Tokamak physics: Pfirsch-Schlüter regime

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1. Introduction

In these notes, we study collisional transport in tokamak plasmas. To simplify the problem, we only study electron transport and we consider a tokamak that is sufficiently collisional to obey Braginskii equations. This collisional regime is known as Pfirsch-Schlüter regime (Pfirsch & Schlüter 1962).

In these notes, we consider a Braginskii plasma composed of one ion species with charge e and mass m_i, and electrons with charge $-e$ and mass m_e. To simplify the derivation, we assume that the plasma flow is small,

$$\frac{\rho_i}{L} v_{ti} \lesssim u_i \sim u_e \ll v_{ti}, \tag{1.1}$$

where L is the characteristic size of the tokamak. Hence, the electric field satisfies

$$|\mathbf{E}_\perp| \ll v_{ti} B. \tag{1.2}$$

We also assume that the parallel current is small and comparable to the perpendicular current,

$$|u_{i\parallel} - u_{e\parallel}| \sim |\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}| \sim \frac{\rho_i}{L} v_{ti} \sim \frac{\rho_e}{L} v_{te}. \tag{1.3}$$

We will see that assumption (1.3) is equivalent to assuming that the ratio β between the thermal energy and the tokamak energy is of order unity,

$$\beta = \frac{2\mu_0 P}{B^2} \sim 1, \tag{1.4}$$

where P is the total pressure. Finally, we order the mass ratio as

$$\sqrt{\frac{m_e}{m_i}} \sim \frac{\rho_e}{L} \tag{1.5}$$

to ensure that the collisional energy exchange between electron and ions is of the order of the rest of the terms in the electron energy equation, thus keeping as much physical effects as is possible. Note that (1.5) implies that $\rho_i/L \sim 1$, and hence it is not relevant to a real tokamak. When $\rho_i/L \ll 1$, $\rho_e/L \ll \sqrt{m_e/m_i}$ and the collisional energy exchange will dominate, forcing the ion and electron temperatures to be equal to each other.

Assumptions (1.1), (1.2), (1.3) and (1.5) are different from Braginskii's ordering $u_i \sim u_e \sim v_{ti}$, $|\mathbf{E}_\perp| \sim v_{ti}B$, $|u_{i\parallel} - u_{e\parallel}| \sim v_{ti}$ and $\sqrt{m_e/m_i} \sim \lambda_{ee}/L \gg \rho_e/L$, but we can use Braginskii's equations if we neglect some of the terms proportional to $u_{i\parallel}$, $u_{e\parallel}$ and m_e/m_i.

2. Tokamak electromagnetic fields

A tokamak is an axisymmetric toroidal magnetic confinement device (see figure 1(a)). One can think of the tokamak as a cylinder that is bent so that its top and its bottom connect to each other.

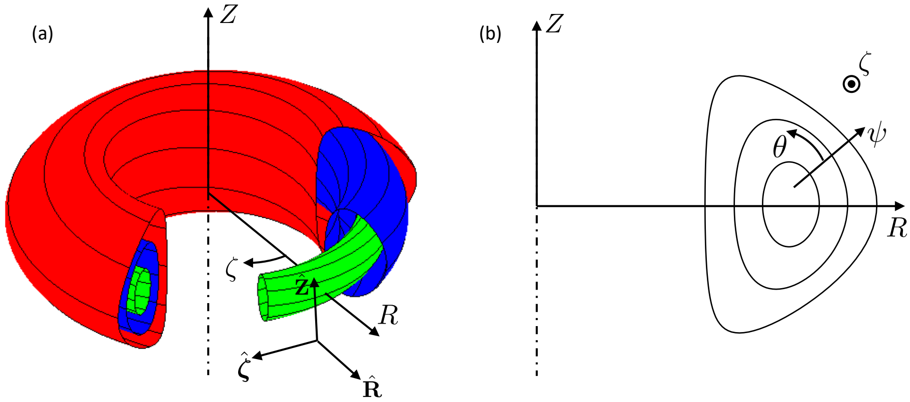

FIGURE 1. (a) Three nested tokamak flux surfaces in green, blue and red. The axis of symmetry of the tokamak is the dash-dot line. A few magnetic field lines tangent to the flux surfaces are plotted in black. The cylindrical coordinates $\{R, Z, \zeta\}$ and their corresponding unit vectors $\{\hat{\mathbf{R}}, \hat{\mathbf{Z}}, \hat{\boldsymbol{\zeta}}\}$ are shown. (b) Cut through a plane of constant ζ of the same three flux surfaces. Both the cylindrical coordinates $\{R, Z, \zeta\}$ and the flux coordinates $\{\psi, \theta, \zeta\}$ are shown. The toroidal angle ζ increases towards the reader.

2.1. *Tokamak magnetic field*

To describe the magnetic field in a tokamak, we use the cylindrical coordinates $\{R, Z, \zeta\}$ centered around the axis of symmetry of the tokamak (see figure 1). The angle ζ is the toroidal angle. The unit vectors $\hat{\mathbf{R}}$, $\hat{\mathbf{Z}}$ and $\hat{\boldsymbol{\zeta}}$ point in the direction of increase of the variables R, Z and ζ.

The magnetic field in a tokamak is axisymmetric, that is, it does not depend on ζ,

$$\mathbf{B} = B_R(R,Z)\hat{\mathbf{R}} + B_Z(R,Z)\hat{\mathbf{Z}} + B_\zeta(R,Z)\hat{\boldsymbol{\zeta}}. \tag{2.1}$$

This magnetic field has to satisfy $\nabla \cdot \mathbf{B} = 0$, leading to

$$\nabla \cdot \mathbf{B} = \frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_Z}{\partial Z} = 0. \tag{2.2}$$

This equation implies that there is a flux function $\psi(R, Z)$ such that

$$B_R = -\frac{1}{R}\frac{\partial \psi}{\partial Z}, \quad B_Z = \frac{1}{R}\frac{\partial \psi}{\partial R}. \tag{2.3}$$

According to this result, the component of the magnetic field perpendicular to the ζ-direction, known as poloidal magnetic field, can be written as

$$\boxed{\mathbf{B}_p(R,Z) = B_R(R,Z)\hat{\mathbf{R}} + B_Z(R,Z)\hat{\mathbf{Z}} = \nabla\zeta \times \nabla\psi,} \tag{2.4}$$

where $\nabla\zeta = \hat{\boldsymbol{\zeta}}/R$. The surfaces of constant ψ are known as flux surfaces, and they are parallel to the magnetic field $\mathbf{B}$, that is, $\mathbf{B}\cdot\nabla\psi = 0$. In figure 1, we show three tokamak flux surfaces. The existence of flux surfaces is a non-trivial property of the tokamak magnetic field. In general, a magnetic field need not have flux surfaces.

In addition to $\nabla\cdot\mathbf{B} = 0$, the magnetic field needs to satisfy plasma force balance. The total momentum equation for the plasma is

$$\underbrace{n_i m_i \left(\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i \right)}_{\text{small in } \frac{u_i^2}{v_{ti}^2}} = -\nabla(p_e + p_i) - \underbrace{\nabla\cdot\boldsymbol{\Pi}_i}_{\text{small in } \frac{\lambda_{ii}}{L}\frac{u_i}{v_{ti}}} + \mathbf{J}\times\mathbf{B}. \tag{2.5}$$

Then, using Ampere's law $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$, we find

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla P, \tag{2.6}$$

where $P = p_i + p_e$ is the total pressure. Equation (2.6) imposes several constraints on $\mathbf{B}$ and the total pressure P. Taking the scalar product of equation (2.6) with $\mathbf{B}$, we obtain

$$\mathbf{B} \cdot \nabla P = 0 \Rightarrow -\frac{\partial \psi}{\partial Z}\frac{\partial P}{\partial R} + \frac{\partial \psi}{\partial R}\frac{\partial P}{\partial Z} = 0. \tag{2.7}$$

Thus, P is function of the flux function $\psi(R, Z)$ only,

$$\boxed{P(R, Z) = P(\psi(R, Z)).} \tag{2.8}$$

The pressure is constant along flux surfaces because there is no net force along the magnetic field (recall that we have neglected inertia and viscosity). Note that we can confine the plasma because P can change with ψ and hence the pressure can be zero at the external flux surface and different from zero at the magnetic axis, the degenerate flux surface in the middle of the tokamak that is one single toroidal field line.

Using equation (2.8), we can write equation (2.6) as

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \frac{\mathrm{d}P}{\mathrm{d}\psi} \nabla \psi. \tag{2.9}$$

Taking the scalar product of equation (2.9) with $\nabla \times \mathbf{B}$, we find

$$(\nabla \times \mathbf{B}) \cdot \nabla \psi = 0, \tag{2.10}$$

that is, the plasma current has to be parallel to the flux surfaces. This constraint on the current has important consequences for the toroidal magnetic field B_ζ. Indeed, using

$$\nabla \times \mathbf{B} = \frac{\partial B_\zeta}{\partial Z}\hat{\mathbf{R}} - \frac{1}{R}\frac{\partial}{\partial R}(RB_\zeta)\hat{\mathbf{Z}} + \left(\frac{\partial B_Z}{\partial R} - \frac{\partial B_R}{\partial Z}\right)\hat{\boldsymbol{\zeta}}, \tag{2.11}$$

we find that $(\nabla \times \mathbf{B}) \cdot \nabla \psi = 0$ becomes

$$\frac{\partial \psi}{\partial R}\frac{\partial}{\partial Z}(RB_\zeta) - \frac{\partial \psi}{\partial Z}\frac{\partial}{\partial R}(RB_\zeta) = 0. \tag{2.12}$$

Thus, RB_ζ, like the total pressure P, is a function of the flux function ψ only,

$$\boxed{RB_\zeta(R, Z) = I(\psi(R, Z)).} \tag{2.13}$$

Using equations (2.4) and (2.13), the magnetic field in a tokamak can be written as

$$\boxed{\mathbf{B} = \nabla \zeta \times \nabla \psi + I \nabla \zeta.} \tag{2.14}$$

and $\nabla \times \mathbf{B}$ becomes

$$\begin{aligned} \nabla \times \mathbf{B} &= \frac{1}{R}\frac{\mathrm{d}I}{\mathrm{d}\psi}\left(\frac{\partial \psi}{\partial Z}\hat{\mathbf{R}} - \frac{\partial \psi}{\partial R}\hat{\mathbf{Z}}\right) + \left[\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) + \frac{1}{R}\frac{\partial^2 \psi}{\partial Z^2}\right]\hat{\boldsymbol{\zeta}} \\ &= -\mathbf{B}\frac{\mathrm{d}I}{\mathrm{d}\psi} + \left[\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) + \frac{1}{R}\frac{\partial^2 \psi}{\partial Z^2} + \frac{I}{R}\frac{\mathrm{d}I}{\mathrm{d}\psi}\right]\hat{\boldsymbol{\zeta}}. \end{aligned} \tag{2.15}$$

Using equations (2.14) and (2.15), the force balance equation (2.9) finally becomes

$$\boxed{R^2 \nabla \cdot \left(\frac{\nabla \psi}{R^2}\right) = -I\frac{\mathrm{d}I}{\mathrm{d}\psi} - \mu_0 R^2 \frac{\mathrm{d}P}{\mathrm{d}\psi}.} \tag{2.16}$$

where

$$R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right)=R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial Z^2}. \tag{2.17}$$

This nonlinear equation for $\psi(R,Z)$ is known as the Grad-Shafranov equation, and it determines the magnetic field in a tokamak given $I(\psi)$, $P(\psi)$ and the shape of one of the flux surfaces (if we know one of the flux surfaces, we can impose that $\psi(R,Z)=0$ on that flux surface as a boundary condition for equation (2.16)). Combining equations (2.15) and (2.16), we also obtain the tokamak current

$$\boxed{\mathbf{J}=\frac{\nabla\times\mathbf{B}}{\mu_0}=-\frac{\mathbf{B}}{\mu_0}\frac{\mathrm{d}I}{\mathrm{d}\psi}-\frac{\mathrm{d}P}{\mathrm{d}\psi}R\hat{\boldsymbol{\zeta}}.} \tag{2.18}$$

This is a common form for flows in a tokamak: a rigid rotation (piece proportional to $R\hat{\boldsymbol{\zeta}}$) plus a parallel flow without divergence (piece proportional to $\mathbf{B}$).

If we assume that all the terms in equation (2.16) are of the same order, we find that $\psi\sim IL\sim L^2B$ and that $\beta\sim 1$, as we assumed in equation (1.4). Using these estimates in (2.18), we obtain that $\mathbf{J}\sim(\rho_i/L)en_e v_{ti}$, which is consistent with our assumption (1.3). In general, β is small in a tokamak to avoid instabilities (tokamak β ranges from a few percent to tens of percent), but the results that we will obtain are robust, and we can perform a subsidiary expansion in $\beta\ll 1$ if necessary.

2.2. *Tokamak electric field*

The tokamak electric field must satisfy $\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t$. In a steady state plasma, this equation becomes $\nabla\times\mathbf{E}=0$, and its solution is

$$\mathbf{E}=-\nabla\phi+R_0E_\zeta\nabla\zeta, \tag{2.19}$$

where $\phi(R,Z)$ is the electrostatic potential and R_0 and E_ζ are constants. The toroidal electric field E_ζ represents the electric field generated at radius R_0 by a solenoid with a time-varying current wound around the axis of symmetry of the tokamak (see figure 2). This solenoid is also known as transformer. Applying the induction equation to a toroidal loop C of radius R_C that is inside the plasma, we find

$$2\pi R_0E_\zeta=\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_C}\mathbf{B}\cdot\hat{\mathbf{Z}}\,\mathrm{d}^2S, \tag{2.20}$$

where S_C is the circle whose boundary is the toroidal loop C. The left side of equation (2.20) is independent of the radius R_C due to the $1/R$ in $\nabla\zeta$. Physically, the result is independent of the radius R_C as long as C is inside the plasma because we assume that the change in magnetic flux is confined to the solenoid outside of the plasma ($\mathbf{B}_{\mathrm{transf}}(t)$ in figure 2).

3. Flux coordinates

In tokamak theory, we use flux coordinates $\{\psi,\theta,\zeta\}$. The flux function ψ designates a flux surface, and the angle ζ a cut of that flux surface with a plane of constant ζ. To locate a point within the resulting 2D figure, we use a third coordinate θ (see figure 1(b)). Since the curve determined by given values of ψ and θ is a closed curve, it is natural to define θ to be an angle, that is, a coordinate that can take values between 0 an 2π and is periodic.

The variable ζ is perpendicular to ψ and θ, that is,

$$\nabla\zeta\cdot\nabla\psi=0,\quad \nabla\zeta\cdot\nabla\theta=0, \tag{3.1}$$

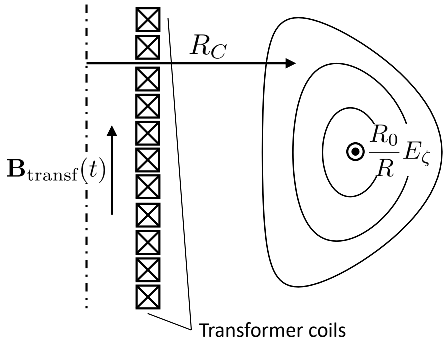

FIGURE 2. Sketch of a tokamak transformer. The axis of symmetry is the dash-dot line. The current in the transformer coils is directed into the page, and the toroidal electric field $(R_0/R)E_\zeta$ points towards the reader. The vertical magnetic field inside the transformer is denoted by $\mathbf{B}_{\rm transf}(t)$. If $\mathbf{B}_{\rm transf}(t)\cdot\hat{\mathbf{Z}}$ increases in time, E_ζ is positive.

but ψ and θ need not be perpendicular, $\nabla\psi\cdot\nabla\theta\neq 0$. Thus, one needs to be careful with the formulas for coordinate changes given in textbooks.

The determinant of the Jacobian matrix of the transformation between $\mathbf{r}$ and $\{\psi,\theta,\zeta\}$ is

$$\mathcal{J}=\det\left(\frac{\partial(\mathbf{r})}{\partial(\psi,\theta,\zeta)}\right)=\frac{1}{\nabla\psi\cdot(\nabla\theta\times\nabla\zeta)}=\frac{1}{\mathbf{B}\cdot\nabla\theta}, \tag{3.2}$$

where, to obtain the final result, we have used equation (2.14). With the determinant of the Jacobian, we can calculate divergences in flux coordinates,

$$\begin{aligned}\nabla\cdot\mathbf{\Gamma}&=\frac{1}{\mathcal{J}}\frac{\partial}{\partial\psi}(\mathcal{J}\mathbf{\Gamma}\cdot\nabla\psi)+\frac{1}{\mathcal{J}}\frac{\partial}{\partial\theta}(\mathcal{J}\mathbf{\Gamma}\cdot\nabla\theta)\\&=\mathbf{B}\cdot\nabla\theta\frac{\partial}{\partial\psi}\left(\frac{\mathbf{\Gamma}\cdot\nabla\psi}{\mathbf{B}\cdot\nabla\theta}\right)+\mathbf{B}\cdot\nabla\theta\frac{\partial}{\partial\theta}\left(\frac{\mathbf{\Gamma}\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\theta}\right).\end{aligned} \tag{3.3}$$

Using the determinant of the Jacobian, we can also calculate volume integrals in flux coordinates,

$$\int f\,\mathrm{d}^3r=\int\frac{f}{\mathbf{B}\cdot\nabla\theta}\,\mathrm{d}\psi\,\mathrm{d}\theta\,\mathrm{d}\zeta. \tag{3.4}$$

With the volume integral in flux coordinates, we can define the flux surface average. The flux surface average is a volume average over a volume contained between two flux surfaces separated by an infinitesimal small $\Delta\psi$. Thus,

$$\langle f\rangle_\psi=\lim_{\Delta\psi\to 0}\frac{\int_{\Delta\psi}\mathrm{d}\psi\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\zeta\,\frac{f}{\mathbf{B}\cdot\nabla\theta}}{\int_{\Delta\psi}\mathrm{d}\psi\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\zeta\,\frac{1}{\mathbf{B}\cdot\nabla\theta}}=\frac{2\pi}{V'}\int_0^{2\pi}\frac{f}{\mathbf{B}\cdot\nabla\theta}\,\mathrm{d}\theta, \tag{3.5}$$

where

$$V'=2\pi\int_0^{2\pi}\frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta} \tag{3.6}$$

is the derivative with respect to ψ of $V(\psi)=2\pi\int^\psi\mathrm{d}\psi'\int_0^{2\pi}\mathrm{d}\theta\,(\mathbf{B}\cdot\nabla\theta)^{-1}$, which is the volume contained by the flux surface ψ. The flux surface average of the divergence in equation (3.3) takes the form

$$\langle\nabla\cdot\mathbf{\Gamma}\rangle_\psi=\frac{1}{V'}\frac{\mathrm{d}}{\mathrm{d}\psi}\left(V'\langle\mathbf{\Gamma}\cdot\nabla\psi\rangle_\psi\right). \tag{3.7}$$

4. Electron collisional transport in a tokamak

The tokamak magnetic field, density and temperature profiles must be sustained by injecting particles, momentum and energy. By studying the electrons, we will be able determine the toroidal electric field E_ζ, the source of particles $S_n(\psi, \theta)$ and the source of electron energy $S_{Ee}(\psi, \theta)$ needed to maintain the tokamak in operation. We will use the parallel electron momentum equation,

$$0 = -\hat{\mathbf{b}} \cdot \nabla p_e + en_e \hat{\mathbf{b}} \cdot \nabla \phi - en_e R_0 E_\zeta \hat{\mathbf{b}} \cdot \nabla \zeta + \underbrace{\frac{0.51 m_e \nu_{ei}}{e} J_\parallel - 0.71 n_e \hat{\mathbf{b}} \cdot \nabla T_e}_{= F_{ei\parallel}}, \quad (4.1)$$

the electron continuity equation,

$$\nabla \cdot (n_e \mathbf{u}_e) = S_n \quad (4.2)$$

and the electron total energy equation,

$$\nabla \cdot \left[\left(\frac{5}{2} p_e + \cancelto{\text{small in } \frac{m_e}{m_i}\frac{u_e^2}{v_{ti}^2}}{\frac{1}{2} n_e m_e u_e^2} \right) \mathbf{u}_e + \mathbf{q}_e \right] = -en_e \mathbf{u}_e \cdot \mathbf{E} + \mathbf{u}_i \cdot \mathbf{F}_{ei} + \frac{3 n_e m_e \nu_{ei}}{m_i} (T_i - T_e) + S_{Ee}. \quad (4.3)$$

The sizes of E_ζ, S_n and S_{Ee} needed to maintain the tokamak in operation are of order

$$E_\zeta \sim \frac{\nu_{ei}}{\Omega_e} \frac{T_e}{eL}, \quad S_n \sim \frac{\nu_{ei}}{\Omega_e} \frac{\rho_e}{L} \frac{n_e v_{te}}{L}, \quad S_{Ee} \sim \frac{\nu_{ei}}{\Omega_e} \frac{\rho_e}{L} \frac{p_e v_{te}}{L}, \quad (4.4)$$

as we will show.

In a tokamak, the density, the electrostatic potential and the temperature depend mostly on ψ,

$$\begin{aligned} n_e(\psi, \theta) &= n_{e0}(\psi) + n_{e1}(\psi, \theta), \\ \phi(\psi, \theta) &= \phi_0(\psi) + \phi_1(\psi, \theta), \\ T_e(\psi, \theta) &= T_{e0}(\psi) + T_{e1}(\psi, \theta), \\ T_i(\psi, \theta) &= T_{i0}(\psi) + T_{i1}(\psi, \theta). \end{aligned}$$

Using the electron continuity equation, the electron parallel momentum equation and the electron energy equation, we are going to show that

$$\frac{n_{e1}}{n_{e0}} - \frac{e\phi_1}{T_{e0}} \sim \frac{\nu_{ei}}{\Omega_e} \ll 1, \quad \frac{T_{e1}}{T_{e0}} \sim \frac{\nu_{ei}}{\Omega_e} \ll 1. \quad (4.5)$$

Using the ion energy equation with a source $S_{Ei} \sim (\nu_{ii}/\Omega_i)(\rho_i/L)(p_i v_{ti}/L)$ and following a procedure similar to the one we are going to follow for the electron energy equation, it is possible to show that $T_{i1}/T_{i0} \sim \nu_{ii}/\Omega_i \ll 1$. Using the total momentum equation (2.5) but keeping the parallel ion viscosity, the ion inertia and a source $S_{Mi} \sim (\nu_{ii}/\Omega_i)(\rho_i/L)(u_i/v_{ti})(p_i/L)$, one can show that the perturbation to the total pressure $P_1 = n_{e1}(T_{e0} + T_{i0}) + n_{e0}(T_{e1} + T_{i1})$ is smaller than the lowest order total pressure $P \simeq n_{e0}(T_{e0} + T_{i0})$ by a factor of $u_i^2/v_{ti}^2 \ll 1$ or $(\lambda_{ii}/L)(\rho_i/L) \ll 1$, whichever is larger (the terms of order u_i^2/v_{ti}^2 are due to inertia, and the terms of order $(\lambda_{ii}/L)(\rho_i/L)$ are due to viscosity; it may seem that the terms due to the viscosity should be of order $(\lambda_{ii}/L)(u_i/v_{ti})$, but a lowest order cancelation leads to $(\lambda_{ii}/L)(\rho_i/L)$).

4.1. *Parallel electron momentum equation*

The parallel momentum equation (4.1) gives to lowest order

$$\frac{T_{e0}}{e}\hat{\mathbf{b}}\cdot\nabla\theta\frac{\partial}{\partial\theta}\left(\frac{n_{e1}}{n_{e0}}-\frac{e\phi_1}{T_{e0}}+1.71\frac{T_{e1}}{T_{e0}}\right)=\eta_0 J_\parallel-\frac{IR_0E_\zeta}{R^2B}. \tag{4.6}$$

where

$$\eta_0=\frac{0.51m_e\nu_{ei0}}{e^2n_{e0}} \tag{4.7}$$

is the lowest order resistivity and

$$\nu_{ei0}=\frac{4\sqrt{2\pi}}{3}\frac{e^4n_{e0}\ln\Lambda_{ei}}{(4\pi\epsilon_0)^2m_e^{1/2}T_{e0}^{3/2}} \tag{4.8}$$

is the lowest order electron-ion collision frequency. By dividing equation (4.6) by $\hat{\mathbf{b}}\cdot\nabla\theta$ and integrating in θ, we find

$$0=\eta_0\langle J_\parallel B\rangle_\psi-\left\langle\frac{1}{R^2}\right\rangle_\psi IR_0E_\zeta. \tag{4.9}$$

This equation indicates that only the transformer electric field can drive an average current in the parallel direction. The electrostatic potential, the pressure and the temperature are periodic, and hence the forces exerted by their gradients average to zero.

Equation (4.9) can be used to determine the transformer electric field needed to sustain the tokamak magnetic field. From equation (2.15), we obtain

$$J_\parallel B=\frac{(\nabla\times\mathbf{B})\cdot\mathbf{B}}{\mu_0}=-\frac{|\nabla\psi|^2}{\mu_0R^2}\frac{\mathrm{d}I}{\mathrm{d}\psi}+\frac{I}{\mu_0}\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right). \tag{4.10}$$

Flux surface averaging this equation and using equation (3.7), the average parallel current becomes

$$\begin{aligned}\langle J_\parallel B\rangle_\psi&=-\left\langle\frac{|\nabla\psi|^2}{R^2}\right\rangle_\psi\frac{1}{\mu_0}\frac{\mathrm{d}I}{\mathrm{d}\psi}+\frac{I}{\mu_0V'}\frac{\mathrm{d}}{\mathrm{d}\psi}\left(V'\left\langle\frac{|\nabla\psi|^2}{R^2}\right\rangle_\psi\right)\\&=\frac{I^2}{\mu_0V'}\frac{\mathrm{d}}{\mathrm{d}\psi}\left(V'\left\langle\frac{|\nabla\psi|^2}{IR^2}\right\rangle_\psi\right).\end{aligned} \tag{4.11}$$

Thus, equation (4.9) gives

$$\boxed{R_0E_\zeta=\frac{\eta_0I}{\mu_0\langle R^{-2}\rangle_\psi V'}\frac{\mathrm{d}}{\mathrm{d}\psi}\left(V'\left\langle\frac{|\nabla\psi|^2}{IR^2}\right\rangle_\psi\right)\sim\frac{B^2}{2\mu_0P}\frac{\nu_{ei}}{\Omega_e}\frac{T_e}{e}.} \tag{4.12}$$

Equation (4.12) is an equation for $I(\psi)$ (recall that, according to equation (2.16), $\psi(R,Z)$ is a functional of $I(\psi)$). Note that equation (4.12) gives the expected order of magnitude estimate (4.4) for E_ζ (recall that we are assuming $\beta=2\mu_0P/B^2\sim1$). Note as well that equation (4.12) gives little flexibility for shaping the tokamak magnetic field because E_ζ is a constant. Moreover, one cannot maintain E_ζ for long times because it would require an ever increasing transformer magnetic field. For these reasons, current tokamaks use radiofrequency waves to drive current. The effect of these waves cannot be included as a simple force in the fluid equations.

But equation (4.6) does not only give R_0E_ζ. It also partially determines $n_{e1}/n_{e0}-e\phi_1/T_{e0}$ and T_{e1}/T_{e0}. Expression (4.11) emphasizes that the current is the curl of the magnetic field, and hence it is a function of the gradients of $\mathbf{B}$. However, according to equation (2.18), the current is also a function of the pressure gradient. Expression (2.18) is the preferred choice to determine $n_{e1}/n_{e0} - e\phi_1/T_{e0}$ and T_{e1}/T_{e0} for reasons that will become clear soon. Using equation (2.18), we find

$$J_\| B = -\frac{B^2}{\mu_0}\frac{\mathrm{d}I}{\mathrm{d}\psi} - I\frac{\mathrm{d}P}{\mathrm{d}\psi}. \tag{4.13}$$

Hence, equation (4.9) gives

$$\frac{\mathrm{d}I}{\mathrm{d}\psi} = -\frac{\mu_0 I}{\langle B^2\rangle_\psi}\frac{\mathrm{d}P}{\mathrm{d}\psi} - \left\langle \frac{1}{R^2}\right\rangle_\psi \frac{\mu_0 I R_0 E_\zeta}{\eta_0\langle B^2\rangle_\psi}, \tag{4.14}$$

and the tokamak parallel current can be finally written as

$$J_\| = \left\langle \frac{1}{R^2}\right\rangle_\psi \frac{IBR_0E_\zeta}{\eta_0\langle B^2\rangle_\psi} - \frac{I}{B}\frac{\mathrm{d}P}{\mathrm{d}\psi}\left(1 - \frac{B^2}{\langle B^2\rangle_\psi}\right). \tag{4.15}$$

Using this result in (4.6), we obtain

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\frac{n_{e1}}{n_{e0}} - \frac{e\phi_1}{T_{e0}} + 1.71\frac{T_{e1}}{T_{e0}}\right) = &-\frac{0.51 I m_e \nu_{ei0}}{e p_{e0}\mathbf{B}\cdot\nabla\theta}\left(1 - \frac{B^2}{\langle B^2\rangle_\psi}\right)\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0}+p_{i0}) \\ &- \frac{eIR_0E_\zeta}{T_{e0}\mathbf{B}\cdot\nabla\theta}\left(\frac{1}{R^2} - \left\langle\frac{1}{R^2}\right\rangle_\psi \frac{B^2}{\langle B^2\rangle_\psi}\right),\end{aligned} \tag{4.16}$$

where we have used $P \simeq p_{e0} + p_{i0}$, $p_{e0} = n_{e0}T_{e0}$ and $p_{i0} = n_{e0}T_{i0}$.

Equation (4.16) shows why expression (2.18) is the preferred form of the current to calculate $n_{e1}/n_{e0} - e\phi_1/T_{e0}$ and T_{e1}/T_{e0}. Expression (4.16) emphasizes that $n_{e1}/n_{e0} - e\phi_1/T_{e0}$ and T_{e1}/T_{e0} are driven by the background pressure gradient and the non-uniformity of the transformer electric field $R_0E_\zeta\nabla\zeta$. Physically, the current must satisfy $\nabla\cdot\mathbf{J} = 0$ so that the charge does not accumulate in parts of the tokamak. However, the perpendicular current is completely determined by equation (2.5),

$$\mathbf{J}_\perp = en_e(\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}) \simeq \frac{1}{B}\hat{\mathbf{b}}\times\nabla(p_e + p_i), \tag{4.17}$$

and in general $\nabla\cdot\mathbf{J}_\perp \neq 0$. Moreover, the divergence of the parallel current driven by the toroidal electric field, $R_0E_\zeta\hat{\mathbf{b}}\cdot\nabla\zeta/\eta_0$, does not vanish either, $\nabla\cdot[(R_0E_\zeta\hat{\mathbf{b}}\cdot\nabla\zeta/\eta_0)\hat{\mathbf{b}}] \neq 0$. As a result, the tokamak must establish a parallel current (known as Pfirsch-Schlüter current) to ensure that $\nabla\cdot\mathbf{J} = 0$. This current is driven by pressure, temperature and electrostatic potential gradients parallel to the magnetic field, that is, by $n_{e1}/n_{e0} - e\phi_1/T_{e0}$ and T_{e1}/T_{e0}.

4.2. *Continuity equation*

We average equation (4.2) over flux surfaces to determine the particle balance in each flux surface. Flux surface averaging equation (4.2) and using equation (3.7), we find

$$\boxed{\frac{1}{V'}\frac{\mathrm{d}}{\mathrm{d}\psi}\left(V'\langle n_e\mathbf{u}_e\cdot\nabla\psi\rangle_\psi\right) = \langle S_n\rangle_\psi.} \tag{4.18}$$

To calculate the average electron flux $\langle n_e\mathbf{u}_e\cdot\nabla\psi\rangle_\psi$ across flux surfaces, we need the electron velocity $\mathbf{u}_e = u_{e\|}\hat{\mathbf{b}} + \mathbf{u}_{e\perp}$. The perpendicular electron velocity can be calculated using the perpendicular electron momentum equation,

$$\underbrace{n_e m_e (\mathbf{u}_e \cdot \nabla \mathbf{u}_e)_\perp}_{\sim \frac{m_e}{m_i}\frac{u_e^2}{v_{ti}^2}\frac{p_e}{L}} \xrightarrow{} \text{small} \quad = \quad \underbrace{e n_e \nabla\phi - e n_e \mathbf{u}_e \times \mathbf{B}}_{\sim \frac{L}{\rho_i}\frac{u_e}{v_{ti}}\frac{p_e}{L}} \underbrace{- \nabla p_e}_{\sim \frac{p_e}{L}}$$

$$\underbrace{+ n_e m_e \nu_{ei}(\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}) - \frac{3}{2}\frac{n_e \nu_{ei}}{\Omega_e}\hat{\mathbf{b}} \times \nabla T_e - e n_e R_0 E_\zeta \nabla\zeta.}_{\sim \frac{\nu_{ei}}{\Omega_e}\frac{p_e}{L} \ll \frac{p_e}{L}} \quad (4.19)$$

We can use equation (4.19) to solve for $\mathbf{u}_e$ if we can calculate $\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}$. Since $\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}$ appears in a small term, we can use the lowest order result in (4.17). Thus,

$$e n_e \mathbf{u}_e \times \mathbf{B} \simeq e n_e \nabla\phi - \nabla p_e + \frac{\nu_{ei}}{\Omega_e}\hat{\mathbf{b}} \times \left[\nabla(p_e + p_i) - \frac{3}{2} n_e \nabla T_e\right] - e n_e R_0 E_\zeta \nabla\zeta. \quad (4.20)$$

We can then solve for $\mathbf{u}_{e\perp}$, finding

$$\mathbf{u}_{e\perp} = \underbrace{-\frac{1}{B}\nabla\phi \times \hat{\mathbf{b}}}_{\sim u_i} \underbrace{- \frac{1}{e n_e B}\hat{\mathbf{b}} \times \nabla p_e}_{\sim \frac{\rho_i}{L} v_{ti}}$$

$$\underbrace{- \frac{\nu_{ei}}{e n_e B \Omega_e}\left[\nabla_\perp(p_e + p_i) - \frac{3}{2} n_e \nabla_\perp T_e\right] + \frac{R_0 E_\zeta}{B}\nabla\zeta \times \hat{\mathbf{b}}.}_{\sim \frac{\nu_{ei}}{\Omega_e}\frac{\rho_i}{L} v_{ti}} \quad (4.21)$$

Formula (4.21) for the perpendicular electron velocity gives to lowest order

$$\mathbf{u}_e \cdot \nabla\psi = \frac{T_{e0}(\hat{\mathbf{b}} \times \nabla\psi)\cdot\nabla\theta}{eB}\frac{\partial}{\partial\theta}\left(\frac{n_{e1}}{n_{e0}} - \frac{e\phi_1}{T_{e0}} + \frac{T_{e1}}{T_{e0}}\right) + \frac{R_0 E_\zeta}{B}(\nabla\zeta \times \hat{\mathbf{b}})\cdot\nabla\psi$$
$$- \frac{\nu_{ei0}|\nabla\psi|^2}{e n_{e0} B \Omega_e}\left[\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0} + p_{i0}) - \frac{3}{2} n_{e0}\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi}\right] \sim \frac{\nu_{ei}}{\Omega_e}\frac{\rho_e}{L} v_{te}|\nabla\psi|. \quad (4.22)$$

This result can be simplified using equation (2.14) to write

$$(\hat{\mathbf{b}} \times \nabla\psi)\cdot\nabla\theta = I\hat{\mathbf{b}}\cdot\nabla\theta, \quad (\nabla\zeta \times \hat{\mathbf{b}})\cdot\nabla\psi = -\frac{|\nabla\psi|^2}{BR^2}. \quad (4.23)$$

Then, the average particle flux across a flux surface becomes

$$\boxed{\langle n_e \mathbf{u}_e \cdot \nabla\psi\rangle_\psi \simeq n_{e0}\langle \mathbf{u}_e \cdot \nabla\psi\rangle_\psi,} \quad (4.24)$$

where

$$\langle \mathbf{u}_e \cdot \nabla\psi\rangle_\psi = \frac{I T_{e0}}{e}\left\langle \frac{\mathbf{B}\cdot\nabla\theta}{B^2}\frac{\partial}{\partial\theta}\left(\frac{n_{e1}}{n_{e0}} - \frac{e\phi_1}{T_{e0}} + \frac{T_{e1}}{T_{e0}}\right)\right\rangle_\psi - \left\langle \frac{|\nabla\psi|^2}{B^2 R^2}\right\rangle_\psi R_0 E_\zeta$$
$$- \frac{m_e \nu_{ei0}}{e^2 n_{e0}}\left\langle \frac{|\nabla\psi|^2}{B^2}\right\rangle_\psi \left[\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0} + p_{i0}) - \frac{3}{2} n_{e0}\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi}\right]. \quad (4.25)$$

From this equation, it is clear that we need to know $n_{e1}/n_{e0} - e\phi_1/T_{e0}$ and T_{e1}/T_{e0} to calculate the particle flux. We will find $n_{e1}/n_{e0} - e\phi_1/T_{e0}$ and T_{e1}/T_{e0} and hence $\langle \mathbf{u}_e \cdot \nabla\psi\rangle_\psi$ in subsection 4.4.

To be able to obtain $n_{e1}/n_{e0} - e\phi_1/T_{e0}$ and T_{e1}/T_{e0}, we need information from the continuity equation. To lowest order, equation (4.21) gives

$$\mathbf{u}_{e\perp} \simeq \mathbf{u}_{e\perp 0} = \underbrace{\frac{1}{B}\frac{\mathrm{d}\phi_0}{\mathrm{d}\psi}\hat{\mathbf{b}}\times\nabla\psi}_{\sim u_i} - \underbrace{\frac{1}{en_{e0}B}\frac{\mathrm{d}p_{e0}}{\mathrm{d}\psi}\hat{\mathbf{b}}\times\nabla\psi}_{\sim\frac{\rho_i}{L}v_{ti}}. \tag{4.26}$$

As a result, the lowest order electron velocity $\mathbf{u}_e \simeq \mathbf{u}_{e0} = u_{e\|0}\hat{\mathbf{b}} + \mathbf{u}_{e\perp 0}$ is of the order of the ion velocity $u_i \gtrsim (\rho_e/L)v_{te}$, and hence, in equation (4.2), the term $\nabla\cdot(n_e\mathbf{u}_e)$ is much larger than the source S_n (recall equation (4.4)). Thus, equation (4.2) gives $\nabla\cdot(n_{e0}\mathbf{u}_{e0}) \simeq 0$ to lowest order. We can then write

$$n_{e0}\nabla\cdot\mathbf{u}_{e0} + \cancelto{0}{\mathbf{u}_{e0}\cdot\nabla\psi}\,\frac{\mathrm{d}n_{e0}}{\mathrm{d}\psi} = 0 \Rightarrow \nabla\cdot\mathbf{u}_{e0} = 0. \tag{4.27}$$

This equation can be used to determine $u_{e\|0}$, but we will not need the parallel velocity. It is sufficient to know that the divergence of the lowest order electron velocity vanishes.

4.3. *Energy equation*

Flux surface averaging equation (4.3) and using equation (3.7), we find

$$\begin{aligned}\frac{1}{V'}\frac{\mathrm{d}}{\mathrm{d}\psi}\left[V'\left(\frac{5}{2}\langle p_e\mathbf{u}_e\cdot\nabla\psi\rangle_\psi + \langle\mathbf{q}_e\cdot\nabla\psi\rangle_\psi\right)\right] &= \underbrace{\langle\mathbf{u}_e\cdot(-en_e\mathbf{E}+\mathbf{F}_{ei})\rangle_\psi}_{\sim\frac{L}{\rho_i}\frac{u_i^2}{v_{ti}^2}\frac{p_e v_{ti}}{L}\gg\frac{\nu_{ei}}{\Omega_e}\frac{\rho_e}{L}\frac{p_e v_{te}}{L}} \\ &+ \underbrace{\left\langle\frac{\mathbf{J}\cdot\mathbf{F}_{ei}}{en_e}\right\rangle_\psi}_{\sim\frac{\nu_{ei}}{\Omega_e}\frac{\rho_e}{L}\frac{p_e v_{te}}{L}} + \underbrace{\frac{3n_{e0}m_e\nu_{ei0}}{m_i}(T_{i0}-T_{e0})}_{\sim\frac{\nu_{ei}}{\Omega_e}\frac{\rho_e}{L}\frac{p_e v_{te}}{L}} + \underbrace{\langle S_{Ee}\rangle_\psi}_{\sim\frac{\nu_{ei}}{\Omega_e}\frac{\rho_e}{L}\frac{p_e v_{te}}{L}}.\end{aligned} \tag{4.28}$$

The terms on the right side of equation (4.28) (other than S_{Ee}) describe the energy injected by the transformer and the energy exchange between electrons and ions. This energy exchange can be due to collisions or the electric field generated by the plasma. The energy injected by the transformer is given by

$$H_{\mathrm{transf}} = R_0E_\zeta\langle\mathbf{J}\cdot\nabla\zeta\rangle_\psi. \tag{4.29}$$

Using equation (2.18) and equation (4.14), the heating due to the transformer becomes

$$\boxed{H_{\mathrm{transf}} = \frac{(\langle R^{-2}\rangle_\psi)^2I^2R_0^2E_\zeta^2}{\eta_0\langle B^2\rangle_\psi} - \left\langle\frac{|\nabla\psi|^2}{R^2}\right\rangle_\psi\frac{R_0E_\zeta}{\langle B^2\rangle_\psi}\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0}+p_{i0}).} \tag{4.30}$$

After several manipulations (see Appendix A), we rewrite the energy equation (4.28) as

$$\boxed{\begin{aligned}\frac{1}{V'}\frac{\mathrm{d}}{\mathrm{d}\psi}\left[V'\left(\frac{5}{2}\langle p_e\mathbf{u}_e\cdot\nabla\psi\rangle_\psi + \langle\mathbf{q}_e\cdot\nabla\psi\rangle_\psi\right)\right] &= H_{\mathrm{transf}} - \langle\mathbf{u}_e\cdot\nabla\psi\rangle_\psi\frac{\mathrm{d}p_{i0}}{\mathrm{d}\psi} \\ &+ \frac{3n_{e0}m_e\nu_{ei0}}{m_i}(T_{i0}-T_{e0}) + \langle S_{Ee}\rangle_\psi.\end{aligned}} \tag{4.31}$$

The term $\langle\mathbf{u}_e\cdot\nabla\psi\rangle_\psi(\mathrm{d}p_{i0}/\mathrm{d}\psi)$ is the energy transfer between electrons and ions due to the electric field.

To find the energy flux, we have to calculate the convective energy flux and the heat flux through a flux surface. Using (4.22), we write the convective energy flux as

$$\boxed{\frac{5}{2}\langle p_e \mathbf{u}_e \cdot \nabla\psi\rangle_\psi \simeq \frac{5}{2} p_{e0} \langle \mathbf{u}_e \cdot \nabla\psi\rangle_\psi .} \tag{4.32}$$

To calculate the heat flux, we need the diamagnetic heat flux

$$\mathbf{q}_{e\times} = -\frac{5p_e}{2m_e\Omega_e}\hat{\mathbf{b}} \times \nabla T_e \tag{4.33}$$

and the perpendicular heat flux

$$\mathbf{q}_{e\perp} = -\left(\sqrt{2} + \frac{13}{4}\right)\frac{p_e\nu_{ei}}{m_e\Omega_e^2}\nabla_\perp T_e - \frac{3}{2}\frac{p_e\nu_{ei}}{\Omega_e}\hat{\mathbf{b}} \times (\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}). \tag{4.34}$$

Using the lowest order result (4.17), the perpendicular heat flux simplifies to

$$\mathbf{q}_{e\perp} = -\frac{T_e\nu_{ei}}{m_e\Omega_e^2}\left[\left(\sqrt{2} + \frac{13}{4}\right) n_e \nabla_\perp T_e - \frac{3}{2}\nabla_\perp (p_e + p_i)\right]. \tag{4.35}$$

Using equations (4.23), (4.33) and (4.35), we find

$$\begin{aligned}\langle \mathbf{q}_e \cdot \nabla\psi\rangle_\psi =& \frac{5Ip_{e0}T_{e0}}{2e}\left\langle \frac{\mathbf{B}\cdot\nabla\theta}{B^2}\frac{\partial}{\partial\theta}\left(\frac{T_{e1}}{T_{e0}}\right)\right\rangle_\psi - \frac{5}{2}\left\langle\frac{|\nabla\psi|^2}{B^2R^2}\right\rangle_\psi p_{e0}R_0E_\zeta \\ &- \frac{m_e\nu_{ei0}T_{e0}}{e^2}\left\langle\frac{|\nabla\psi|^2}{B^2}\right\rangle_\psi \left[\left(\sqrt{2} + \frac{13}{4}\right) n_{e0}\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi} - \frac{3}{2}\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0} + p_{i0})\right].\end{aligned} \tag{4.36}$$

As in the case of the particle flux, we need to calculate $n_{e1}/n_{e0} - e\phi_1/T_{e0}$ and T_{e1}/T_{e0}. This is done in subsection 4.4. To calculate $n_{e1}/n_{e0} - e\phi_1/T_{e0}$ and T_{e1}/T_{e0}, we need information from the energy equation. To lowest order, the electron heat flux is $\mathbf{q}_e \simeq q_{e\|}\hat{\mathbf{b}} + \mathbf{q}_{e\times}$. The diamagnetic heat flux $\mathbf{q}_{e\times}$, defined in (4.33), is to lowest order

$$\mathbf{q}_{e\times} \simeq -\frac{5p_{e0}}{2m_e\Omega_e}\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi}\hat{\mathbf{b}} \times \nabla\psi \sim \frac{\rho_e}{L}\frac{p_e v_{te}}{L}. \tag{4.37}$$

The parallel heat flux is

$$q_{e\|} = -\frac{3.16p_e}{m_e\nu_{ee}}\hat{\mathbf{b}}\cdot\nabla T_e - 0.71p_e(u_{i\|} - u_{e\|}). \tag{4.38}$$

Using equation (4.6) to obtain $J_\| = en_e(u_{i\|} - u_{e\|})$, the parallel heat flux becomes

$$q_{e\|} \simeq -\frac{T_{e0}^2}{e^2\eta_0}\hat{\mathbf{b}}\cdot\nabla\theta\frac{\partial}{\partial\theta}\left[0.71\left(\frac{n_{e1}}{n_{e0}} - \frac{e\phi_1}{T_{e0}}\right) + 2.83\frac{T_{e1}}{T_{e0}}\right] - \frac{0.71T_{e0}IR_0E_\zeta}{e\eta_0R^2B} \sim \frac{\rho_e}{L}\frac{p_e v_{te}}{L}, \tag{4.39}$$

where we have used the fact that $\nu_{ee} \simeq \nu_{ei}$. Hence $\mathbf{q}_e \sim (\rho_e/L)(p_e v_{te}/L)$. We have also seen that the electron velocity is of order $\mathbf{u}_e \simeq \mathbf{u}_{e0} \sim u_i \gg (\rho_e/L)v_{te}$. Thus, the largest terms in equation (4.3) are

$$\nabla\cdot\left(\frac{5}{2}p_{e0}\mathbf{u}_{e0}\right) + \nabla\cdot\mathbf{q}_e \simeq en_{e0}\frac{\mathrm{d}\phi_0}{\mathrm{d}\psi}\cancelto{0}{\mathbf{u}_{e0}\cdot\nabla\psi}. \tag{4.40}$$

Equation (4.27) implies that $\nabla\cdot(p_{e0}\mathbf{u}_{e0}) = 0$, leaving

$$\nabla\cdot(q_{e\|}\hat{\mathbf{b}}) + \nabla\cdot\mathbf{q}_{e\times} = \mathbf{B}\cdot\nabla\theta\frac{\partial}{\partial\theta}\left(\frac{q_{e\|}}{B}\right) - \mathbf{B}\cdot\nabla\theta\frac{\partial}{\partial\theta}\left[\frac{5p_{e0}}{2e}\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi}\frac{(\hat{\mathbf{b}}\times\nabla\psi)\cdot\nabla\theta}{B\mathbf{B}\cdot\nabla\theta}\right] = 0. \tag{4.41}$$

Using equation (4.23), we finally obtain

$$\mathbf{B}\cdot\nabla\theta\frac{\partial}{\partial\theta}\left(\frac{q_{e\|}}{B}-\frac{5Ip_{e0}}{2eB^2}\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi}\right)=0. \tag{4.42}$$

This equation gives

$$q_{e\|}=K_q(\psi)B+\frac{5Ip_{e0}}{2eB}\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi}, \tag{4.43}$$

where $K_q(\psi)$ is a function only of ψ. We can determine $K_q(\psi)$ by dividing equation (4.39) by $\hat{\mathbf{b}}\cdot\nabla\theta$ and integrating over θ to obtain

$$\langle q_{e\|}B\rangle_\psi=-\left\langle\frac{1}{R^2}\right\rangle_\psi\frac{0.71T_{e0}IR_0E_\zeta}{e\eta_0}. \tag{4.44}$$

Using this result in equation (4.43), we find

$$K_q(\psi)=-\left\langle\frac{1}{R^2}\right\rangle_\psi\frac{0.71T_{e0}IR_0E_\zeta}{e\eta_0\langle B^2\rangle_\psi}-\frac{5Ip_{e0}}{2e\langle B^2\rangle_\psi}\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi}, \tag{4.45}$$

leading to

$$q_{e\|}=-\left\langle\frac{1}{R^2}\right\rangle_\psi\frac{0.71T_{e0}IR_0BE_\zeta}{e\eta_0\langle B^2\rangle_\psi}+\frac{5Ip_{e0}}{2eB}\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi}\left(1-\frac{B^2}{\langle B^2\rangle_\psi}\right). \tag{4.46}$$

Using result (4.46) in equation (4.39), we obtain

$$\begin{aligned}\frac{\partial}{\partial\theta}\left[0.71\left(\frac{n_{e1}}{n_{e0}}-\frac{e\phi_1}{T_{e0}}\right)+2.83\frac{T_{e1}}{T_{e0}}\right]=&-\frac{1.28Im_e\nu_{ei0}}{eT_{e0}\mathbf{B}\cdot\nabla\theta}\left(1-\frac{B^2}{\langle B^2\rangle_\psi}\right)\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi}\\&-\frac{0.71eIR_0E_\zeta}{T_{e0}\mathbf{B}\cdot\nabla\theta}\left(\frac{1}{R^2}-\left\langle\frac{1}{R^2}\right\rangle_\psi\frac{B^2}{\langle B^2\rangle_\psi}\right).\end{aligned} \tag{4.47}$$

4.4. *Particle and energy fluxes*

In this section, we calculate $n_{e1}/n_{e0}-e\phi_1/T_{e0}$ and T_{e1}/T_{e0} and then use them to calculate the particle and energy fluxes. Solving for

$$\frac{\partial}{\partial\theta}\left(\frac{n_{e1}}{n_{e0}}-\frac{e\phi_1}{T_{e0}}\right),\quad\frac{\partial}{\partial\theta}\left(\frac{T_{e1}}{T_{e0}}\right) \tag{4.48}$$

in equations (4.16) and (4.47), we obtain

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\frac{n_{e1}}{n_{e0}}-\frac{e\phi_1}{T_{e0}}\right)=&-\frac{Im_e\nu_{ei0}}{e\mathbf{B}\cdot\nabla\theta}\left(1-\frac{B^2}{\langle B^2\rangle_\psi}\right)\left[\frac{0.89}{p_{e0}}\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0}+p_{i0})-\frac{1.35}{T_{e0}}\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi}\right]\\&-\frac{eIR_0E_\zeta}{T_{e0}\mathbf{B}\cdot\nabla\theta}\left(\frac{1}{R^2}-\left\langle\frac{1}{R^2}\right\rangle_\psi\frac{B^2}{\langle B^2\rangle_\psi}\right),\end{aligned} \tag{4.49}$$

$$\frac{\partial}{\partial\theta}\left(\frac{T_{e1}}{T_{e0}}\right)=-\frac{Im_e\nu_{ei0}}{e\mathbf{B}\cdot\nabla\theta}\left(1-\frac{B^2}{\langle B^2\rangle_\psi}\right)\left[-\frac{0.22}{p_{e0}}\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0}+p_{i0})+\frac{0.79}{T_{e0}}\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi}\right]. \tag{4.50}$$

We could integrate these equations to obtain $n_{e1}/n_{e0}-e\phi_1/T_{e0}$ and T_{e1}/T_{e0}, but their derivative with respect to θ is sufficient to calculate the particle and energy fluxes.

Substituting equations (4.49) and (4.50) into equation (4.25), we find

$$\boxed{\begin{aligned}\langle \mathbf{u}_e \cdot \nabla\psi\rangle_\psi = &-\frac{D_{\rm PS}+D_{\rm cl}}{n_{e0}T_{e0}}\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0}+p_{i0}) + \frac{K_{\rm PS}+K_{\rm cl}}{T_{e0}}\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi} \\ &-\left\langle \frac{|\nabla\psi|^2}{R^2}\right\rangle_\psi \frac{R_0 E_\zeta}{\langle B^2\rangle_\psi},\end{aligned}} \tag{4.51}$$

where the Pfirsch-Schlüter and classical particle diffusion coefficients are

$$D_{\rm PS} = \frac{0.67 I^2 T_{e0} m_e \nu_{ei0}}{e^2}\left(\left\langle\frac{1}{B^2}\right\rangle_\psi - \frac{1}{\langle B^2\rangle_\psi}\right), \quad D_{\rm cl} = \frac{T_{e0} m_e \nu_{ei0}}{e^2}\left\langle \frac{|\nabla\psi|^2}{B^2}\right\rangle_\psi, \tag{4.52}$$

and the Pfirsch-Schlüter and classical thermodiffusion coefficients are

$$K_{\rm PS} = \frac{0.56 I^2 T_{e0} m_e \nu_{ei0}}{e^2}\left(\left\langle\frac{1}{B^2}\right\rangle_\psi - \frac{1}{\langle B^2\rangle_\psi}\right), \quad K_{\rm cl} = \frac{3T_{e0} m_e \nu_{ei0}}{2e^2}\left\langle \frac{|\nabla\psi|^2}{B^2}\right\rangle_\psi. \tag{4.53}$$

The quantity $\langle B^{-2}\rangle_\psi - (\langle B^2\rangle_\psi)^{-1}$ is always positive. To prove it, we use that $(B - \langle B^2\rangle_\psi/B)^2 \geqslant 0$. Expanding the square and flux surface averaging it, we find $\langle B^{-2}\rangle_\psi \geqslant (\langle B^2\rangle_\psi)^{-1}$.

Substituting equations (4.49) and (4.50) into equation (4.36), we obtain

$$\boxed{\langle \mathbf{q}_e \cdot \nabla\psi\rangle_\psi = -(\chi_{\rm PS}+\chi_{\rm cl})\frac{\mathrm{d}T_{e0}}{\mathrm{d}\psi} + (\kappa_{\rm PS}+\kappa_{\rm cl})\frac{1}{n_{e0}}\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0}+p_{i0}),} \tag{4.54}$$

where the Pfirsch-Schlüter and classical heat diffusion coefficients are

$$\chi_{\rm PS} = \frac{1.98 I^2 p_{e0} m_e \nu_{ei0}}{e^2}\left(\left\langle\frac{1}{B^2}\right\rangle_\psi - \frac{1}{\langle B^2\rangle_\psi}\right), \chi_{\rm cl} = \left(\sqrt{2}+\frac{13}{4}\right)\frac{p_{e0} m_e \nu_{ei0}}{e^2}\left\langle \frac{|\nabla\psi|^2}{B^2}\right\rangle_\psi, \tag{4.55}$$

and

$$\kappa_{\rm PS} = \frac{0.56 I^2 p_{e0} m_e \nu_{ei0}}{e^2}\left(\left\langle\frac{1}{B^2}\right\rangle_\psi - \frac{1}{\langle B^2\rangle_\psi}\right), \quad \kappa_{\rm cl} = \frac{3p_{e0} m_e \nu_{ei0}}{2e^2}\left\langle \frac{|\nabla\psi|^2}{B^2}\right\rangle_\psi. \tag{4.56}$$

The classical diffusion coefficients $D_{\rm cl}$, $K_{\rm cl}$, $\chi_{\rm cl}$ and $\kappa_{\rm cl}$ would appear in an infinite cylinder that is not "bended" like the tokamak. This would have been the naive expectation for particle and energy transport, but the transport is enhanced by the Pfirsch-Schlüter diffusion coefficients. The last term in equation (4.24) is the $\mathbf{E}\times\mathbf{B}$ drift due to the electric field of the transformer modified by diamagnetic flows driven by n_{e1}, ϕ_1 and T_{e1}.

5. Summary

To summarize, the tokamak equilibrium is determined in the following way:

- Equations (2.16) and (4.12) determine the magnetic field for given pressure profile and transformer electric field.
- Equation (4.18) (with equations (4.24) and (4.51)) determines the density profile for given electron temperature profile, transformer electric field and source of particles.
- Equation (4.31) (with equations (4.30), (4.32), (4.51) and (4.54)) determines the electron temperature profile for given density profile, ion pressure profile, transformer electric field and source of energy.

- The ion equations will determine the ion temperature and the tokamak flows, and ultimately, the tokamak flows determine the electrostatic potential (note, for example, that the electron perpendicular flow (4.26) depends on $\mathrm{d}\phi_0/\mathrm{d}\psi$).

Thus, we have obtained a formulation that determines the particles, momentum and energy that we need to inject into a tokamak to obtain the desired profiles. This formulation is not adequate to most tokamaks for two reasons: we have assumed that collisions are very frequent, whereas in most of the tokamak collisions are rare; and what is more important, tokamak plasmas are not usually quiescent, but turbulent. It is possible to formulate equations for turbulence in the Braginskii limit. The difference with the Pfirsch-Schlüter formulation is that one needs to allow the perturbations n_{e1}, ϕ_1, T_{e1} and T_{i1} to be non-axisymmetric, that is, they depend on ζ. These non-axisymmetric perturbations are not steady-state solutions and one needs to allow them to depend on time t as well.

REFERENCES

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Appendix A. Energy exchange terms in the electron energy equation

In this Appendix, we derive equation (4.31) from equation (4.28).

In equation (4.28), the term $\langle \mathbf{J}\cdot\mathbf{F}_{ei}/en_e\rangle_\psi$ is of the same order as S_{Ee} and hence we can use the lowest order expressions (2.18) and (4.14) to write,

$$\left\langle \frac{\mathbf{J}\cdot\mathbf{F}_{ei}}{en_e}\right\rangle_\psi = \left\langle \frac{1}{R^2}\right\rangle_\psi \frac{I\langle BF_{ei\|}\rangle_\psi R_0 E_\zeta}{en_{e0}\eta_0\langle B^2\rangle_\psi} - \frac{I\langle F_{ei\|}B\rangle_\psi}{en_{e0}\langle B^2\rangle_\psi}\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0}+p_{i0}) - \frac{\langle R\hat{\boldsymbol{\zeta}}\cdot\mathbf{F}_{ei}\rangle_\psi}{en_{e0}}\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0}+p_{i0}). \tag{A 1}$$

Here, we can use

$$F_{ei,\|} = \hat{\mathbf{b}}\cdot(\nabla p_e - en_e\nabla\phi + en_e R_0 E_\zeta \nabla\zeta) \tag{A 2}$$

$$\simeq p_{e0}\hat{\mathbf{b}}\cdot\nabla\theta\frac{\partial}{\partial\theta}\left(\frac{n_{e1}}{n_{e0}} - \frac{e\phi_1}{T_{e0}} + \frac{T_{e1}}{T_{e0}}\right) + \frac{en_{e0}IR_0E_\zeta}{R^2} \tag{A 3}$$

to find

$$\langle F_{ei\|}B\rangle_\psi = \left\langle\frac{1}{R^2}\right\rangle_\psi en_{e0}IR_0E_\zeta. \tag{A 4}$$

With this result, equation (A 1) becomes

$$\left\langle \frac{\mathbf{J}\cdot\mathbf{F}_{ei}}{en_e}\right\rangle_\psi = \frac{(\langle R^{-2}\rangle_\psi)^2 I^2 R_0^2 E_\zeta^2}{\eta_0\langle B^2\rangle_\psi} + \left\langle\frac{I^2}{R^2}\right\rangle_\psi \frac{R_0E_\zeta}{\langle B^2\rangle_\psi}\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0}+p_{i0}) - \frac{\langle R\hat{\boldsymbol{\zeta}}\cdot\mathbf{F}_{ei}\rangle_\psi}{en_{e0}}\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0}+p_{i0}). \tag{A 5}$$

Finally, multiplying $0 = -\nabla p_e - en_e(-\nabla\phi + R_0E_\zeta\nabla\zeta + \mathbf{u}_e\times\mathbf{B}) + \mathbf{F}_{ei}$ by $R\hat{\boldsymbol{\zeta}}$ and using $R\mathbf{B}\times\hat{\boldsymbol{\zeta}} = \nabla\psi$, we find $R\hat{\boldsymbol{\zeta}}\cdot\mathbf{F}_{ei} = en_{e0}(\mathbf{u}_e\cdot\nabla\psi + R_0E_\zeta)$. Hence, equation (A 5) finally becomes

$$\left\langle \frac{\mathbf{J}\cdot\mathbf{F}_{ei}}{en_e}\right\rangle_\psi = H_{\mathrm{transf}} - \langle\mathbf{u}_e\cdot\nabla\psi\rangle_\psi\frac{\mathrm{d}}{\mathrm{d}\psi}(p_{e0}+p_{i0}), \tag{A 6}$$

where we have used the energy injected by the transformer H_{transf}, defined in (4.30)

Using $-en_e\mathbf{E} + \mathbf{F}_{ei} = en_e\mathbf{u}_e\times\mathbf{B} + \nabla p_e$, we obtain

$$\langle\mathbf{u}_e\cdot(-en_e\mathbf{E}+\mathbf{F}_{ei})\rangle_\psi = \langle\mathbf{u}_e\cdot\nabla p_e\rangle_\psi \sim \frac{p_e u_i}{L} \gg \frac{\nu_{ei}}{\Omega_e}\frac{\rho_i}{L}\frac{p_e v_{ti}}{L}. \tag{A 7}$$

We need to keep the higher order corrections to p_e because this term is much larger than S_{Ee},

$$\langle\mathbf{u}_e\cdot(-en_e\mathbf{E}+\mathbf{F}_{ei})\rangle_\psi = \langle\mathbf{u}_e\cdot\nabla\psi\rangle_\psi\frac{\mathrm{d}p_{e0}}{\mathrm{d}\psi} + \langle\mathbf{u}_{e0}\cdot\nabla(n_{e1}T_{e0}+n_{e0}T_{e1})\rangle_\psi. \tag{A 8}$$

Using the fact that the divergence of $\mathbf{u}_{e0}$ vanishes (see equation (4.27)), we find

$$\langle\mathbf{u}_{e0}\cdot\nabla(n_{e1}T_{e0}+n_{e0}T_{e1})\rangle_\psi = \langle\nabla\cdot[(n_{e1}T_{e0}+n_{e0}T_{e1})\mathbf{u}_{e0}]\rangle_\psi = \frac{1}{V'}\frac{\mathrm{d}}{\mathrm{d}\psi}\left(V'\langle(n_{e1}T_{e0}+n_{e0}T_{e1})\cancelto{0}{\mathbf{u}_{e0}\cdot\nabla\psi}\rangle_\psi\right) = 0. \tag{A 9}$$

Hence, equation (A 8) becomes

$$\langle\mathbf{u}_e\cdot(-en_e\mathbf{E}+\mathbf{F}_{ei})\rangle_\psi = \langle\mathbf{u}_e\cdot\nabla\psi\rangle_\psi\frac{\mathrm{d}p_{e0}}{\mathrm{d}\psi}. \tag{A 10}$$

With equations (A 6) and (A 10), we convert equation (4.28) in equation (4.31).