equations (4.21) and (4.22) fail. For this reason, we need to solve the equations in an inner region in which the resistivity becomes important.

4.2.2. *Inner region*

In the outer region solution, the first derivative $\tilde{\psi}_o'$ is discontinuous. This discontinuity is indicative of a fast rate of change of $\tilde{\psi}_i'$ in the inner region, that is, $\tilde{\psi}_i''$ is large (see figure 4). Here the subindex $_i$ refers to quantities in the inner region. If the size of the inner region is of order $\delta \ll L$,

$$\tilde{\psi}_i'' \sim \frac{\psi}{\delta L}. \tag{4.30}$$

The function $\tilde{\phi}_i$, given in (4.21) for the outer region, must be of the form shown in figure 4. Thus, we expect its second derivative to satisfy

$$\tilde{\phi}_i'' \sim \frac{\phi}{\delta^2}. \tag{4.31}$$

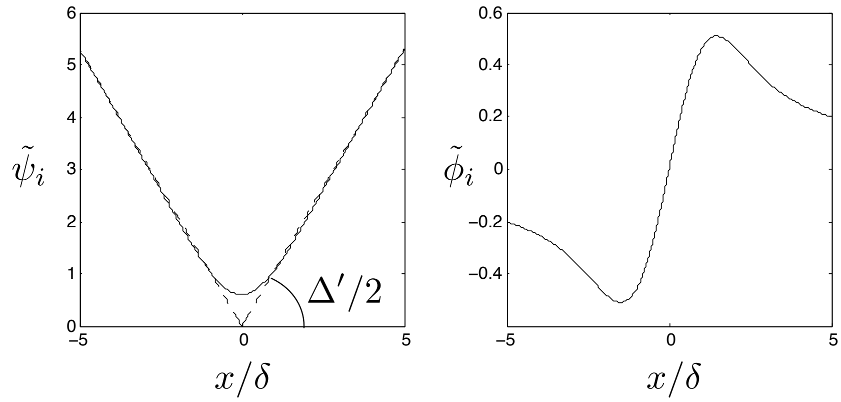

FIGURE 4. The functions $\tilde{\psi}(x)$ and $\tilde{\phi}(x)$ in the inner region for $\Delta' > 0$.

Using the estimates (4.30) and (4.31), and employing $B_y(x) \simeq B_y'(0)x$, $B_y''(x) \simeq B_y'''(0)x$ and $\rho(x) \simeq \rho(0)$, equations (4.18) and (4.19) become

$$\underbrace{\gamma\tilde{\psi}_i}_{\sim\gamma\tau_A\frac{\psi}{\tau_A}} - \underbrace{\frac{\mathrm{i}k_y B_y'(0)x}{B_z}\tilde{\phi}_i}_{\sim\frac{\phi}{v_A\psi}\frac{\delta}{L}\frac{\psi}{\tau_A}} = \underbrace{\frac{\eta}{\mu_0}\tilde{\psi}_i''}_{\sim S^{-1}\frac{L}{\delta}\frac{\psi}{\tau_A}} - \underbrace{\cancelto{\text{small by } \delta/L \ll 1}{\frac{\eta}{\mu_0}k_y^2\tilde{\psi}_i}}_{\sim S^{-1}\frac{\psi}{\tau_A}} \tag{4.32}$$

and

$$\underbrace{\frac{\mu_0\gamma\rho(0)}{\mathrm{i}k_y B_z B_y'(0)x}\tilde{\phi}_i''}_{\sim\gamma\tau_A\frac{\phi}{v_A\psi}\left(\frac{L}{\delta}\right)^3\frac{\psi}{L^2}} - \underbrace{\cancelto{\text{small by } (\delta/L)^2 \ll 1}{\frac{\mu_0\gamma\rho(0)}{\mathrm{i}k_y B_z B_y'(0)x}k_y^2\tilde{\phi}_i}}_{\sim\gamma\tau_A\frac{\phi}{v_A\psi}\frac{L}{\delta}\frac{\psi}{L^2}} = \underbrace{\tilde{\psi}_i''}_{\sim\frac{L}{\delta}\frac{\psi}{L^2}} - \underbrace{\cancelto{\text{small by } \delta/L \ll 1}{\left(k_y^2 + \frac{B_y''(0)}{B_y'(0)}\right)\tilde{\psi}_i}}_{\sim\frac{\psi}{L^2}}. \tag{4.33}$$

By balancing all the terms in equations (4.32) and (4.33), we find the order of magnitude of $\gamma\tau_A$, δ/L and $\phi/v_A\psi$,

$$\gamma\tau_A \sim S^{-3/5}, \ \frac{\phi}{v_A\psi} \sim S^{-1/5}, \ \frac{\delta}{L} \sim S^{-2/5}. \tag{4.34}$$

Note that the result for γ is consistent with our assumption (4.20), needed to solve the outer region.

According to the estimates in (4.34), $\delta \sim S^{-2/5}L$, and hence, since $\tilde{\psi}_i' \sim \psi/L$, the total change in $\tilde{\psi}_i$ across the inner region is

$$\Delta\tilde{\psi}_i \sim \delta\tilde{\psi}_i' \sim \frac{\delta}{L}\psi \sim S^{-2/5}\psi \ll \psi. \tag{4.35}$$

Thus, to lowest order in $S^{-2/5} \ll 1$, $\tilde{\psi}_i(x) \simeq \tilde{\psi}_o(0)$, where $\tilde{\psi}_o(0)$ is determined by the outer region. The fact that $\tilde{\psi}(x)$ does not change much across the inner region justifies the boundary condition (4.23) for the outer region. The approximation $\tilde{\psi}_i(x) \simeq \tilde{\psi}_o(0)$ is known as constant flux approximation.

Using $\tilde{\psi}_i(x) \simeq \tilde{\psi}_o(0)$, equation (4.32) becomes

$$\gamma\tilde{\psi}_o(0) - \frac{\mathrm{i}k_y B_y'(0)x}{B_z}\tilde{\phi}_i = \frac{\eta}{\mu_0}\tilde{\psi}_i''. \tag{4.36}$$

Using equation (4.33) to obtain $\tilde{\psi}_i''$ as a function of $\tilde{\phi}_i''$, we can rewrite equation (4.36)

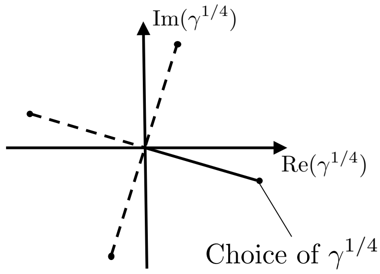

FIGURE 5. Sketch showing our choice of $\gamma^{1/4}$.

as

$$\gamma\tilde{\psi}_o(0) - \frac{\mathrm{i}k_y B_y'(0)x}{B_z}\tilde{\phi}_i = -\frac{\mathrm{i}\eta\gamma\rho(0)}{k_y B_z B_y'(0)x}\tilde{\phi}_i''. \tag{4.37}$$

For large x, the solution for $\tilde{\phi}_i$ must match with the outer region solution. The outer region solution, given in (4.21), is

$$\tilde{\phi}_o \simeq -\frac{\mathrm{i}\gamma B_z}{k_y B_y'(0)x}\tilde{\psi}_o(0) \tag{4.38}$$

for x around zero. Thus, the boundary condition for equation (4.37) is that

$$\tilde{\phi}_i \simeq -\frac{\mathrm{i}\gamma B_z}{k_y B_y'(0)x}\tilde{\psi}_o(0) \tag{4.39}$$

for $x \to \pm\infty$.

Once $\tilde{\phi}_i(x)$ is obtained from equation (4.37) with boundary condition (4.39), we can find $\tilde{\psi}_i'(x)$ by using (4.33),

$$\tilde{\psi}_i'' = -\frac{\mathrm{i}\mu_0\gamma\rho(0)}{k_y B_z B_y'(0)x}\tilde{\phi}_i''. \tag{4.40}$$

We can integrate this equation across the inner region to find the size of the jump in $\tilde{\psi}_i'$,

$$\tilde{\psi}_i'(\infty) - \tilde{\psi}_i'(-\infty) = \int_{-\infty}^{\infty}\tilde{\psi}_i''\,\mathrm{d}x = -\frac{\mathrm{i}\mu_0\gamma\rho(0)}{k_y B_z B_y'(0)}\int_{-\infty}^{\infty}\frac{\tilde{\phi}_i''}{x}\,\mathrm{d}x. \tag{4.41}$$

This jump in $\tilde{\psi}_i'$ must be equal to the discontinuity observed in the outer region solution, characterized by the parameter Δ'. Then,

$$\Delta' = -\frac{\mathrm{i}\mu_0\gamma\rho(0)}{k_y B_z B_y'(0)\tilde{\psi}_o(0)}\int_{-\infty}^{\infty}\frac{\tilde{\phi}_i''}{x}\,\mathrm{d}x. \tag{4.42}$$

This condition is an equation for the growth rate γ.

4.2.3. *Final solution*

To solve the equation for $\tilde{\phi}_i$ in (4.37), we use the normalized variables

$$U = \frac{\tilde{\phi}_i}{\Phi},\ X = \frac{x}{\delta}, \tag{4.43}$$

where the quantities Φ and δ need to be determined. With this normalization, equation (4.37) becomes

$$\gamma\tilde{\psi}_o(0) - \frac{\mathrm{i}k_y B_y'(0)\delta\Phi}{B_z}XU = -\frac{\mathrm{i}\eta\gamma\rho(0)\Phi}{k_y B_z B_y'(0)\delta^3}\frac{1}{X}\frac{\mathrm{d}^2U}{\mathrm{d}X^2}. \tag{4.44}$$

Imposing

$$\gamma\tilde{\psi}_o(0) = -\frac{\mathrm{i}k_y B_y'(0)\delta\Phi}{B_z} = -\frac{\mathrm{i}\eta\gamma\rho(0)\Phi}{k_y B_z B_y'(0)\delta^3}, \tag{4.45}$$

we obtain the normalization factors

$$\delta = \frac{(\eta\rho(0))^{1/4}\gamma^{1/4}}{(k_y B_y'(0))^{1/2}},\ \Phi = \frac{\mathrm{i}\gamma^{3/4}B_z}{(k_y B_y'(0))^{1/2}(\eta\rho(0))^{1/4}}\tilde{\psi}_o(0), \tag{4.46}$$

and the equation for $U(X)$,

$$\frac{\mathrm{d}^2 U}{\mathrm{d}X^2} = X(1 + XU). \tag{4.47}$$

With the results in (4.46), the boundary condition in (4.39) becomes

$$U(X) \simeq -\frac{1}{X} \tag{4.48}$$

for $X \to \pm\infty$.

In (4.46), there are four possible choices for $\gamma^{1/4}$: $|\gamma|^{1/4}\exp(\mathrm{i}\theta_\gamma/4 + \mathrm{i}k\pi/2)$, with $k = 0, 1, 2, 3$. Here θ_γ is the argument of γ, defined to be between $-\pi$ and π. To be consistent with the solution for $U(X)$ that we will obtain in (4.49), we choose the root of $\gamma^{1/4} = |\gamma|^{1/4}\exp(\mathrm{i}\theta_\gamma/4)$ with positive real part and the imaginary part with smallest absolute value, that is, the root with argument between $-\pi/4$ and $\pi/4$ (see figure 5).

The solution to equation (4.47) with boundary conditions (4.48) is the Rutherford-Furth solution

$$U(X) = -\frac{X}{2}\int_0^1 \exp\left(-\frac{\mu X^2}{2}\right)(1-\mu^2)^{-1/4}\,\mathrm{d}\mu. \tag{4.49}$$

(see Appendix B for a proof). This solution satisfies the boundary condition if X^2 has positive real part, $\mathrm{Re}(X^2) > 0$, because then we can change variable of integration to $s = \mu X^2/2$ to find that for $|X|$ large

$$\begin{aligned} U(X) &= -\frac{1}{X}\int_0^{X^2/2}\exp(-s)\left(1 - \overset{\text{small when } \exp(-s)\sim 1}{\cancel{\frac{4s^2}{X^4}}}\right)^{-1/4}\mathrm{d}s \simeq -\frac{1}{X}\int_0^{X^2/2}\exp(-s)\,\mathrm{d}s \\ &= -\frac{1}{X}\left[1 - \overset{\text{small for } \mathrm{Re}(X^2) > 0}{\cancel{\exp\left(-\frac{X^2}{2}\right)}}\right] \simeq -\frac{1}{X}. \end{aligned} \tag{4.50}$$

We note that $X^2 = x^2/\delta^2$, and that the argument of δ is the argument of $\gamma^{1/4}$. Thus, to ensure that $\mathrm{Re}(X^2) > 0$, we choose the root $\gamma^{1/4} = |\gamma|^{1/4}\exp(\mathrm{i}\theta_\gamma/4)$ as in figure 5. Note that the root $-|\gamma|^{1/4}\exp(\mathrm{i}\theta_\gamma/4)$ would also give $\mathrm{Re}(X^2) > 0$, but using this root would be equivalent to a simple change of sign of x, and it would give the same final $\tilde{\phi}_i(x)$.

With (4.46) and (4.49), the condition in (4.42) becomes

$$\Delta' = \frac{\mu_0(\rho(0))^{1/4}\gamma^{5/4}}{(k_y B_y'(0))^{1/2}\eta^{3/4}}\int_{-\infty}^{\infty}\frac{1}{X}\frac{\mathrm{d}^2U}{\mathrm{d}X^2}\,\mathrm{d}X = \frac{\mu_0\rho^{1/4}\gamma^{5/4}}{(k_y B_y'(0))^{1/2}\eta^{3/4}}\sqrt{2}[\Gamma(3/4)]^2. \tag{4.51}$$

where $\Gamma(\nu) = \int_0^\infty t^{\nu-1}\exp(-t)\,\mathrm{d}t$ is the Euler Gamma function (in Appendix C we show how to take the integral of the Rutherford-Furth solution). According to (4.51), Δ' is proportional to $(\gamma^{1/4})^5$. From our choice $\gamma^{1/4} = |\gamma|^{1/4}\exp(\mathrm{i}\theta_\gamma/4)$ (see figure 5), the argument of Δ' must be between $-5\pi/4$ and $5\pi/4$. Thus, positive Δ' corresponds to

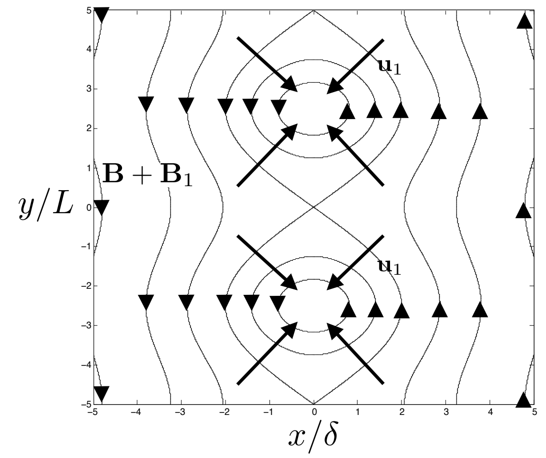

FIGURE 6. Sketch of the perturbed magnetic field $\mathbf{B}+\mathbf{B}_1$ and perturbed velocity $\mathbf{u}_1$ in the inner region. Note the different scale of x and y.

$\Delta'\exp(\mathrm{i}0)$, and negative Δ' to $|\Delta'|\exp(\pm\mathrm{i}\pi)$. Solving for γ, we obtain

$$\gamma = \left(\frac{1}{\sqrt{2}[\Gamma(3/4)]^2}\right)^{4/5}\frac{(k_yB_y'(0))^{2/5}\eta^{3/5}}{\mu_0^{4/5}(\rho(0))^{1/5}}(\Delta')^{4/5}. \tag{4.52}$$

From this formula, we see that the current sheet is unstable for $\Delta' > 0$ ($\equiv k_yL$ small) since $\gamma \propto \exp(\mathrm{i}0)$, and it is stable for $\Delta' < 0$ ($\equiv k_yL$ large) since $\gamma \propto \exp(\pm 4\mathrm{i}\pi/5)$.

The perturbed magnetic field $\mathbf{B}+\mathbf{B}_1$ and the perturbed velocity $\mathbf{u}_1$, defined in (4.10) and (4.13), are sketched in figure 6 for the unstable case. For k_yL small, the current sheet breaks into a series of individual current channels. The drive for the instability is the fact that currents of the same sign attract. Resistivity is necessary because without it, the magnetic field cannot reconnect and form islands. In the case k_yL large, in which $\Delta' < 0$, the instability is quenched by the tension of the lines that prevents reconnection from happening by making it hard to bend magnetic field lines.

REFERENCES

FURTH, H.P., KILLEEN, J. & ROSENBLUTH, M.N. 1963 Finite-resistivity instabilities of a sheet pinch. *Phys. Fluids* **6**, 459.

SCHEKOCHIHIN, A.A. 2015 Lecture notes on Kinetic Theory and Magnetohydrodynamics of Plasmas. http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/KT/.

Appendix A. Limits of $\tilde{\psi}_o$ for $k_y L \ll 1$ and $k_y L \gg 1$

In this Appendix, we show that Δ' scales as $1/k_y$ for $k_y L \ll 1$, and as $-k_y$ for $k_y L \gg 1$, hence proving that Δ' ranges from $-\infty$ to ∞ as k_y decreases from ∞ to 0..

- For $k_y L \ll 1$, we solve equation (4.22) by trying solutions of the form $\tilde{\psi}_o = \Psi \exp(-k_y |x|)$. Then, equation (4.22) becomes

$$\left[B_y^2(x) \left(\frac{\Psi}{B_y(x)} \right)' \right]' = \frac{2x}{|x|} k_y B_y(x) \Psi'. \tag{A 1}$$

The function Ψ must be continuous at $x = 0$ and it has to become constant for $x \to \pm\infty$ (recall that $\tilde{\psi}_o \sim \exp(-k_y |x|)$ for large x). Expanding Ψ in $k_y L \ll 1$, we find

$$\Psi(x) = \Psi_0(x) + \underbrace{\Psi_1(x)}_{\sim k_y L \Psi_0 \ll \Psi_0} + \ldots \tag{A 2}$$

To lowest order, equation (A 1) becomes $[B_y^2 (\Psi_0 / B_y)']' = 0$. One possible solution to this equation that satisfies the boundary conditions is $\Psi_0(x) = B_y(|x|)$. It will not become obvious why we choose this symmetric form until we solve the next order of equation (A 1), given by

$$\left[B_y^2(x) \left(\frac{\Psi_1}{B_y(x)} \right)' \right]' = \frac{2x}{|x|} k_y B_y(x) \Psi_0' = 2 k_y B_y(x) B_y'(x). \tag{A 3}$$

We can integrate this equation imposing first that $\Psi_1' \to 0$ for $x \to \pm\infty$ to find

$$\left(\frac{\Psi_1}{B_y(x)} \right)' = -k_y \left(\frac{B_{y\infty}^2}{B_y^2(x)} - 1 \right), \tag{A 4}$$

where $B_{y\infty} = B_y(x \to \infty)$ is the value of B_y at infinity. We integrate equation (A 4) again imposing that $\Psi_1 \to 0$ for $x \to \pm\infty$ to obtain

$$\Psi_1(x) = k_y B_y(|x|) \int_{|x|}^{\infty} \left(\frac{B_{y\infty}^2}{B_y^2(s)} - 1 \right) \mathrm{d}s. \tag{A 5}$$

(we could have chosen that Ψ_1 tend to a constant different from zero, but this constant can be absorbed into Ψ_0). The solution (A 5) is continuous at $x = 0$ because we chose the lowest order symmetric function $\Psi_0 = B_y(|x|)$. Adding Ψ_0 and Ψ_1, we find that $\tilde{\psi}_o$ is given by

$$\tilde{\psi}_o(x) = B_y(|x|) \exp(-k_y |x|) \left[1 + k_y \int_{|x|}^{\infty} \left(\frac{B_{y\infty}^2}{B_y^2(s)} - 1 \right) \mathrm{d}s + O(k_y^2 L^2) \right]. \tag{A 6}$$

The solution in (A 6) gives $\Delta' = 2[B_y'(0)]^2 / k_y B_{y\infty}^2 \to +\infty$ for $k_y \to 0$.

- For $k_y L \gg 1$, equation (4.22) becomes $\tilde{\psi}_o'' - k_y^2 \tilde{\psi}_o = 0$. The solution to this equation that goes to zero at $x \to \pm\infty$ and is continuous at $x = 0$ is

$$\tilde{\psi}_o = \exp(-k_y |x|). \tag{A 7}$$

This solution leads to $\Delta' = -2k_y \to -\infty$.
We will see shortly that the mode is unstable when $\Delta' > 0$, and stable when $\Delta' < 0$. Thus, the mode will be unstable for small k_y, and stable for large k_y.

Appendix B. Rutherford-Furth solution

In this Appendix we check by direct substitution that the function $U(X)$ in (4.49) is a solution to equation (4.47). By differentiating with respect to X twice, we find

$$\frac{\mathrm{d}^2U}{\mathrm{d}X^2} = \frac{3X}{2}\int_0^1 \exp\left(-\frac{\mu X^2}{2}\right)\mu(1-\mu^2)^{-1/4}\,\mathrm{d}\mu - \frac{X^3}{2}\int_0^1 \exp\left(-\frac{\mu X^2}{2}\right)\mu^2(1-\mu^2)^{-1/4}\,\mathrm{d}\mu. \tag{B 1}$$

The first integral can be rewritten in a convenient form by integrating by parts in μ,

$$\begin{aligned}\frac{3X}{2}\int_0^1 \exp\left(-\frac{\mu X^2}{2}\right)\mu(1-\mu^2)^{-1/4}\,\mathrm{d}\mu &= -X\left[\exp\left(-\frac{\mu X^2}{2}\right)(1-\mu^2)^{3/4}\right]_{\mu=0}^{\mu=1}\\ &\quad -\frac{X^3}{2}\int_0^1 \exp\left(-\frac{\mu X^2}{2}\right)(1-\mu^2)^{3/4}\,\mathrm{d}\mu\\ &= X - \frac{X^3}{2}\int_0^1 \exp\left(-\frac{\mu X^2}{2}\right)(1-\mu^2)(1-\mu^2)^{-1/4}\,\mathrm{d}\mu.\end{aligned} \tag{B 2}$$

Using this result in (B 1), we find

$$\frac{\mathrm{d}^2U}{\mathrm{d}X^2} = X - \frac{X^3}{2}\int_0^1 \exp\left(-\frac{\mu X^2}{2}\right)(1-\mu^2)^{-1/4}\,\mathrm{d}\mu = X(1-XU), \tag{B 3}$$

proving that the function $U(X)$ in (4.49) is a solution to equation (4.47).

Appendix C. Integral of the Rutherford-Furth solution to obtain Δ'

In this Appendix, we take the integral $\int_{-\infty}^{\infty} X^{-1}(\mathrm{d}^2U/\mathrm{d}X^2)\,\mathrm{d}X$. Using (B 1), we find

$$\int_{-\infty}^{\infty}\frac{1}{X}\frac{\mathrm{d}^2U}{\mathrm{d}X^2}\,\mathrm{d}X = \int_{-\infty}^{\infty}\mathrm{d}X\int_0^1 \mathrm{d}\mu\,\exp\left(-\frac{\mu X^2}{2}\right)\left(\frac{3}{2}-\frac{\mu X^2}{2}\right)\mu(1-\mu^2)^{-1/4}. \tag{C 1}$$

We can take the integral in X first. Using the usual Gaussian integrals, we obtain

$$\int_{-\infty}^{\infty}\frac{1}{X}\frac{\mathrm{d}^2U}{\mathrm{d}X^2}\,\mathrm{d}X = \sqrt{2\pi}\int_0^1 \mu^{1/2}(1-\mu^2)^{-1/4}\,\mathrm{d}\mu. \tag{C 2}$$

Changing to the new variable $t=\mu^2$, the integral becomes

$$\int_{-\infty}^{\infty}\frac{1}{X}\frac{\mathrm{d}^2U}{\mathrm{d}X^2}\,\mathrm{d}X = \sqrt{\frac{\pi}{2}}\int_0^1 t^{-1/4}(1-t)^{-1/4}\,\mathrm{d}t = \sqrt{\frac{\pi}{2}}\mathrm{B}(3/4,3/4), \tag{C 3}$$

where $\mathrm{B}(\nu,\mu)=\int_0^1 t^{\nu-1}(1-t)^{\mu-1}\,\mathrm{d}t$ is the Euler Beta function. Since

$$\mathrm{B}(\nu,\mu) = \frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}, \tag{C 4}$$

we finally obtain

$$\int_{-\infty}^{\infty}\frac{1}{X}\frac{\mathrm{d}^2U}{\mathrm{d}X^2}\,\mathrm{d}X = \sqrt{\frac{\pi}{2}}\frac{[\Gamma(3/4)]^2}{\Gamma(3/2)} = \sqrt{2}[\Gamma(3/4)]^2. \tag{C 5}$$