Fokker-Planck collision operator

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1. Introduction

In these notes, we study the effect on the plasma of close encounters between charged particles (electrons and ions). These close encounters are the collisions between charge particles that we have ignored in Collisionless Plasma Physics. We can naively estimate how close the particles have to be to notice one another. The characteristic potential energy of a system of two charged particles with charges of the order of the proton charge e separated by a distance b is $e^2/4\pi\epsilon_0 b$, where ϵ_0 is the vacuum permittivity. For the particles to be affected by their mutual interaction, this potential energy must be of the order of the typical kinetic energy of the particles, that is, of the order of the temperature of the plasma T. Thus, the distance b between two particles that have a significant collision is

$$b = \frac{e^2}{4\pi\epsilon_0 T}. \tag{1.1}$$

The naive estimation in (1.1) is incorrect in weakly coupled plasmas. To define weakly coupled plasmas, we need to define the coupling parameter: the ratio of the characteristic potential energy between two typical particles in the plasma and the characteristic kinetic energy of the plasma. The typical distance between two particles is $n^{-1/3}$, where n is the particle density (number of particles per unit volume). Then, the potential energy between two typical particles is $e^2 n^{1/3}/4\pi\epsilon_0$, and the coupling parameter is

$$\Gamma = \frac{e^2 n^{1/3}}{4\pi\epsilon_0 T}. \tag{1.2}$$

By definition, in weakly coupled plasmas, the coupling parameter is small,

$$\Gamma \ll 1. \tag{1.3}$$

As a result, particles are rarely within a distance b of each other, that is, the probablity of finding a particle within a sphere of radius b is very small,

$$\frac{4\pi}{3} n b^3 \sim \Gamma^3 \ll 1. \tag{1.4}$$

Importantly, the electric force on a given particle is dominated by particles that are at a distance of the order of the Debye length,

$$\lambda_D = \sqrt{\frac{\epsilon_0 T}{e^2 n}}. \tag{1.5}$$

In weakly coupled plasmas, the Debye length is much larger than the impact parameter b. Indeed, the ratio between the two, also known as plasma parameter, is

$$\Lambda = \frac{\lambda_D}{b} \sim \frac{1}{\Gamma^{3/2}} \gg 1. \tag{1.6}$$

Thus, the electric force on a particle exerted by another particle at a Debye length is much smaller than the force exerted by another particle at a distance b. Even though the force due to each individual particle is small, the collective force due to all the particles within a Debye length is comparable to (and as we will see, larger than) the force exerted by the small number of particles that on rare occasions happen to be at a distance b. Indeed, the number of particles within a Debye sphere is very large,

$$\frac{4\pi}{3} n\lambda_D^3 \sim \frac{1}{\Gamma^{3/2}} \gg 1. \tag{1.7}$$

The total potential energy due to all these particles is

$$\frac{4\pi}{3} n\lambda_D^3 \frac{e^2}{4\pi\epsilon_0\lambda_D} \sim \frac{4\pi}{3} n\lambda_D^3 \frac{1}{\Lambda} \frac{e^2}{4\pi\epsilon_0 b} \sim \frac{e^2}{4\pi\epsilon_0 b} \sim T, \tag{1.8}$$

Thus, the collective potential energy due to all the particles within the Debye sphere is comparable to the potential energy of a particle that is sufficiently close (a distance b) to modify the kinetic energy significantly. We will show that in fact, the force due to the particles within the Debye sphere dominates over the close encounters. This is in contrast to the collisions considered in the kinetic theory of neutral gases. Collisions in neutral gases are dominated by close encounters, and the collective force of particles far from the particle of interest is negligible.

Based on these considerations, we derive the collision operator for charged particles, known as Fokker-Planck collision operator, starting from the Boltzmann operator for binary collisions derived in the Kinetic Theory (Dellar 2015). As we have indicated above, the collisional events are not composed of just two particles interacting with each other, but we can still use a binary collision operator. We can consider the individual interaction of the particle of interest with every other particle within a Debye length, and then using the fact that these interactions are small and hence the effects are simply additive, we can sum over all of them.

2. Boltzmann collision operator

Before we give the Boltzmann collision operator, we discuss binary collisions in detail.

2.1. *Binary collisions*

In a binary collision between particle 1 of species s and particle 2 of species s', we need to determine the final velocities of both particles, $\mathbf{v}_{1f}$ and $\mathbf{v}_{2f}$, from their initial velocities, $\mathbf{v}_{1i}$ and $\mathbf{v}_{2i}$ (see figure 1). For simplicity, we only consider particles that interact via a potential $V(r)$ that only depends on the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$ between the position of the two particles. The potential $V(r)$ vanishes for $r \to \infty$. The equations of motion for particles 1 and 2 are

$$m_s \frac{\mathrm{d}^2\mathbf{r}_1}{\mathrm{d}t^2} = -\nabla_{\mathbf{r}_1} V = -\frac{\mathrm{d}V}{\mathrm{d}r} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{2.1}$$

$$m_{s'} \frac{\mathrm{d}^2\mathbf{r}_2}{\mathrm{d}t^2} = -\nabla_{\mathbf{r}_2} V = -\frac{\mathrm{d}V}{\mathrm{d}r} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}. \tag{2.2}$$

Assuming a potential that only depends on the separation between the particles simplifies the problem considerably. There are three important simplifications:

(*a*) **Conservation of momentum.** Adding equations (2.1) and (2.2), we find that

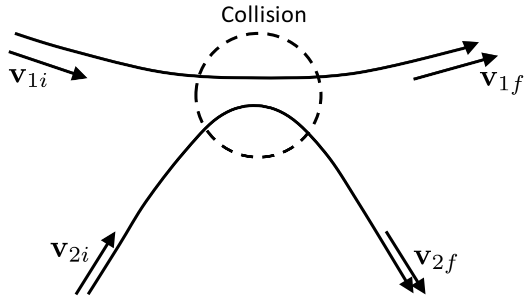

FIGURE 1. Schematic collision between particles 1 and 2.

the total momentum of both particles is conserved during the collision, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_s\mathbf{v}_1 + m_{s'}\mathbf{v}_2) = 0. \tag{2.3}$$

This property indicates that the center of mass velocity

$$\mathbf{v}_{CM} = \frac{m_s\mathbf{v}_1 + m_{s'}\mathbf{v}_2}{m_s + m_{s'}} \tag{2.4}$$

is constant. This is a convenient result because it means that we can write both velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ as functions of a constant, $\mathbf{v}_{CM}$, and the difference between the two velocities

$$\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2. \tag{2.5}$$

Indeed,

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{v}_{CM} + \frac{m_{s'}}{m_s + m_{s'}}\mathbf{g},\\ \mathbf{v}_2 &= \mathbf{v}_{CM} - \frac{m_s}{m_s + m_{s'}}\mathbf{g}.\end{aligned} \tag{2.6}$$

Thus, we just need to determine $\mathbf{g}_f = \mathbf{v}_{1f} - \mathbf{v}_{2f}$ from the initial condition $\mathbf{g}_i = \mathbf{v}_{1i} - \mathbf{v}_{2i}$. To find an equation for $\mathbf{g}(t)$, we subtract equation (2.2) divided by $m_{s'}$ from equation (2.1) divided by m_s. The result is an equation for the separation between the particles $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$: the reduced particle equation

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\frac{1}{\mu_{ss'}}\frac{\mathrm{d}V}{\mathrm{d}r}\frac{\mathbf{r}}{r}, \tag{2.7}$$

where

$$\mu_{ss'} = \frac{m_s m_{s'}}{m_s + m_{s'}} \tag{2.8}$$

is the reduced mass. Once $\mathbf{r}(t)$ is calculated using (2.7), we can find $\mathbf{g} = \mathrm{d}\mathbf{r}/\mathrm{d}t$ by taking a time derivative.

(*b*) **Planar motion.** The separation between particles $\mathbf{r}(t)$ and the velocity difference $\mathbf{g}(t)$ remain within a plane during the collision. By taking the cross product of (2.7) with $\mu_{ss'}\mathbf{r}$, we find that the reduced particle's angular momentum

$$\mathbf{L} = \mu_{ss'}\mathbf{r} \times \mathbf{g} \tag{2.9}$$

is a constant of the motion, $\mathrm{d}\mathbf{L}/\mathrm{d}t = 0$. Since $\mathbf{L}$ is a constant and $\mathbf{L}\cdot\mathbf{r} = 0$, the separation $\mathbf{r}$ and the velocity difference $\mathbf{g} = \mathrm{d}\mathbf{r}/\mathrm{d}t$ are in the plane perpendicular to $\mathbf{L}$ for all time t. The effects of the collision are independent of the orientation of the plane $\mathbf{L}\cdot\mathbf{r} = 0$ because equation (2.7) is the same independently of the plane in which it is projected.

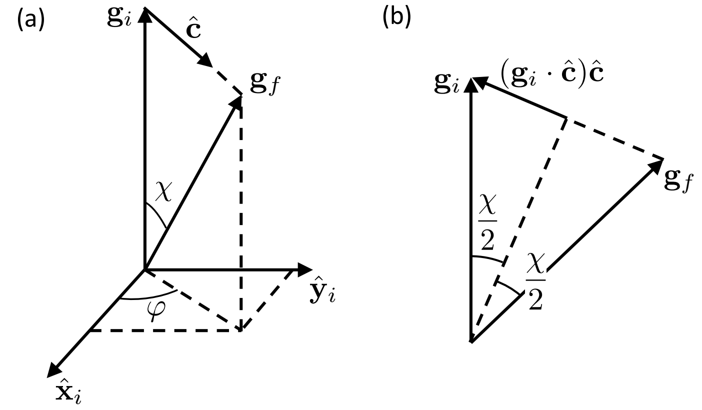

FIGURE 2. (a) 3D sketch of the relative position of the final velocity difference $\mathbf{g}_f$ with respect to the initial velocity difference $\mathbf{g}_i$. The angles χ and φ and the vector $\hat{\mathbf{c}}$ are sketched. (b) Relative position of the final velocity difference $\mathbf{g}_f$ with respect to the initial velocity $\mathbf{g}_i$ in the plane of the collision. The projection $(\mathbf{g}_i \cdot \hat{\mathbf{c}})\hat{\mathbf{c}}$ is sketched.

(*c*) **Elastic collisions.** Since we have assumed that the force is derived from a potential, the collision is elastic, that is, the total kinetic energy is the same before and after the collision. To show this, we can take the scalar product of (2.7) with $\mu_{ss'}\mathbf{g} = \mu_{ss'}(\mathrm{d}\mathbf{r}/\mathrm{d}t)$ and use $(\mathrm{d}V/\mathrm{d}r)(\mathbf{r}/|\mathbf{r}|) = \nabla_{\mathbf{r}} V$ to obtain that the reduced particle's total energy

$$E = \frac{1}{2}\mu_{ss'} g^2 + V(r) \tag{2.10}$$

is a constant of the motion, $\mathrm{d}E/\mathrm{d}t = 0$. Since $V(r) \to 0$ for $r \to \infty$, the magnitudes of $\mathbf{g}_i$ and $\mathbf{g}_f$ are the same, and only the direction is different.

The properties described above imply that we just need to determine the change of direction of the velocity difference $\mathbf{g}$ due to a collision. Since the plane of the collision is mostly irrelevant, the angle χ between the initial velocity difference $\mathbf{g}_i$ and final velocity difference $\mathbf{g}_f$ is the important parameter. We use an orthonormal basis $\{\hat{\mathbf{x}}_i, \hat{\mathbf{y}}_i, \mathbf{g}_i/g_i\}$ aligned with the initial velocity difference $\mathbf{g}_i$ (see figure 2). Note that $\hat{\mathbf{x}}_i$ and $\hat{\mathbf{y}}_i$ are only determined up to a rotation since we only know that they must be in the plane perpendicular to $\mathbf{g}_i$. In the basis $\{\hat{\mathbf{x}}_i, \hat{\mathbf{y}}_i, \mathbf{g}_i/g_i\}$, the final velocity difference $\mathbf{g}_f$ is determined by two angles, χ and φ,

$$\mathbf{g}_f = \underline{\mathbf{g}_i} \equiv g_i \sin\chi(\cos\varphi\, \hat{\mathbf{x}}_i + \sin\varphi\, \hat{\mathbf{y}}_i) + \cos\chi\, \mathbf{g}_i. \tag{2.11}$$

The angle φ determines the plane in which the collision takes place and χ gives the deflection within that plane. Note that we have defined the operator $\underline{Q}$ that relates quantities before and after the collision.

Instead of the two angles χ and φ, we can determine the direction of $\mathbf{g}_f$ using the unit vector $\hat{\mathbf{c}}$ (see figure 2),

$$\hat{\mathbf{c}} = \frac{\mathbf{g}_f - \mathbf{g}_i}{|\mathbf{g}_f - \mathbf{g}_i|} = \cos\left(\frac{\chi}{2}\right)(\cos\varphi\, \hat{\mathbf{x}}_i + \sin\varphi\, \hat{\mathbf{y}}_i) - \sin\left(\frac{\chi}{2}\right)\frac{\mathbf{g}_i}{g_i}. \tag{2.12}$$

With this unit vector we can define the operator $\underline{Q}$ as

$$\mathbf{g}_f = \underline{\mathbf{g}_i} \equiv (\mathbf{I} - 2\hat{\mathbf{c}}\hat{\mathbf{c}}) \cdot \mathbf{g}_i, \tag{2.13}$$

The advantage of using the vector $\hat{\mathbf{c}}$ is that it shows that the operator $\underline{Q}$ is its own inverse, and hence it gives the conditions before the collision if we know the conditions

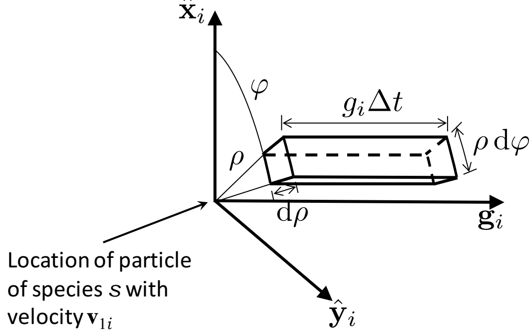

FIGURE 3. Sketch of the volume that a particle of species s' and velocity within the infinitesimal volume $\mathrm{d}^3 v_{2i}$ around $\mathbf{v}_{2i}$ must occupy during the time interval Δt to pass at a distance $\in [\rho, \rho + \mathrm{d}\rho]$ and at an angle $\in [\varphi, \varphi + \mathrm{d}\varphi]$ of a particle of species s with velocity $\mathbf{v}_{1i}$. The time Δt is assumed to be sufficiently long that $g_i \Delta t \gg \rho$. As a result, we can ignore particles that are a distance $\sim \rho$ ahead of the particle of species s with velocity $\mathbf{v}_{1i}$ even though they interacting with the particle of species s with velocity $\mathbf{v}_{1i}$.

after,

$$\mathbf{g}_i = \underline{\mathbf{g}_f} \equiv (\mathbf{I} - 2\hat{\mathbf{c}}\hat{\mathbf{c}}) \cdot \mathbf{g}_f. \tag{2.14}$$

Indeed,

$$\underline{\underline{\mathbf{g}_i}} = (\mathbf{I} - 2\hat{\mathbf{c}}\hat{\mathbf{c}}) \cdot (\mathbf{I} - 2\hat{\mathbf{c}}\hat{\mathbf{c}}) \cdot \mathbf{g}_i = [\mathbf{I} - 4\hat{\mathbf{c}}\hat{\mathbf{c}} + 4(\hat{\mathbf{c}} \cdot \hat{\mathbf{c}})\hat{\mathbf{c}}\hat{\mathbf{c}}] \cdot \mathbf{g}_i = \mathbf{g}_i. \tag{2.15}$$

Once $\hat{\mathbf{c}}$ or the angles χ and φ are given, we can obtain the final $\mathbf{g}_f$, and using (2.6), determine $\mathbf{v}_{1f}$ and $\mathbf{v}_{2f}$,

$$\begin{aligned} \mathbf{v}_{1f} = \underline{\mathbf{v}_{1i}} &\equiv \mathbf{v}_{1i} - \frac{2m_{s'}}{m_s + m_{s'}} \hat{\mathbf{c}}(\hat{\mathbf{c}} \cdot \mathbf{g}_i), \\ \mathbf{v}_{2f} = \underline{\mathbf{v}_{2i}} &\equiv \mathbf{v}_{2i} + \frac{2m_s}{m_s + m_{s'}} \hat{\mathbf{c}}(\hat{\mathbf{c}} \cdot \mathbf{g}_i). \end{aligned} \tag{2.16}$$

The relation (2.16) is its own inverse,

$$\begin{aligned} \mathbf{v}_{1i} = \underline{\mathbf{v}_{1f}} &\equiv \mathbf{v}_{1f} - \frac{2m_{s'}}{m_s + m_{s'}} \hat{\mathbf{c}}(\hat{\mathbf{c}} \cdot \mathbf{g}_f), \\ \mathbf{v}_{2i} = \underline{\mathbf{v}_{2f}} &\equiv \mathbf{v}_{2f} + \frac{2m_s}{m_s + m_{s'}} \hat{\mathbf{c}}(\hat{\mathbf{c}} \cdot \mathbf{g}_f). \end{aligned} \tag{2.17}$$

We have again used the operator $\underline{Q}$ that gives the final (initial) velocities for a collision whose vector $\hat{\mathbf{c}}$ and initial (final) velocities are known.

2.2. *Boltzmann collision operator*

The collision operator $C_{ss'}$ gives the rate of change in time of the distribution function of species s, $f_s(\mathbf{r}, \mathbf{v}, t)$, as a result of collisions with species s'. The kinetic equation for the distribution function $f_s(\mathbf{r}, \mathbf{v}, t)$ in a collisional plasma is

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{Z_s e}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_s = \sum_{s'} C_{ss'}. \tag{2.18}$$

The Boltzmann operator for binary collisions between species s and s' was derived using the BBGKY approach in (Dellar 2015). Here we use a heuristic derivation. The

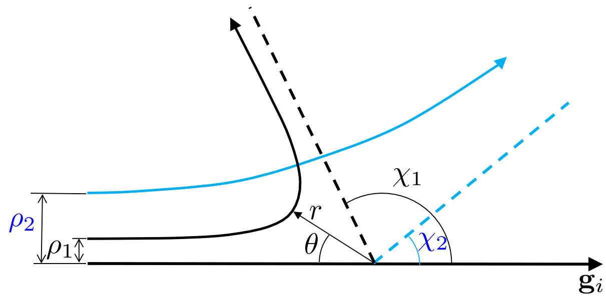

FIGURE 4. Impact parameter and its relation to the angle χ. Two impact parameters, ρ_1 (black) and ρ_2 (blue), are given with their respective trajectories and angles, χ_1 and χ_2. This figure assumes a repelling force that increases when the particles gets closer.

collision operator is composed of two terms,

$$C_{ss'}(\mathbf{r},\mathbf{v},t)\,\mathrm{d}^3v\,\mathrm{d}^3r = (\text{collisions/time with } \mathbf{v}_{1f} \text{ within volume } \mathrm{d}^3v \text{ around } \mathbf{v}) \\ -(\text{collisions/time with } \mathbf{v}_{1i} \text{ within volume } \mathrm{d}^3v \text{ around } \mathbf{v}). \quad (2.19)$$

To estimate the number of collisions per unit time, we use the construction in figure 3. The impact parameter ρ is the distance of closest approach between the two particles if we assume that they do not interact. The impact parameter determines the angle χ that characterizes the collision (see figure 4). The number of collisions

- characterized by the angles $\in [\varphi, \varphi + \mathrm{d}\varphi]$ and $\in [\chi, \chi + \mathrm{d}\chi]$
- of a particle of species s with velocity $\mathbf{v}_{1i}$
- in the time interval Δt
- with particles of species s' with velocity within the infinitesimal volume d^3v_{2i} around $\mathbf{v}_{2i}$

corresponds to the number

- of particles of species s' with velocity within the infinitesimal volume d^3v_{2i} around $\mathbf{v}_{2i}$
- that pass at a distance $\in [\rho(\chi) - \mathrm{d}\rho, \rho(\chi)]$ (with $\mathrm{d}\rho = |\partial\rho/\partial\chi|\,\mathrm{d}\chi$) and an angle $\in [\varphi, \varphi + \mathrm{d}\varphi]$ of a particle of species s with velocity $\mathbf{v}_{1i}$
- in the time interval Δt.

The number of particles of species s' that satisfy these conditions is

$$(\text{density of particles of species } s' \text{ within the volume } \mathrm{d}^3v_{2i} \text{ around } \mathbf{v}_{2i}) \\ \times(\text{volume sketched in figure 3}) = (f_{s'}(\mathbf{r},\mathbf{v}_{2i},t)\,\mathrm{d}^3v_{2i})(g_i\Delta t\,\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi). \quad (2.20)$$

Dividing by Δt, multiplying by the number of particles of species s with velocity within the infinitesimal volume d^3v_{1i} around $\mathbf{v}_{1i}$, $f_s(\mathbf{r},\mathbf{v}_{1i},t)\,\mathrm{d}^3v_{1i}\,\mathrm{d}^3r$, and integrating over all possible initial velocities of particles of species s', and over all possible impact parameters ρ and angles φ, we obtain the number of collisions per unit time of particles of species s with initial velocity within an infinitesimal volume d^3v_{1i} around $\mathbf{v}_{2i}$,

$$\text{collisions/time with initial velocity within an infinitesimal volume } \mathrm{d}^3v_{1i} \text{ around } \mathbf{v}_{1i} \\ = \mathrm{d}^3r\,\mathrm{d}^3v_{1i}\int \mathrm{d}^3v_{2i}\int_0^\infty \mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\varphi\, f_s(\mathbf{r},\mathbf{v}_{1i},t)f_{s'}(\mathbf{r},\mathbf{v}_{2i},t)g_i\rho. \quad (2.21)$$

Before continuing, it is convenient to define the differential cross section $\sigma_{ss'}(g_i,\chi)$. The differential cross section can be calculated if we know the relation between the impact parameter ρ and the angle χ, $\rho(\chi)$ (see figure 4),

$$\sigma_{ss'} = \frac{\rho}{\sin\chi}\left|\frac{\partial\rho}{\partial\chi}\right|. \tag{2.22}$$

For collisions satisfying equations (2.1)-(2.2), the differential cross section can only depend on the magnitude of the initial velocity difference, g_i, and the angle χ due to the symmetry of the equations (see (2.7)). With the definition (2.22), equation (2.21) becomes

$$\text{collisions/time with initial velocity within an infinitesimal volume } \mathrm{d}^3v_{1i} \text{ of } \mathbf{v}_{1i}$$
$$= \mathrm{d}^3r\,\mathrm{d}^3v_{1i}\int \mathrm{d}^3v_{2i}\int_0^\pi \mathrm{d}\chi\int_0^{2\pi}\mathrm{d}\varphi\, f_s(\mathbf{r},\mathbf{v}_{1i},t)f_{s'}(\mathbf{r},\mathbf{v}_{2i},t)g_i\sigma_{ss'}(g_i,\chi)\sin\chi. \tag{2.23}$$

Using (2.23), we can obtain the two terms in (2.19). Since the operator $\underline{Q}$ gives the initial conditions for some given final conditions, we use it in (2.23) to rewrite the first term in (2.19) as

$$\text{collisions/time with } \mathbf{v}_{1f} \text{ within volume } \mathrm{d}^3v \text{ of } \mathbf{v}$$
$$= \mathrm{d}^3r\,\mathrm{d}^3\underline{v}\int \mathrm{d}^3\underline{v}'\int_0^\pi \mathrm{d}\chi\int_0^{2\pi}\mathrm{d}\varphi\, f_s(\mathbf{r},\underline{\mathbf{v}},t)f_{s'}(\mathbf{r},\underline{\mathbf{v}}',t)g\sigma_{ss'}(g,\chi)\sin\chi, \tag{2.24}$$

where $\mathbf{v}_{2i} = \underline{\mathbf{v}}'$, and $g = g_i = \underline{g}$. Using (2.23), the second term in (2.19) simply becomes

$$\text{collisions/time with } \mathbf{v}_{1i} \text{ within volume } \mathrm{d}^3v \text{ of } \mathbf{v}$$
$$= \mathrm{d}^3r\,\mathrm{d}^3v\int \mathrm{d}^3v'\int_0^\pi \mathrm{d}\chi\int_0^{2\pi}\mathrm{d}\varphi\, f_s(\mathbf{r},\mathbf{v},t)f_{s'}(\mathbf{r},\mathbf{v}',t)g\sigma_{ss'}(g,\chi)\sin\chi. \tag{2.25}$$

With these results, equation (2.19) can be rewritten as

$$\begin{aligned} C_{ss'}(\mathbf{r},\mathbf{v},t)\,\mathrm{d}^3v\,\mathrm{d}^3r = {} & \mathrm{d}^3r\,\mathrm{d}^3\underline{v}\int \mathrm{d}^3\underline{v}'\int_0^\pi \mathrm{d}\chi\int_0^{2\pi}\mathrm{d}\varphi\, f_s(\mathbf{r},\underline{\mathbf{v}},t)f_{s'}(\mathbf{r},\underline{\mathbf{v}}',t)g\sigma_{ss'}(g,\chi)\sin\chi \\ & -\mathrm{d}^3r\,\mathrm{d}^3v\int \mathrm{d}^3v'\int_0^\pi \mathrm{d}\chi\int_0^{2\pi}\mathrm{d}\varphi\, f_s(\mathbf{r},\mathbf{v},t)f_{s'}(\mathbf{r},\mathbf{v}',t)g\sigma_{ss'}(g,\chi)\sin\chi. \end{aligned} \tag{2.26}$$

We can simplify further using the fact that the operator $\underline{Q}$ is its own inverse. The determinant of the Jacobian of the operator $\underline{Q}$ is then ± 1, giving $\mathrm{d}^3\underline{v}\,\mathrm{d}^3\underline{v}' = \mathrm{d}^3v\,\mathrm{d}^3v'$. Using this result, equation (2.26) finally becomes

$$\begin{aligned} C_{ss'}[f_s,f_{s'}](\mathbf{r},\mathbf{v},t) = \int \mathrm{d}^3v'\int_0^\pi \mathrm{d}\chi\int_0^{2\pi}\mathrm{d}\varphi\, g\sigma_{ss'}(g,\chi)\sin\chi\Big[& f_s(\mathbf{r},\underline{\mathbf{v}},t)f_{s'}(\mathbf{r},\underline{\mathbf{v}}',t) \\ & -f_s(\mathbf{r},\mathbf{v},t)f_{s'}(\mathbf{r},\mathbf{v}',t)\Big]. \end{aligned} \tag{2.27}$$

To deduce the Fokker-Planck collision operator, we will not use (2.27). Instead, it is convenient to use moments of the collision operator. For any function $X(\mathbf{r},\mathbf{v},t)$, we calculate the corresponding moment

$$\int X(\mathbf{r},\mathbf{v},t)\, C_{ss'}[f_s, f_{s'}](\mathbf{r},\mathbf{v},t)\, \mathrm{d}^3v =$$

$$\int \mathrm{d}^3v\, X(\mathbf{v}) \int \mathrm{d}^3v' \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\chi\, g\sigma_{ss'}(g,\chi) \sin\chi f_s(\underline{\mathbf{v}}) f_{s'}(\underline{\mathbf{v}}')$$

$$- \int \mathrm{d}^3v\, X(\mathbf{v}) \int \mathrm{d}^3v' \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\chi\, g\sigma_{ss'}(g,\chi) \sin\chi f_s(\mathbf{v}) f_{s'}(\mathbf{v}'), \tag{2.28}$$

where we have omitted the dependence on $\mathbf{r}$ and t in the right side for brevity. Using the fact that the operator $\underline{Q}$ is its own inverse (and hence $\mathrm{d}^3v\, \mathrm{d}^3v' = \mathrm{d}^3\underline{v}\, \mathrm{d}^3\underline{v}'$), and the fact that $\underline{g} = g$, the first integral in (2.28) can be written as

$$\int \mathrm{d}^3v\, X(\mathbf{v}) \int \mathrm{d}^3v' \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\chi\, g\sigma_{ss'}(g,\chi) \sin\chi f_s(\underline{\mathbf{v}}) f_{s'}(\underline{\mathbf{v}}') =$$

$$\int \mathrm{d}^3\underline{v}\, X(\underline{\underline{\mathbf{v}}}) \int \mathrm{d}^3\underline{v}' \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\chi\, \underline{g}\sigma_{ss'}(\underline{g},\chi) \sin\chi f_s(\underline{\mathbf{v}}) f_{s'}(\underline{\mathbf{v}}'). \tag{2.29}$$

Using (2.29), and changing the dummy integration variables $\underline{\mathbf{v}}$ and $\underline{\mathbf{v}}'$ to $\mathbf{v}$ and $\mathbf{v}'$, equation (2.28) becomes

$$\int X(\mathbf{r},\mathbf{v},t)\, C_{ss'}[f_s, f_{s'}](\mathbf{r},\mathbf{v},t)\, \mathrm{d}^3v =$$

$$\int \mathrm{d}^3v\, f_s(\mathbf{v}) \int \mathrm{d}^3v'\, f_{s'}(\mathbf{v}') \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\chi\, g\sigma_{ss'}(g,\chi) \sin\chi\, [X(\underline{\mathbf{v}}) - X(\mathbf{v})]. \tag{2.30}$$

3. Coulomb collisions

We need to calculate the differential cross section for Coulomb interactions. To do so, we first calculate the relation between the impact parameter ρ and the angle χ. The equations of motion are (2.1)-(2.2) with the potential

$$V(r) = \frac{Z_s Z_{s'} e^2}{4\pi\epsilon_0 r}. \tag{3.1}$$

As we have already explained, the motion throughout the collision is described by equation (2.7) for the reduced particle. To solve this equation, we recall that the motion remains within the same plane, and we use the polar coordinates r and θ in figure 4. We also use that the reduced particle angular momentum is conserved,

$$|\mathbf{L}(t)| \equiv \mu_{ss'} r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = |\mathbf{L}(t \to -\infty)| \equiv \mu_{ss'} \rho g, \tag{3.2}$$

and that the reduced particle total energy is constant,

$$E(t) \equiv \frac{1}{2}\mu_{ss'} \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{1}{2}\mu_{ss'} r^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + \frac{Z_s Z_{s'} e^2}{4\pi\epsilon_0 r} = E(t \to -\infty) \equiv \frac{1}{2}\mu_{ss'} g^2. \tag{3.3}$$

Solving for $\mathrm{d}\theta/\mathrm{d}t$ from (3.2) and for $\mathrm{d}r/\mathrm{d}t$ from (3.3), we find an equation for $r(\theta)$,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}r/\mathrm{d}t}{\mathrm{d}\theta/\mathrm{d}t} = \pm\frac{r^2}{\rho}\sqrt{1 - \frac{2b_{ss'}}{r} - \frac{\rho^2}{r^2}}, \tag{3.4}$$

where

$$b_{ss'} = \frac{Z_s Z_s' e^2}{4\pi\epsilon_0 \mu_{ss'} g^2}. \tag{3.5}$$

Due to the symmetry of the problem (see figure 4), the solution from $\theta = \pi/2 - \chi/2$ to $\theta = \pi - \chi$ is the reflection of the solution from $\theta = 0$ to $\theta = \pi/2 - \chi/2$. The sign of $\mathrm{d}r/\mathrm{d}\theta$ determines whether the equation is for the portion of the orbit in which r decreases as θ increases (in figure 4, from $r = \infty$ at $\theta = 0$ to $r = r_{\mathrm{min}}$ at $\theta = \pi/2 - \chi/2$) or it is for the section in which r increases as θ increases (in figure 4, from $r = r_{\mathrm{min}}$ at $\theta = \pi/2 - \chi/2$ to $r = \infty$ at $\theta = \pi - \chi$). We integrate (3.4) from $\theta = 0$ to $\theta = \pi/2 - \chi/2$,

$$-\int_{\infty}^{r_{\mathrm{min}}} \frac{\rho\, \mathrm{d}r}{r^2\sqrt{1 - 2b_{ss'}/r - \rho^2/r^2}} = \int_0^{\pi/2-\chi/2} \mathrm{d}\theta. \tag{3.6}$$

This integral gives

$$\left[\arcsin\left(\frac{\rho/r + b_{ss'}/\rho}{\sqrt{1 + b_{ss'}^2/\rho^2}}\right)\right]_{r=\infty}^{r=r_{\mathrm{min}}} = \frac{\pi}{2} - \frac{\chi}{2}. \tag{3.7}$$

The minimum r is given by setting $\mathrm{d}r/\mathrm{d}\theta$ in (3.4) equal to zero,

$$r_{\mathrm{min}} = \frac{\rho}{\sqrt{1 + b_{ss'}^2/\rho^2} - b_{ss'}/\rho}. \tag{3.8}$$

Using this result in (3.7), we find

$$\frac{\pi}{2} - \arcsin\left(\frac{b_{ss'}/\rho}{\sqrt{1 + b_{ss'}^2/\rho^2}}\right) = \frac{\pi}{2} - \frac{\chi}{2}, \tag{3.9}$$

and it can be rewritten as

$$\rho = \frac{b_{ss'}}{\tan(\chi/2)}. \tag{3.10}$$

Using (3.10) in (2.22), we obtain the differential cross section for Coulomb collisions, also known as Rutherford cross section,

$$\sigma_{ss'} = \frac{b_{ss'}^2}{4\sin^4(\chi/2)}. \tag{3.11}$$

4. Fokker-Planck collision operator

We have argued in the introduction that the interaction of a given charged particle with other charged particles is dominated by small collisions with all the particles within the Debye sphere around the particle of interest. For a weakly coupled plasma, particles in the Debye sphere are mostly at a distance $\rho \sim \lambda_D \gg b_{ss'}$. As a result, equation (3.10) implies that

$$\chi = 2\arctan\left(\frac{b_{ss'}}{\rho}\right) \simeq \frac{2b_{ss'}}{\rho} \sim \frac{1}{\Lambda} \ll 1. \tag{4.1}$$

Then, we can assume $\chi \ll 1$ and use the approximation

$$\sigma_{ss'} \simeq \frac{4b_{ss'}^2}{\chi^4} \tag{4.2}$$

for the Rutherford cross section.

The fact that χ is small can be used to simplify the operator $\underline{Q}$ in (2.16). For example, the final velocity of a particle of species s that started with velocity $\mathbf{v}$ after a collision with a particle of species s' with velocity $\mathbf{v}'$ becomes

$$\underline{\mathbf{v}} \equiv \mathbf{v} + \Delta\mathbf{v} + O(\chi^3\mathbf{v}), \tag{4.3}$$

where

$$\Delta\mathbf{v} = \frac{m_{s'}}{m_s + m_{s'}}\left[\chi g\,(\cos\varphi\,\hat{\mathbf{x}}_i + \sin\varphi\,\hat{\mathbf{y}}_i) - \frac{\chi^2}{2}\mathbf{g}\right]. \tag{4.4}$$

We have kept corrections of order χ^2 because we will need them due to several cancellations.

Using equation (4.3) for particles of species s and a similar expression for particles of species s', we can expand the Boltzmann operator in (2.27). Performing the expansion directly is tedious, but there is a shortcut that simplifies the calculation. We will use the expression for the moments of the Boltzman collision operator given in (2.30). We are going to look for an approximate operator $C_{ss'}^{\mathrm{FP}}$ that satisfies

$$\int X(\mathbf{r},\mathbf{v},t)\,C_{ss'}^{\mathrm{FP}}[f_s, f_{s'}](\mathbf{r},\mathbf{v},t)\,\mathrm{d}^3v \simeq \int X(\mathbf{r},\mathbf{v},t)\,C_{ss'}[f_s, f_{s'}](\mathbf{r},\mathbf{v},t)\,\mathrm{d}^3v \tag{4.5}$$

for any functions $X(\mathbf{r},\mathbf{v},t)$, $f_s(\mathbf{r},\mathbf{v},t)$ and $f_{s'}(\mathbf{r},\mathbf{v},t)$. This operator must then satisfy

$$C_{ss'}^{\mathrm{FP}}[f_s, f_{s'}] \simeq C_{ss'}[f_s, f_{s'}]. \tag{4.6}$$

Using (4.3), we obtain

$$X(\underline{\mathbf{v}}) - X(\mathbf{v}) = \Delta\mathbf{v}\cdot\nabla_v X + \frac{1}{2}\Delta\mathbf{v}\Delta\mathbf{v} : \nabla_v\nabla_v X + O(\chi^3 X). \tag{4.7}$$

Using this approximation in (2.30), we find

$$\begin{aligned}
&\int X(\mathbf{r},\mathbf{v},t)\,C_{ss'}[f_s, f_{s'}](\mathbf{r},\mathbf{v},t)\,\mathrm{d}^3v = \\
&\quad\int \mathrm{d}^3v\, f_s(\mathbf{v}) \int \mathrm{d}^3v'\, f_{s'}(\mathbf{v}') \int_0^{2\pi} \mathrm{d}\varphi \Bigg[\int_0^{\chi_{\mathrm{small}}} \mathrm{d}\chi\, g\sigma_{ss'}(g,\chi) \Bigg(\chi\Delta\mathbf{v}\cdot\nabla_v X \\
&\quad + \frac{\chi}{2}\Delta\mathbf{v}\Delta\mathbf{v} : \nabla_v\nabla_v X + O(\chi^4 X)\Bigg) + \underbrace{\int_{\chi_{\mathrm{small}}}^{\pi} \mathrm{d}\chi\, g\sigma_{ss'}(g,\chi)\sin\chi\,\Big(X(\underline{\mathbf{v}}) - X(\mathbf{v})\Big)}_{\sim g b_{ss'}^2 X}\Bigg].
\end{aligned} \tag{4.8}$$

Note that we have split the integral over χ into two intervals: the interval $[0, \chi_{\mathrm{small}}]$, with $\chi_{\mathrm{small}} \ll 1$, in which the approximation $\chi \ll 1$ is valid, and the interval $[\chi_{\mathrm{small}}, \pi]$ that accounts for the collisions with large deflection angles.

Using (4.2) and (4.4), we take the integrals over the angles φ and χ in the region $[0, \chi_{\mathrm{small}}]$, finding

$$\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\chi_{\mathrm{small}}} \mathrm{d}\chi\, g\sigma_{ss'}(g,\chi)\,\chi\Delta\mathbf{v} = -\frac{4\pi m_{s'} g b_{ss'}^2}{m_s + m_{s'}}\mathbf{g} \int_0^{\chi_{\mathrm{small}}} \frac{\mathrm{d}\chi}{\chi} \tag{4.9}$$

and

$$\begin{aligned}
&\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\chi_{\mathrm{small}}} \mathrm{d}\chi\, g\sigma_{ss'}(g,\chi)\,\frac{\chi}{2}\Delta\mathbf{v}\Delta\mathbf{v} \\
&\qquad = \frac{2\pi m_{s'}^2 g^3 b_{ss'}^2}{(m_s + m_{s'})^2}(\hat{\mathbf{x}}_i\hat{\mathbf{x}}_i + \hat{\mathbf{y}}_i\hat{\mathbf{y}}_i) \int_0^{\chi_{\mathrm{small}}} \left(\frac{1}{\chi} + O(1)\right)\mathrm{d}\chi.
\end{aligned} \tag{4.10}$$

Both of these integrals diverge due to the lower limit $\chi = 0$. The Coulomb interaction is long range, and it may seem that as a result, every charged particle interacts with all the other charged particles in the system. Fortunately, there is a scale, the Debye length,

λ_D, beyond which the Coulomb potential is shielded. To obtain the Boltzmann collision operator, we had to assume that we can consider the interaction between charged particles as many uncorrelated binary collisions. Particles beyond the Debye length respond to the motion of the charged particle of interest in a correlated manner. We can only use the Boltzmann collision operator for $\rho \lesssim \lambda_D$. Then, according to equation (4.1), the Botzmann operator can only be used for collisions that satisfy $\chi \gtrsim \chi_{\min} = 2b_{ss'}/\lambda_D$. For χ of order $\chi_{\min} = 2b_{ss'}/\lambda_D$ or smaller, we need to take into account the correlations between particles. It can be done rigorously using the BBGKY formalism, and the result is the Balescu-Lenard collision operator (Balescu 1960; Lenard 1960; Hazeltine & Waelbroeck 2004). The Boltzmann collision operator and the Balescu-Lenard operator can be combined into a single collision operator (Frieman & Book 1963) that is rarely used.

Using the fact that the Botzmann operator can only be used for collisions that satisfy $\chi \gtrsim \chi_{\min} = 2b_{ss'}/\lambda_D$, the integral $\int_0^{\chi_{\text{small}}} \chi^{-1}\,\mathrm{d}\chi$ that appears in (4.9) and (4.10) should be

$$\int_{\chi_{\min}}^{\chi_{\text{small}}} \frac{\mathrm{d}\chi}{\chi} = \ln\left(\frac{\chi_{\text{small}}}{\chi_{\min}}\right) = \ln\Lambda_{ss'} \gg 1, \tag{4.11}$$

where we have assumed χ_{small} to be larger than $\chi_{\min} = 2b_{ss'}/\lambda_D$. The function $\ln\Lambda_{ss'}$ is the <u>Coulomb</u> <u>logarithm</u>. We have used the notation $\ln\Lambda_{ss'}$ even though, as we will see shortly, the quantity $\chi_{\text{small}}/\chi_{\min}$ will not be the plasma parameter Λ defined in (1.6). For $\ln\Lambda_{ss'} \gg 1$, the collision operator for charged particles is greatly simplified. Using (4.11) in (4.9) and (4.10), we obtain

$$\int_0^{2\pi} \mathrm{d}\varphi \int_{\chi_{\min}}^{\chi_{\text{small}}} \mathrm{d}\chi\, g\sigma_{ss'}(g,\chi)\,\chi\Delta\mathbf{v} = -\frac{4\pi m_{s'} g b_{ss'}^2 \ln\Lambda_{ss'}}{m_s + m_{s'}}\mathbf{g} \tag{4.12}$$

and

$$\begin{aligned}\int_0^{2\pi} \mathrm{d}\varphi \int_{\chi_{\min}}^{\chi_{\text{small}}} \mathrm{d}\chi\, g\sigma_{ss'}(g,\chi)\,\frac{\chi}{2}\Delta\mathbf{v}\Delta\mathbf{v}& \\ \simeq \frac{2\pi m_{s'}^2 g^3 b_{ss'}^2 \ln\Lambda_{ss'}}{(m_s + m_{s'})^2}(\hat{\mathbf{x}}_i\hat{\mathbf{x}}_i + \hat{\mathbf{y}}_i\hat{\mathbf{y}}_i).&\end{aligned} \tag{4.13}$$

The term due to $\chi \in [\chi_{\text{small}}, \pi]$ in (4.8) can be neglected as small in $1/\ln\Lambda_{ss'} \ll 1$ compared to the terms in (4.12) and (4.13). The contribution from collisions with $\chi \lesssim \chi_{\min} = 2b_{ss'}/\lambda_D$, included in the Balescu-Lenard collision operator, turns out to be small in $1/\ln\Lambda_{ss'} \ll 1$ as well. Thus, as long as $1/\ln\Lambda_{ss'} \gg 1$, we can neglect the collisions that do not satisfy $\chi_{\min} \lesssim \chi \lesssim \chi_{\text{small}}$.

With the results in (4.12) and (4.13), and using $\hat{\mathbf{x}}_i\hat{\mathbf{x}}_i + \hat{\mathbf{y}}_i\hat{\mathbf{y}}_i = \mathbf{I} - \mathbf{g}\mathbf{g}/g^2$, equation (4.8) becomes

$$\begin{aligned}&\int X(\mathbf{r},\mathbf{v},t)\, C_{ss'}[f_s, f_{s'}](\mathbf{r},\mathbf{v},t)\,\mathrm{d}^3v \\ &\qquad \simeq \frac{\gamma_{ss'}}{m_s}\int \mathrm{d}^3v\, f_s(\mathbf{v}) \int \mathrm{d}^3v'\, f_{s'}(\mathbf{v}')\Bigg[-2\left(\frac{1}{m_s} + \frac{1}{m_{s'}}\right)\frac{\mathbf{g}}{g^3}\cdot\nabla_v X \\ &\qquad + \frac{1}{m_s}\frac{g^2\mathbf{I} - \mathbf{g}\mathbf{g}}{g^3} : \nabla_v\nabla_v X\Bigg],\end{aligned} \tag{4.14}$$

where

$$\gamma_{ss'} = \frac{2\pi Z_s^2 Z_{s'}^2 e^4 \ln\Lambda_{ss'}}{(4\pi\epsilon_0)^2}. \tag{4.15}$$

To write the right side of (4.14) as $\int X(\mathbf{r},\mathbf{v},t)\, C^{\mathrm{FP}}_{ss'}[f_s,f_{s'}](\mathbf{r},\mathbf{v},t)\,\mathrm{d}^3v$, we integrate by parts in velocity space to find

$$\int f_s(\mathbf{v})\,\frac{\mathbf{g}}{g^3}\cdot\nabla_v X\,\mathrm{d}^3v = -\int X\nabla_v\cdot\left(f_s(\mathbf{v})\,\frac{\mathbf{g}}{g^3}\right)\mathrm{d}^3v \tag{4.16}$$

and

$$\int f_s(\mathbf{v})\,\frac{g^2\mathbf{I}-\mathbf{g}\mathbf{g}}{g^3}:\nabla_v\nabla_v X\,\mathrm{d}^3v = \int X\nabla_v\cdot\left[\nabla_v\cdot\left(f_s(\mathbf{v})\,\frac{g^2\mathbf{I}-\mathbf{g}\mathbf{g}}{g^3}\right)\right]\mathrm{d}^3v. \tag{4.17}$$

With these results, equation (4.14) becomes (4.5) with

$$C^{\mathrm{FP}}_{ss'}[f_s,f_{s'}] = \frac{\gamma_{ss'}}{m_s}\nabla_v\cdot\left\{\int f_{s'}(\mathbf{v}')\left[\frac{1}{m_s}\nabla_v\cdot\left(\frac{g^2\mathbf{I}-\mathbf{g}\mathbf{g}}{g^3}f_s(\mathbf{v})\right)\right.\right.$$
$$\left.\left.+2\left(\frac{1}{m_s}+\frac{1}{m_{s'}}\right)\frac{\mathbf{g}}{g^3}f_s(\mathbf{v})\right]\mathrm{d}^3v'\right\}. \tag{4.18}$$

This is the Fokker-Planck collision operator.

5. The Coulomb logarithm

The Fokker-Planck collision operator is based on the expansion in $1/\ln\Lambda_{ss'}\ll 1$. To find $\ln\Lambda_{ss'}$, we need equations for χ_{min} and χ_{small}. Fortunately, both χ_{min} and χ_{small} appear inside logarithm. As long as $\ln\Lambda_{ss'}\gg 1$, we do not need to worry about factors of order unity in χ_{min} and χ_{small}.

(*a*) For $\chi_{\mathrm{min}} = 2b_{ss'}/\lambda_D$, we need to evaluate both $b_{ss'}$ and λ_D. Conventionally (Huba 2013), the Debye length is defined to be

$$\lambda_D = \left(\sum_{\sqrt{T_{s''}/m_{s''}}\gtrsim g}\frac{Z^2_{s''}e^2n_{s''}}{\epsilon_0 T_{s''}}\right)^{-1/2}, \tag{5.1}$$

where the summation is done only over species whose thermal $v_{ts''}=\sqrt{T_{s''}/m_{s''}}$ is faster or of the order of the typical velocity difference $g\sim\sqrt{T_s/m_s+T_{s'}/m_{s'}}$. The idea is that species with thermal speeds much slower than g cannot respond in the time scale of the collision and will not be able to shield the electric field. Note that the formula in (5.1) is defined such that having more species reduces the Debye length due to additional shielding.

The impact parameter $b_{ss'}$ depends on the relative velocity g. Since $b_{ss'}$ appears within the logarithm, we can replace g by an approximate average value,

$$g\approx\langle g\rangle = \sqrt{\frac{3T_s}{m_s}+\frac{3T_{s'}}{m_{s'}}}. \tag{5.2}$$

The approximate average value of $b_{ss'}$ calculated using $\langle g\rangle$ is usually employed (Huba 2013),

$$b_{ss'}\approx\langle b_{ss'}\rangle = \frac{Z_sZ_{s'}e^2(m_s+m_{s'})}{12\pi\epsilon_0(m_sT_{s'}+m_{s'}T_s)}. \tag{5.3}$$

(*b*) If the plasma is sufficiently hot, the characteristic quantum wavelength,

$$\ell_{ss'} = \frac{\hbar}{2\mu_{ss'}g}, \tag{5.4}$$

with $\hbar = h/2\pi$ and h Planck's constant, becomes large and one needs to include quantum effects. In the classical limit, the smallest scattering angle is $\chi_{\min} = 2b_{ss'}/\lambda_D$. When quantum effects are included, the particle is a wave that diffracts due to perturbations to the charge density of characteristic length λ_D. The diffraction angle in this case is of the order of $\ell_{ss'}/\lambda_D$. When this angle is larger than $2b_{ss'}/\lambda_D$, quantum effects dominate small scattering angles. Thus, we take $\chi_{\min}$ to be

$$\chi_{\min} = \max\left(\frac{2b_{ss'}}{\lambda_D}, \frac{2\ell_{ss'}}{\lambda_D}\right). \tag{5.5}$$

The angle $\chi_{\min}$ depends on g, but it appears inside the logarithm. Thus, we can use the approximation in (5.2) to find

$$\chi_{\min} \approx \langle\chi_{\min}\rangle = \max\left(\frac{2\langle b_{ss'}\rangle}{\lambda_D}, \frac{2\langle\ell_{ss'}\rangle}{\lambda_D}\right), \tag{5.6}$$

where

$$\langle\ell_{ss'}\rangle = \frac{\hbar(m_s + m_{s'})}{2\sqrt{3m_s m_{s'}(m_s T_{s'} + m_{s'} T_s)}}. \tag{5.7}$$

(c) The limit χ_{small} was imposed in (4.8) so that the $\chi \ll 1$ approximations were valid. It is then natural to choose a number of order unity for χ_{small}. Conventionally, $\chi_{\text{small}} = 2$.

Thus, conventionally (Huba 2013), we define the Coulomb logarithm to be

$$\ln\Lambda_{ss'} = \min\left(\ln\left(\frac{\lambda_D}{\langle b_{ss'}\rangle}\right), \ln\left(\frac{\lambda_D}{\langle\ell_{ss'}\rangle}\right)\right), \tag{5.8}$$

where λ_D, $\langle b_{ss'}\rangle$ and $\langle\ell_{ss'}\rangle$ are defined in (5.1), (5.3) and (5.7), respectively. As an example, we can calculate $\ln\Lambda_{ss'}$ for electron-electron, electron-ion and ion-ion collisions. For electron-electron collisions, since $m_i \gg m_e$, only the electrons are sufficiently fast to satisfy the condition $v_{te} \sim g \gg v_{ti}$, giving $\lambda_D = \sqrt{\epsilon_0 T_e/e^2 n_e}$. The impact parameter is $\langle b_{ee}\rangle = e^2/12\pi\epsilon_0 T_e$, and the characteristic quantum wavelength is $\langle\ell_{ee}\rangle = \hbar/\sqrt{6m_e T_e}$. Then,

$$\ln\Lambda_{ee} = \left\{\begin{array}{ll} 18 - (1/2)\ln(n_e[10^{20}\text{m}^{-3}]) + (3/2)\ln(T_e[1\text{ keV}]) & \text{for } T_e \lesssim 20\text{ eV} \\ 16 - (1/2)\ln(n_e[10^{20}\text{m}^{-3}]) + \ln(T_e[1\text{ keV}]) & \text{for } T_e \gtrsim 20\text{ eV} \end{array}\right. . \tag{5.9}$$

For electron-ion collisions, the electrons are the only species that has a thermal speed $v_{te} \sim g \gg v_{ti}$, giving $\lambda_D = \sqrt{\epsilon_0 T_e/e^2 n_e}$. Neglecting $m_e \ll m_i$, we find that the impact parameter is $\langle b_{ei}\rangle = Ze^2/12\pi\epsilon_0 T_e$, and the characteristic quantum wavelength is $\langle\ell_{ei}\rangle = \hbar/2\sqrt{3m_e T_e}$. Then,

$$\ln\Lambda_{ei} = \left\{\begin{array}{ll} 18 - (1/2)\ln(n_e[10^{20}\text{m}^{-3}]) - \ln Z + (3/2)\ln(T_e[1\text{ keV}]) & \text{for } T_e \lesssim 40Z^2\text{ eV} \\ 16 - (1/2)\ln(n_e[10^{20}\text{m}^{-3}]) + \ln(T_e[1\text{ keV}]) & \text{for } T_e \gtrsim 40Z^2\text{ eV} \end{array}\right. . \tag{5.10}$$

Finally, for ion-ion collisions, both electron and ions satisfy $v_{ts} \gtrsim g$, giving the Debye length $\lambda_D = (e^2 n_e/\epsilon_0 T_e + \sum_{i''} Z_{i''}^2 e^2 n_{i''}/\epsilon_0 T_{i''})^{-1/2}$. Then,

$$\ln\Lambda_{ii'} =$$

$$\left\{\begin{array}{ll} 18 - \ln\left[\frac{A_i T_{i'} + A_{i'} T_i}{Z_i Z_{i'}(A_i + A_{i'})}\left(\frac{n_e}{T_e} + \sum_{i''}\frac{Z_{i''}^2 n_{i''}}{T_{i''}}\right)^{1/2}\right] & \text{for } \frac{T_i}{A_i} + \frac{T_{i'}}{A_{i'}} \lesssim 70 Z_i^2 Z_{i'}^2\text{ keV} \\ 20 - \ln\left[\frac{\sqrt{A_i A_{i'}(A_i T_{i'} + A_{i'} T_i)}}{A_i + A_{i'}}\left(\frac{n_e}{T_e} + \sum_{i''}\frac{Z_{i''}^2 n_{i''}}{T_{i''}}\right)^{1/2}\right] & \text{for } \frac{T_i}{A_i} + \frac{T_{i'}}{A_{i'}} \gtrsim 70 Z_i^2 Z_{i'}^2\text{ keV} \end{array}\right. , \tag{5.11}$$

where the densities n_s must be given in 10^{20} m^{-3}, the temperatures T_s in keV, and $A_s = m_s/m_p$ is the mass of species s divided by the mass of the proton.

6. Landau form of the Fokker-Planck collision operator

The Fokker-Planck collision operator can be written in a form that makes the symmetry between species s and s' explicit: the Landau form. This form is very convenient to show that the Fokker-Planck collision operator conserves particles, momentum and energy and that it has an H-theorem.

To obtain the Landau form, we first need to prove two useful relationships,

$$\nabla_g \nabla_g g = \frac{g^2\mathbf{I} - \mathbf{g}\mathbf{g}}{g^3} \tag{6.1}$$

and

$$\nabla_g^2 \nabla_g g = -\frac{2\mathbf{g}}{g^3}. \tag{6.2}$$

We proceed to prove them. In an orthonormal basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$, $\mathbf{g} = g_x\hat{\mathbf{x}} + g_y\hat{\mathbf{y}} + g_z\hat{\mathbf{z}}$ and the gradient ∇_g is defined to be

$$\nabla_g f = \hat{\mathbf{x}}\frac{\partial f}{\partial g_x} + \hat{\mathbf{y}}\frac{\partial f}{\partial g_y} + \hat{\mathbf{z}}\frac{\partial f}{\partial g_z}. \tag{6.3}$$

Then, for $g = |\mathbf{g}| = \sqrt{g_x^2 + g_y^2 + g_z^2}$, we obtain

$$\nabla_g g = \frac{g_x\hat{\mathbf{x}} + g_y\hat{\mathbf{y}} + g_z\hat{\mathbf{z}}}{\sqrt{g_x^2 + g_y^2 + g_z^2}} = \frac{\mathbf{g}}{g}. \tag{6.4}$$

Taking a second gradient of this expression, we find

$$\nabla_g \nabla_g g = \nabla_g\left(\frac{\mathbf{g}}{g}\right) = \frac{\nabla_g \mathbf{g}}{g} + \nabla_g\left(\frac{1}{g}\right)\mathbf{g} = \frac{\mathbf{I}}{g} - \frac{(\nabla_g g)\mathbf{g}}{g^2} = \frac{\mathbf{I}}{g} - \frac{\mathbf{g}\mathbf{g}}{g^3}. \tag{6.5}$$

This expression proves (6.1). Taking the trace of (6.1), we obtain

$$\nabla_g^2 g = \mathrm{Trace}(\nabla_g\nabla_g g) = \mathrm{Trace}\left(\frac{g^2\mathbf{I} - \mathbf{g}\mathbf{g}}{g^3}\right) = \frac{3}{g} - \frac{\mathbf{g}\cdot\mathbf{g}}{g^3} = \frac{2}{g}. \tag{6.6}$$

Taking a gradient of this expression, we find

$$\nabla_g \nabla_g^2 g = \nabla_g\left(\frac{2}{g}\right) = -\frac{2\nabla_g g}{g^2} = -\frac{2\mathbf{g}}{g^3}. \tag{6.7}$$

This expression proves (6.2).

From equation (6.2), we deduce the useful relationships

$$\nabla_v \cdot (\nabla_g \nabla_g g) = -\nabla_{v'} \cdot (\nabla_g \nabla_g g) = -\frac{2\mathbf{g}}{g^3}. \tag{6.8}$$

We proceed to prove them. Using Einstein's repeated index notation and the chain rule, we obtain

$$[\nabla_v \cdot (\nabla_g \nabla_g g)]_i = \frac{\partial}{\partial v_j}\left(\frac{\partial^2 g}{\partial g_j \partial g_i}\right) = \frac{\partial g_k}{\partial v_j}\frac{\partial^3 g}{\partial g_k \partial g_j \partial g_i} \tag{6.9}$$

and

$$[\nabla_{v'} \cdot (\nabla_g \nabla_g g)]_i = \frac{\partial}{\partial v'_j}\left(\frac{\partial^2 g}{\partial g_j \partial g_i}\right) = \frac{\partial g_k}{\partial v'_j}\frac{\partial^3 g}{\partial g_k \partial g_j \partial g_i}. \tag{6.10}$$

Since $\mathbf{g} = \mathbf{v} - \mathbf{v}'$, we find that $\partial g_k/\partial v_j = \partial v_k/\partial v_j = \delta_{jk}$ and $\partial g_k/\partial v'_j = -\partial v'_k/\partial v'_j = -\delta_{jk}$, where δ_{jk} is Kronecker's delta. Using these expressions in equations (6.9) and (6.10), we obtain

$$[\nabla_v \cdot (\nabla_g \nabla_g g)]_i = \frac{\partial^3 g}{\partial g_j \partial g_j \partial g_i} = (\nabla_g^2 \nabla_g g)_i \tag{6.11}$$

and

$$[\nabla_{v'} \cdot (\nabla_g \nabla_g g)]_i = -\frac{\partial^3 g}{\partial g_j \partial g_j \partial g_i} = -(\nabla_g^2 \nabla_g g)_i. \tag{6.12}$$

These expressions and (6.2) give (6.8).

Using (6.1) and (6.8), we can rewrite (4.18) as

$$C_{ss'}[f_s, f_{s'}] = \frac{\gamma_{ss'}}{m_s} \nabla_v \cdot \left\{ \int f_{s'}(\mathbf{v}') \left[\frac{1}{m_s} \nabla_v \cdot (\nabla_g \nabla_g g f_s(\mathbf{v})) - \frac{1}{m_s} \nabla_v \cdot (\nabla_g \nabla_g g) f_s(\mathbf{v}) \right. \right.$$
$$\left. \left. + \frac{1}{m_{s'}} \nabla_{v'} \cdot (\nabla_g \nabla_g g) f_s(\mathbf{v}) \right] \mathrm{d}^3 v' \right\}, \tag{6.13}$$

Here we can simplify

$$\nabla_v \cdot (\nabla_g \nabla_g g f_s(\mathbf{v})) - \nabla_v \cdot (\nabla_g \nabla_g g) f_s(\mathbf{v}) = \nabla_g \nabla_g g \cdot \nabla_v f_s(\mathbf{v}). \tag{6.14}$$

We can also integrate by parts to find

$$\int f_{s'}(\mathbf{v}') \nabla_{v'} \cdot (\nabla_g \nabla_g g) \, \mathrm{d}^3 v' = -\int \nabla_g \nabla_g g \cdot \nabla_{v'} f_{s'}(\mathbf{v}') \, \mathrm{d}^3 v'. \tag{6.15}$$

Using (6.14) and (6.15) in (6.13), we obtain the Landau form of the Fokker-Planck collision operator

$$C_{ss'}[f_s, f_{s'}] = \frac{\gamma_{ss'}}{m_s} \nabla_v \cdot \left\{ \int \nabla_g \nabla_g g \cdot \left[\frac{f_{s'}(\mathbf{v}')}{m_s} \nabla_v f_s(\mathbf{v}) - \frac{f_s(\mathbf{v})}{m_{s'}} \nabla_{v'} f_{s'}(\mathbf{v}') \right] \mathrm{d}^3 v' \right\}. \tag{6.16}$$

7. Conservation properties of the Fokker-Planck collision operator

The Fokker-Planck collision operator conserves particles, momentum and energy. We proceed to show these properties using the Landau form in (6.16).

7.1. *Conservation of particles*

The Landau collision operator satisfies

$$\int C_{ss'}[f_s, f_{s'}](\mathbf{r}, \mathbf{v}, t) \, \mathrm{d}^3 v = 0. \tag{7.1}$$

This property is a consequence of the fact that the collision operator in (6.16) is a divergence and the fact that the distribution functions f_s and $f_{s'}$ vanish for large velocities.

7.2. *Conservation of momentum*

The friction force on species s due to collision with s' is given by

$$\mathbf{F}_{ss'} = \int m_s \mathbf{v} \, C_{ss'}[f_s, f_{s'}](\mathbf{r}, \mathbf{v}, t) \, \mathrm{d}^3 v. \tag{7.2}$$

The Fokker-Planck collision operator conserves momentum, that is, the force of species s' on species s is equal and opposite to the force exerted by species s on s',

$$\mathbf{F}_{ss'} + \mathbf{F}_{s's} = 0. \tag{7.3}$$

For collisions between particle of the same species s, this property implies that

$$\mathbf{F}_{ss} = 0, \tag{7.4}$$

that is, there is no net collisional force of species s on itself.

We proceed to prove (7.3). We use the Landau form in (7.2) that can be written as

$$C_{ss'}[f_s, f_{s'}] = \frac{\gamma_{ss'}}{m_s} \nabla_v \cdot \mathbf{\Gamma}_{ss'}, \tag{7.5}$$

where

$$\mathbf{\Gamma}_{ss'} = \int \nabla_g \nabla_g g \cdot \left[\frac{f_{s'}(\mathbf{v}')}{m_s} \nabla_v f_s(\mathbf{v}) - \frac{f_s(\mathbf{v})}{m_{s'}} \nabla_{v'} f_{s'}(\mathbf{v}') \right] \mathrm{d}^3 v'. \tag{7.6}$$

Substituting (7.5) into (7.2) and integrating by parts, we obtain

$$\begin{aligned} \mathbf{F}_{ss'} &= \gamma_{ss'} \int \mathbf{v} (\nabla_v \cdot \mathbf{\Gamma}_{ss'})\, \mathrm{d}^3 v = -\gamma_{ss'} \int \mathbf{\Gamma}_{ss'} \cdot \nabla_v \mathbf{v}\, \mathrm{d}^3 v = -\gamma_{ss'} \int \mathbf{\Gamma}_{ss'}\, \mathrm{d}^3 v \\ &= -\gamma_{ss'} \int \mathrm{d}^3 v \int \mathrm{d}^3 v'\, \nabla_g \nabla_g g \cdot \left[\frac{f_{s'}(\mathbf{v}')}{m_s} \nabla_v f_s(\mathbf{v}) - \frac{f_s(\mathbf{v})}{m_{s'}} \nabla_{v'} f_{s'}(\mathbf{v}') \right]. \end{aligned} \tag{7.7}$$

Exchanging s and s' and recalling that $\gamma_{ss'} = \gamma_{s's}$, we find

$$\mathbf{F}_{s's} = -\gamma_{ss'} \int \mathrm{d}^3 v \int \mathrm{d}^3 v'\, \nabla_g \nabla_g g \cdot \left[\frac{f_s(\mathbf{v}')}{m_{s'}} \nabla_v f_{s'}(\mathbf{v}) - \frac{f_{s'}(\mathbf{v})}{m_s} \nabla_{v'} f_s(\mathbf{v}') \right]. \tag{7.8}$$

In this equation, the integration variables $\mathbf{v}$ and $\mathbf{v}'$ are dummy variables. We can exchange their names, finding

$$\mathbf{F}_{s's} = -\gamma_{ss'} \int \mathrm{d}^3 v \int \mathrm{d}^3 v'\, \nabla_g \nabla_g g \cdot \left[\frac{f_s(\mathbf{v})}{m_{s'}} \nabla_{v'} f_{s'}(\mathbf{v}') - \frac{f_{s'}(\mathbf{v}')}{m_s} \nabla_v f_s(\mathbf{v}) \right]. \tag{7.9}$$

Summing (7.7) and (7.9), we obtain the momentum conservation equation in (7.3).

7.3. *Conservation of energy*

The energy gained or lost per unit time by species s due to collision with s' is

$$W_{ss'} = \int \frac{1}{2} m_s v^2\, C_{ss'}[f_s, f_{s'}](\mathbf{r}, \mathbf{v}, t)\, \mathrm{d}^3 \mathbf{v}. \tag{7.10}$$

The Fokker-Planck collision operator conserves energy,

$$W_{ss'} + W_{s's} = 0. \tag{7.11}$$

For collisions between particle of the same species s, this property implies that

$$W_{ss} = 0, \tag{7.12}$$

that is, there is no net collisional energy gain or loss due to collision of species s with itself.

We proceed to prove (7.11). Substituting (7.5) into (7.10) and integrating by parts, we obtain

$$W_{ss'} = \frac{\gamma_{ss'}}{2} \int v^2 (\nabla_v \cdot \mathbf{\Gamma}_{ss'})\, \mathrm{d}^3 v = -\frac{\gamma_{ss'}}{2} \int \mathbf{\Gamma}_{ss'} \cdot \nabla_v v^2\, \mathrm{d}^3 v = -\gamma_{ss'} \int \mathbf{v} \cdot \mathbf{\Gamma}_{ss'}\, \mathrm{d}^3 v$$
$$= -\gamma_{ss'} \int \mathrm{d}^3 v \int \mathrm{d}^3 v'\, \mathbf{v} \cdot \nabla_g \nabla_g g \cdot \left[\frac{f_{s'}(\mathbf{v}')}{m_s} \nabla_v f_s(\mathbf{v}) - \frac{f_s(\mathbf{v})}{m_{s'}} \nabla_{v'} f_{s'}(\mathbf{v}') \right]. \quad (7.13)$$

Exchanging s and s' and the dummy integration variables $\mathbf{v}$ and $\mathbf{v}'$, we find

$$W_{s's} = -\gamma_{ss'} \int \mathrm{d}^3 v \int \mathrm{d}^3 v'\, \mathbf{v}' \cdot \nabla_g \nabla_g g \cdot \left[\frac{f_s(\mathbf{v})}{m_{s'}} \nabla_{v'} f_{s'}(\mathbf{v}') - \frac{f_{s'}(\mathbf{v}')}{m_s} \nabla_v f_s(\mathbf{v}) \right]. \quad (7.14)$$

Summing (7.13) and (7.14), we obtain

$$W_{ss'} + W_{s's} = -\gamma_{ss'} \int \mathrm{d}^3 v \int \mathrm{d}^3 v'\, \mathbf{g} \cdot \nabla_g \nabla_g g \cdot \left[\frac{f_{s'}(\mathbf{v}')}{m_s} \nabla_v f_s(\mathbf{v}) - \frac{f_s(\mathbf{v})}{m_{s'}} \nabla_{v'} f_{s'}(\mathbf{v}') \right]. \quad (7.15)$$

From equation (6.1), we obtain

$$\mathbf{g} \cdot \nabla_g \nabla_g g = \mathbf{g} \cdot \frac{g^2 \mathbf{I} - \mathbf{g}\mathbf{g}}{g^3} = \frac{\mathbf{g}}{g} - \frac{\mathbf{g} \cdot \mathbf{g}}{g^3} \mathbf{g} = 0, \quad (7.16)$$

proving the energy conservation equation in (7.11).

8. H-theorem for the Fokker-Planck collision operator

The Fokker-Planck collision operator satisfies its own H-theorem, that is, it always produces entropy. The species s entropy production due to collisions with species s' is

$$\dot{\sigma}_{ss'} = - \int \ln f_s(\mathbf{r}, \mathbf{v}, t)\, C_{ss'}[f_s, f_{s'}](\mathbf{r}, \mathbf{v}, t)\, \mathrm{d}^3 v. \quad (8.1)$$

The H-theorem for the Fokker-Planck collision operator states that

$$\dot{\sigma}_{ss'} + \dot{\sigma}_{s's} \geqslant 0, \quad (8.2)$$

and $\dot{\sigma}_{ss'} + \dot{\sigma}_{s's}$ is equal to zero only when both f_s and $f_{s'}$ are Maxwellians with the same average velocity $\mathbf{u}$ and temperature T,

$$f_s(\mathbf{r}, \mathbf{v}, t) = n_s(\mathbf{r}, t) \left(\frac{m_s}{2\pi T(\mathbf{r}, t)} \right)^{3/2} \exp \left(-\frac{m_s |\mathbf{v} - \mathbf{u}(\mathbf{r}, t)|^2}{2T(\mathbf{r}, t)} \right),$$
$$f_{s'}(\mathbf{r}, \mathbf{v}, t) = n_{s'}(\mathbf{r}, t) \left(\frac{m_{s'}}{2\pi T(\mathbf{r}, t)} \right)^{3/2} \exp \left(-\frac{m_s |\mathbf{v} - \mathbf{u}(\mathbf{r}, t)|^2}{2T(\mathbf{r}, t)} \right). \quad (8.3)$$

Note that densities n_s and $n_{s'}$ can be different, and that densities, average flow and temperature can be general functions of position $\mathbf{r}$ and time t. For collisions between particles of the same species s, the H-theorem simplifies to

$$\dot{\sigma}_{ss} \geqslant 0, \quad (8.4)$$

and $\dot{\sigma}_{ss}$ is equal to zero only when f_s is a Maxwellian,

$$f_s(\mathbf{r}, \mathbf{v}, t) = n_s(\mathbf{r}, t) \left(\frac{m_s}{2\pi T_s(\mathbf{r}, t)} \right)^{3/2} \exp \left(-\frac{m_s |\mathbf{v} - \mathbf{u}_s(\mathbf{r}, t)|^2}{2T_s(\mathbf{r}, t)} \right). \quad (8.5)$$

In this case, the temperature T_s and the average velocity $\mathbf{u}_s$ do not have to be equal to the temperature or average velocity of any of the other species.

We first prove (8.2), and we then show that $\dot{\sigma}_{ss'} + \dot{\sigma}_{s's} = 0$ only for Maxwellian distribution functions. Substituting (7.5) into (8.1) and integrating by parts, we obtain

$$\begin{aligned}\dot{\sigma}_{ss'} &= -\frac{\gamma_{ss'}}{m_s}\int \ln f_s(\nabla_v \cdot \boldsymbol{\Gamma}_{ss'})\,\mathrm{d}^3v = \frac{\gamma_{ss'}}{m_s}\int \boldsymbol{\Gamma}_{ss'} \cdot \nabla_v \ln f_s\,\mathrm{d}^3v \\ &= \frac{\gamma_{ss'}}{m_s}\int \mathrm{d}^3v \int \mathrm{d}^3v'\,\nabla_v \ln f_s(\mathbf{v}) \cdot \nabla_g\nabla_g g \cdot \left[\frac{f_{s'}(\mathbf{v}')}{m_s}\nabla_v f_s(\mathbf{v}) - \frac{f_s(\mathbf{v})}{m_{s'}}\nabla_{v'} f_{s'}(\mathbf{v}')\right].\end{aligned} \tag{8.6}$$

Exchanging s and s' and the dummy integration variables $\mathbf{v}$ and $\mathbf{v}'$, we find

$$\dot{\sigma}_{s's} = \frac{\gamma_{ss'}}{m_{s'}}\int \mathrm{d}^3v \int \mathrm{d}^3v'\,\nabla_{v'} \ln f_{s'}(\mathbf{v}') \cdot \nabla_g\nabla_g g \cdot \left[\frac{f_s(\mathbf{v})}{m_{s'}}\nabla_{v'} f_{s'}(\mathbf{v}') - \frac{f_{s'}(\mathbf{v}')}{m_s}\nabla_v f_s(\mathbf{v})\right]. \tag{8.7}$$

Summing (8.6) and (8.7), we obtain

$$\dot{\sigma}_{ss'} + \dot{\sigma}_{s's} = \gamma_{ss'}\int \mathrm{d}^3v \int \mathrm{d}^3v'\,f_s(\mathbf{v})f_{s'}(\mathbf{v}')\,\mathbf{a}(\mathbf{v},\mathbf{v}') \cdot \nabla_g\nabla_g g \cdot \mathbf{a}(\mathbf{v},\mathbf{v}'), \tag{8.8}$$

where

$$\mathbf{a}(\mathbf{v},\mathbf{v}') = \frac{1}{m_s}\nabla_v \ln f_s(\mathbf{v}) - \frac{1}{m_{s'}}\nabla_{v'} \ln f_{s'}(\mathbf{v}'). \tag{8.9}$$

From equation (6.1), we obtain

$$\mathbf{a} \cdot \nabla_g\nabla_g g \cdot \mathbf{a} = \frac{1}{g}\left|\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{g}}{g^2}\mathbf{g}\right|^2 \geqslant 0, \tag{8.10}$$

proving that the entropy production in (8.8) is always positive.

We proceed to consider the case in which the entropy production in (8.8) vanishes. Given that the Fokker-Planck collision operator in (6.16) contains second derivatives with respect to velocity of the distribution functions f_s and $f_{s'}$, we assume that the distribution functions are continuous and that their derivatives with respect to the velocities are also continuous. According to (8.10), $\dot{\sigma}_{ss'} + \dot{\sigma}_{s's} = 0$ only if $\mathbf{a} = (\mathbf{a}\cdot\mathbf{g}/g^2)\mathbf{g}$ for all $\mathbf{v}$ and $\mathbf{v}'$ (recall that the derivatives with respect to velocity of f_s and $f_{s'}$ are continuous, and hence $\mathbf{a}(\mathbf{v},\mathbf{v}')$ is continuous). Then, for $\dot{\sigma}_{ss'} + \dot{\sigma}_{s's} = 0$, the vector $\mathbf{a}$, defined in (8.9) must be parallel to $\mathbf{g}$, giving

$$\frac{1}{m_s}\nabla_v \ln f_s(\mathbf{v}) - \frac{1}{m_{s'}}\nabla_{v'} \ln f_{s'}(\mathbf{v}') = K(\mathbf{v},\mathbf{v}')\mathbf{g}, \tag{8.11}$$

where the function $K(\mathbf{v},\mathbf{v}')$ is unknown. Note that, to ensure continuity of the right side of equation (8.11) for $\mathbf{v}' \to \mathbf{v}$, the function $K(\mathbf{v},\mathbf{v}')$ must satisfy $|K(\mathbf{v},\mathbf{v}')| < D/g$ for any constant D when $\mathbf{v}' \to \mathbf{v}$. Indeed, if $|K(\mathbf{v},\mathbf{v}')| \geqslant D/g$ for a particular value of D, the right side of equation (8.11) would diverge for $\mathbf{v}' \to \mathbf{v}$ or it would be discontinuous at $\mathbf{v}' = \mathbf{v}$ because it would depend on the direction of $\mathbf{g}/g$. Thus, for $\mathbf{v}' = \mathbf{v}$, we obtain that

$$\frac{1}{m_s}\nabla_v \ln f_s(\mathbf{v}) = \frac{1}{m_{s'}}\nabla_v \ln f_{s'}(\mathbf{v}). \tag{8.12}$$

Using this result in (8.11), we find

$$\nabla_v \ln f_s(\mathbf{v}) - \nabla_{v'} \ln f_s(\mathbf{v}') = m_s K(\mathbf{v},\mathbf{v}')\mathbf{g}. \tag{8.13}$$

Taking $\mathbf{v}' = 0$ in (8.13), we find

$$\nabla_v \ln f_s(\mathbf{v}) = m_s K(\mathbf{v}, 0)\mathbf{v} + \nabla_v \ln f_s(0). \tag{8.14}$$

Substituting equation (8.14) into (8.13), we obtain

$$K(\mathbf{v}, 0)\mathbf{v} - K(\mathbf{v}', 0)\mathbf{v}' = K(\mathbf{v}, \mathbf{v}')(\mathbf{v} - \mathbf{v}'). \tag{8.15}$$

When $\mathbf{v}$ and $\mathbf{v}'$ are linearly independent, this equation implies that $K(\mathbf{v}, 0) = K(\mathbf{v}, \mathbf{v}')$ and that $K(\mathbf{v}', 0) = K(\mathbf{v}, \mathbf{v}')$, leading to $K(\mathbf{v}, 0) = K(\mathbf{v}', 0)$. Thus, $K(\mathbf{v}, 0)$ and hence $K(\mathbf{v}, \mathbf{v}')$ are constants. This argument only breaks when $\mathbf{v}$ and $\mathbf{v}'$ are colinear, but due to continuity, $K(\mathbf{v}, \mathbf{v}')$ must be a constant in this case as well. The constant K must be negative because $\nabla_v \ln f_s$ in (8.14) must be negative for large $\mathbf{v}$ to ensure that distribution functions vanish at $|\mathbf{v}| \to \infty$. We name the constant $-1/T$,

$$-\frac{1}{T} = K(\mathbf{v}, 0) = K(\mathbf{v}, \mathbf{v}'). \tag{8.16}$$

Using this result and naming the value of $\nabla_v \ln f_s$ at $\mathbf{v} = 0$

$$\frac{m_s \mathbf{u}}{T} = \nabla_v \ln f_s(0), \tag{8.17}$$

equation (8.14) becomes

$$\nabla_v \ln f_s(\mathbf{v}) = -\frac{m_s(\mathbf{v} - \mathbf{u})}{T}. \tag{8.18}$$

Using (8.12), we then obtain

$$\nabla_v \ln f_{s'}(\mathbf{v}) = -\frac{m_{s'}(\mathbf{v} - \mathbf{u})}{T}. \tag{8.19}$$

Integrating equations (8.18) and (8.19) in velocity space gives (8.3).

9. Solutions to the Fokker-Planck collision operator

The H-theorem of the Fokker-Planck collision operator implies that the only solutions to the system of equations

$$C_{ss'}[f_s, f_{s'}] = 0, \tag{9.1}$$
$$C_{s's}[f_{s'}, f_s] = 0, \tag{9.2}$$

are the Maxwellians in (8.3). To prove this, multiply equation (9.1) by $-\ln f_s$ and integrate over velocity space, and multiply equation (9.2) by $-\ln f_{s'}$ and integrate over velocity space. These equations give $\dot{\sigma}_{ss'} = 0 = \dot{\sigma}_{s's}$. Then, according to the H-theorem, f_s and $f_{s'}$ must be the Maxwellians in (8.3). Similarly, we can show that the only solution to

$$C_{ss}[f_s, f_s] = 0 \tag{9.3}$$

is the Maxwellian in (8.5).

The H-theorem is even more powerful. It implies that the steady-state solution of a closed system can only be Maxwellians with the same temperature and velocity for all species. The kinetic equation is

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{Z_s e}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_s = \sum_{s'} C_{ss'}[f_s, f_{s'}]. \tag{9.4}$$

Multiplying this equation by $-\ln f_s$, we obtain

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{Z_s e}{m_s}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_v\right](-f_s \ln f_s + f_s) = -\sum_{s'} \ln f_s\, C_{ss'}[f_s, f_{s'}]. \quad (9.5)$$

Using phase space volume conservation, $\nabla\cdot\mathbf{v} + \nabla_v\cdot[(Z_s e/m_s)(\mathbf{E}+\mathbf{v}\times\mathbf{B})] = 0$, we can rewrite the equation as

$$\frac{\partial}{\partial t}(-f_s \ln f_s + f_s) + \nabla\cdot[\mathbf{v}(-f_s\ln f_s + f_s)] + \nabla_v\cdot\left[\frac{Z_s e}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})(-f_s\ln f_s + f_s)\right] = -\sum_{s'}\ln f_s\, C_{ss'}[f_s, f_{s'}]. \quad (9.6)$$

Integrating over velocity space and the volume V of the plasma, we find

$$\frac{\partial}{\partial t}\int_V \mathrm{d}^3r \int \mathrm{d}^3v\,(-f_s\ln f_s + f_s) + \cancelto{\text{closed system}}{\int_{\partial V}\mathrm{d}^2S\int \mathrm{d}^3v\,\mathbf{v}\cdot\hat{\mathbf{n}}(-f_s\ln f_s + f_s)} = \sum_{s'}\int_V \mathrm{d}^3r\,\dot\sigma_{ss'}, \quad (9.7)$$

where ∂V is the boundary surface of the volume V, and $\hat{\mathbf{n}}$ is the normal to that surface. Summing over species, we finally obtain

$$\sum_s \frac{\partial}{\partial t}\int_V \mathrm{d}^3r\int \mathrm{d}^3v\,(-f_s\ln f_s + f_s) = \sum_{s,s'}\int_V \mathrm{d}^3r\,\dot\sigma_{ss'}. \quad (9.8)$$

Thus, in steady state,

$$\sum_{s,s'}\int_V \mathrm{d}^3r\,\dot\sigma_{ss'} = 0. \quad (9.9)$$

Since $\dot\sigma_{ss'} + \dot\sigma_{s's} \geqslant 0$, the sum of all these entropy productions can only vanish if each one is independently zero. This implies that the only allowed solution are Maxwellians with the same temperature and average velocity.

In most cases, the system is not closed, and it is driven by sources and sinks that prevent the distribution functions from becoming exact Maxwellians.

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