Braginskii fluid equations

Felix I. Parra

Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Oxford OX1 3NP, UK

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1. Introduction

In these notes, we present the Braginskii fluid equations for electrons and ions. We will consider a plasma composed of one ion species with charge e and mass m_i, and electrons with charge $-e$ and mass m_e. Ions and electrons are assumed to have comparable temperatures,

$$T_i \sim T_e, \tag{1.1}$$

and average flows of the order of the ion thermal speed (high flow ordering),

$$\mathbf{u}_i \sim \mathbf{u}_e \sim v_{ti} \sim \sqrt{\frac{m_e}{m_i}} v_{te} \ll v_{te}. \tag{1.2}$$

To derive the fluid equations we will expand in three different small parameters,

$$\frac{\rho_i}{L} \ll 1, \quad \frac{\lambda_{ii}}{L} \sim \frac{\lambda_{ee}}{L} \sim \frac{\lambda_{ei}}{L} \ll 1, \quad \sqrt{\frac{m_e}{m_i}} \ll 1, \tag{1.3}$$

where L is the characteristic size of the system,

$$\rho_i \sim \frac{v_{ti}}{\Omega_i} \sim \frac{\sqrt{m_i T_i}}{eB} \tag{1.4}$$

is the ion thermal gyroradius, and

$$\lambda_{ee} \sim \frac{v_{te}}{\nu_{ee}} \sim \frac{(4\pi\epsilon_0)^2 T_e^2}{e^4 n_e \ln \Lambda_{ee}}, \quad \lambda_{ei} \sim \frac{v_{te}}{\nu_{ei}} \sim \frac{(4\pi\epsilon_0)^2 T_e^2}{e^4 n_i \ln \Lambda_{ei}}, \quad \lambda_{ii} \sim \frac{v_{ti}}{\nu_{ii}} \sim \frac{(4\pi\epsilon_0)^2 T_i^2}{e^4 n_i \ln \Lambda_{ii}} \tag{1.5}$$

are the mean free paths for electron-electron, electron-ion and ion-ion collisions. To simplify the derivation, we will assume the following relative ordering

$$\frac{\rho_i}{L} \ll \frac{\lambda_{ii}}{L} \sim \sqrt{\frac{m_e}{m_i}} \ll 1. \tag{1.6}$$

For ions, it is common to first expand assuming

$$\frac{\rho_i}{L} \sim \frac{\lambda_{ii}}{L} \sim \sqrt{\frac{m_e}{m_i}} \ll 1, \tag{1.7}$$

and then perform the subsidiary expansion in

$$\frac{\rho_i}{\lambda_{ii}} \sim \frac{\nu_{ii}}{\Omega_i} \ll 1 \sim \frac{L}{\lambda_{ii}} \sqrt{\frac{m_e}{m_i}}. \tag{1.8}$$

For electrons, which we will examine in detail, we will start expanding in

$$\frac{\rho_e}{L} \sim \frac{\lambda_{ee}}{L} \sim \frac{\lambda_{ei}}{L} \sim \sqrt{\frac{m_e}{m_i}} \ll 1, \tag{1.9}$$

and then we will perform the subsidiary expansion in

$$\frac{\rho_e}{\lambda_{ee}} \sim \frac{\nu_{ee}}{\Omega_e} \ll 1 \sim \frac{L}{\lambda_{ee}}\sqrt{\frac{m_e}{m_i}}. \tag{1.10}$$

Importantly, the subsidiary expansions suggested above are not the only ones that give Braginskii equations. It is possible to obtain them by first expanding in $\rho_i/L \ll 1$ (assuming $\lambda_{ii}/L \sim 1$) to get drift kinetics (Parra 2017), and then performing a subsidiary expansion in $\lambda_{ii}/L \ll 1$.

2. Fluid equations

The Fokker-Planck kinetic equation for species s is

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s + \frac{Z_s e}{m_s}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_v f_s = \sum_{s'} C_{ss'}[f_s, f_{s'}]. \tag{2.1}$$

Before trying to solve this equation, we start by taking moments of it to obtain fluid equations, as in Kinetic Theory of neutral gases (Dellar 2015). We first calculate the general conservation equations, and we then particularize them for a plasma formed by singly charged ions and electrons.

Due to conservation of phase-space volume, equation (2.1) can be written as

$$\frac{\partial f_s}{\partial t} + \nabla\cdot(\mathbf{v} f_s) + \nabla_v\cdot\left[\frac{Z_s e}{m_s}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) f_s\right] = \sum_{s'} C_{ss'}[f_s, f_{s'}]. \tag{2.2}$$

If we multiply this equation by a function $X(\mathbf{v})$ and we integrate over velocity, we obtain

$$\frac{\partial}{\partial t}\left(\int X f_s \,\mathrm{d}^3v\right) + \nabla\cdot\left(\int X f_s \mathbf{v}\,\mathrm{d}^3v\right) = \frac{Z_s e}{m_s}\int (\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_v X\, f_s\,\mathrm{d}^3v + \sum_{s'}\int X C_{ss'}[f_s, f_{s'}]\,\mathrm{d}^3v. \tag{2.3}$$

Three particular functions, $X = 1, m_s\mathbf{v}, m_s v^2/2$, are of interest because they give the conservation equations for particles, momentum and energy.

2.1. *Continuity equation*

Taking $X = 1$ and recalling that the Fokker-Planck collision operator conserves particles, equation (2.3) becomes

$$\boxed{\frac{\partial n_s}{\partial t} + \nabla\cdot(n_s\mathbf{u}_s) = 0,} \tag{2.4}$$

where

$$n_s = \int f_s\,\mathrm{d}^3v, \quad \mathbf{u}_s = \frac{1}{n_s}\int f_s\mathbf{v}\,\mathrm{d}^3v \tag{2.5}$$

are the density and average velocity.

2.2. *Momentum conservation equation*

Taking $X = m_s\mathbf{v}$, equation (2.3) becomes

$$\frac{\partial}{\partial t}(n_s m_s \mathbf{u}_s) + \nabla\cdot\left(\int f_s m_s \mathbf{v}\mathbf{v}\,\mathrm{d}^3v\right) = Z_s e n_s(\mathbf{E} + \mathbf{u}_s\times\mathbf{B}) + \sum_{s'\neq s}\mathbf{F}_{ss'}, \tag{2.6}$$

where

$$\mathbf{F}_{ss'} = \int m_s \mathbf{v}\, C_{ss'}[f_s, f_{s'}]\, \mathrm{d}^3v \tag{2.7}$$

is the friction force on species s due to collisions with species s'. The integral $\int f_s m_s \mathbf{v}\mathbf{v}\, \mathrm{d}^3v$ in (2.6) can be rewritten in a more familiar form by using the relative velocity $\mathbf{w} = \mathbf{v} - \mathbf{u}_s$. With this relative velocity, we find

$$\int f_s m_s \mathbf{v}\mathbf{v}\, \mathrm{d}^3v = n_s m_s \mathbf{u}_s \mathbf{u}_s + \int f_s m_s \mathbf{w}\mathbf{w}\, \mathrm{d}^3w. \tag{2.8}$$

Moreover, if we define the pressure p_s and the temperature T_s as

$$p_s = n_s T_s = \int f_s \frac{m_s w^2}{3}\, \mathrm{d}^3w, \tag{2.9}$$

equation (2.8) becomes

$$\int f_s m_s \mathbf{v}\mathbf{v}\, \mathrm{d}^3v = n_s m_s \mathbf{u}_s \mathbf{u}_s + p_s \mathbf{I} + \mathbf{\Pi}_s, \tag{2.10}$$

where $\mathbf{I}$ is the unit matrix and

$$\mathbf{\Pi}_s = \int f_s m_s \left(\mathbf{w}\mathbf{w} - \frac{w^2}{3}\mathbf{I} \right) \mathrm{d}^3w \tag{2.11}$$

is the traceless viscosity tensor. With the result in (2.10), equation (2.6) finally becomes

$$\boxed{\frac{\partial}{\partial t}(n_s m_s \mathbf{u}_s) + \nabla \cdot (n_s m_s \mathbf{u}_s \mathbf{u}_s) + \nabla p_s + \nabla \cdot \mathbf{\Pi}_s = Z_s e n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) + \sum_{s' \neq s} \mathbf{F}_{ss'}.} \tag{2.12}$$

Using (2.4), this momentum equation can also be written as

$$\boxed{n_s m_s \left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right) + \nabla p_s + \nabla \cdot \mathbf{\Pi}_s = Z_s e n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) + \sum_{s' \neq s} \mathbf{F}_{ss'}.} \tag{2.13}$$

2.3. *Energy conservation equation*

Taking $X = m_s v^2/2$, equation (2.3) becomes

$$\frac{\partial}{\partial t}\left(\int \frac{1}{2} m_s v^2\, f_s\, \mathrm{d}^3v \right) + \nabla \cdot \left(\int \frac{1}{2} m_s v^2\, f_s \mathbf{v}\, \mathrm{d}^3v \right) = Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s + \sum_{s' \neq s} W_{ss'}, \tag{2.14}$$

where

$$W_{ss'} = \int \frac{1}{2} m_s v^2\, C_{ss'}[f_s, f_{s'}]\, \mathrm{d}^3v \tag{2.15}$$

is the energy gained by species s due to collisions with species s'. The integrals $\int f_s m_s v^2/2\, \mathrm{d}^3v$ and $\int f_s m_s v^2 \mathbf{v}/2\, \mathrm{d}^3v$ in (2.14) can be rewritten in a more familiar form by using the relative velocity $\mathbf{w} = \mathbf{v} - \mathbf{u}_s$. With this relative velocity, we find

$$\int \frac{1}{2} m_s v^2\, f_s\, \mathrm{d}^3v = \int \frac{1}{2} m_s w^2\, f_s\, \mathrm{d}^3w + \frac{1}{2} n_s m_s u_s^2 = \frac{3}{2} n_s T_s + \frac{1}{2} n_s m_s u_s^2 \tag{2.16}$$

and

$$\int \frac{1}{2} m_s v^2 \, f_s \mathbf{v} \, \mathrm{d}^3 v = \int \frac{1}{2} m_s w^2 \, f_s \mathbf{w} \, \mathrm{d}^3 w + \int \frac{1}{2} m_s w^2 \, f_s \mathbf{u}_s \, \mathrm{d}^3 w$$
$$+ \int f_s m_s (\mathbf{w} \cdot \mathbf{u}_s) \mathbf{w} \, \mathrm{d}^3 v + \frac{1}{2} n_s m_s u_s^2 \mathbf{u}_s = \mathbf{q}_s + \left(\frac{5}{2} n_s T_s + \frac{1}{2} n_s m_s u_s^2 \right) \mathbf{u}_s + \mathbf{\Pi}_s \cdot \mathbf{u}_s, \tag{2.17}$$

where

$$\mathbf{q}_s = \int \frac{1}{2} m_s w^2 \, f_s \mathbf{w} \, \mathrm{d}^3 w \tag{2.18}$$

is the heat flux. With the results in (2.16) and (2.17), equation (2.14) finally becomes

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t} \left(\frac{3}{2} n_s T_s + \frac{1}{2} n_s m_s u_s^2 \right) + \nabla \cdot \left[\mathbf{q}_s + \left(\frac{5}{2} n_s T_s + \frac{1}{2} n_s m_s u_s^2 \right) \mathbf{u}_s + \mathbf{\Pi}_s \cdot \mathbf{u}_s \right] \\ &\qquad = Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s + \sum_{s' \neq s} W_{ss'}. \end{aligned}} \tag{2.19}$$

Using (2.4), this energy equation can also be written as

$$n_s \left(\frac{\partial}{\partial t} + \mathbf{u}_s \cdot \nabla \right) \left(\frac{3}{2} T_s + \frac{1}{2} m_s u_s^2 \right) + \nabla \cdot \left(\mathbf{q}_s + n_s T_s \mathbf{u}_s + \mathbf{\Pi}_s \cdot \mathbf{u}_s \right)$$
$$= Z_s e n_s \mathbf{E} \cdot \mathbf{u}_s + \sum_{s' \neq s} W_{ss'}. \tag{2.20}$$

Finally, taking [equation (2.20) $-$ (equation (2.13)) $\cdot\, \mathbf{u}_s$], we find the equation for the thermal energy,

$$\boxed{\frac{3}{2} n_s \left(\frac{\partial T_s}{\partial t} + \mathbf{u}_s \cdot \nabla T_s \right) + \nabla \cdot \mathbf{q}_s + n_s T_s \nabla \cdot \mathbf{u}_s + \mathbf{\Pi}_s : \nabla \mathbf{u}_s = \sum_{s' \neq s} \tilde{W}_{ss'},} \tag{2.21}$$

where

$$\tilde{W}_{ss'} = W_{ss'} - \mathbf{F}_{ss'} \cdot \mathbf{u}_s = \int \frac{1}{2} m_s w^2 \, C_{ss'}[f_s, f_{s'}] \, \mathrm{d}^3 w. \tag{2.22}$$

Note that equations (2.4), (2.13) and (2.21) are almost the same as the equations for a neutral gas. The main apparent difference are the electromagnetic forces and the collisional terms that transfer momentum and energy between species. There will also be differences in the heat fluxes and viscosities, as we will see shortly.

In the next two sections, we particularize these fluid equations for a plasma composed of one ion species with charge e and mass m_i, and electrons with charge $-e$ and mass m_e.

3. Electron equations

To derive the electron fluid equations, we use the ordering in (1.2) and (1.6) (recall that we will first expand assuming (1.9) and we will then perform the subsidiary expansion (1.10)). We need to order several terms with respect to our expansion parameters in (1.6).

• We assume that the electric field is in the high flow ordering, that is, the electric field perpendicular to the magnetic field can balance the magnetic force, whereas the parallel electric field will only be of the order of the pressure gradient,

$$\mathbf{E}_\perp \sim v_{ti}B \sim \frac{L}{\rho_i}\frac{T_e}{eL} \gg E_\| \sim \frac{T_e}{eL}. \tag{3.1}$$

• We assume that the time derivative is of the order of

$$\frac{\partial}{\partial t} \sim \mathbf{u}_s \cdot \nabla \sim \frac{v_{ti}}{L}. \tag{3.2}$$

• Both $\mathbf{u}_e$ and $\mathbf{u}_i$ are of the order of v_{ti}, but their difference need not be. We will allow the velocity difference along the magnetic field to be of the order of v_{ti}, but the difference in the direction perpendicular to the magnetic field must be small, as we will see when we discuss equation (4.4) below. In fact,

$$u_{i\|} - u_{e\|} \sim v_{ti} \gg \mathbf{u}_{i\perp} - \mathbf{u}_{e\perp} \sim \frac{\rho_i}{L} v_{ti}. \tag{3.3}$$

For electrons, we start expanding assuming the orderings in (1.9), and hence

$$\frac{\rho_i}{L} \sim \frac{\rho_e}{L}\sqrt{\frac{m_i}{m_e}} \sim 1. \tag{3.4}$$

Thus, under the assumptions in (1.9), $u_{i\|} - u_{e\|} \sim \mathbf{u}_{i\perp} - \mathbf{u}_{e\perp} \sim v_{ti}$. Once we perform the subsidiary expansion based on (1.10), we recover (3.3).

• The collisional terms are of the size that we estimated when we calculated them in the electron-ion collision notes, that is,

$$\mathbf{F}_{ei} = -\mathbf{F}_{ie} \sim n_e m_e \nu_{ei}(\mathbf{u}_i - \mathbf{u}_e) \sim n_e m_e \nu_{ei} v_{ti} \sim \sqrt{\frac{m_e}{m_i}}\frac{L}{\lambda_{ei}}\frac{p_e}{L} \sim \frac{p_e}{L} \tag{3.5}$$

and

$$W_{ei} = -W_{ie} \sim n_e \frac{3m_e}{m_i}\nu_{ei}(T_i - T_e) \sim n_e \frac{m_e}{m_i}\nu_{ei} T_e \sim \sqrt{\frac{m_e}{m_i}}\frac{L}{\lambda_{ei}}\frac{p_e v_{ti}}{L} \sim \frac{p_e v_{ti}}{L}. \tag{3.6}$$

• Finally from the definitions of the viscosity and the heat flux in (2.11) and (2.18), we find the upper bounds

$$\mathbf{\Pi}_s \lesssim p_s, \quad \mathbf{q}_s \lesssim p_s v_{ts}. \tag{3.7}$$

Equations (2.4), (2.13) and (2.21) for electrons are

$$\underbrace{\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e)}_{\sim \frac{n_e v_{ti}}{L}} = 0, \tag{3.8}$$

$$\underbrace{n_e m_e \left(\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e\right)}_{\sim \frac{n_e m_e v_{ti}^2}{L} \sim \frac{m_e}{m_i}\frac{p_e}{L} \ll \frac{p_e}{L}} = -\underbrace{\nabla p_e}_{\sim \frac{p_e}{L}} - \underbrace{\nabla \cdot \mathbf{\Pi}_e}_{\lesssim \frac{p_e}{L}} - \underbrace{e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B})}_{\sim e n_e v_{ti} B \sim \frac{L}{\rho_i}\frac{p_i}{L} \gg \frac{p_e}{L}} + \underbrace{\mathbf{F}_{ei}}_{\sim \frac{p_e}{L}}, \tag{3.9}$$

$$\underbrace{\frac{3}{2} n_e \left(\frac{\partial T_e}{\partial t} + \mathbf{u}_e \cdot \nabla T_e\right)}_{\sim \frac{p_e v_{ti}}{L}} = -\underbrace{\nabla \cdot \mathbf{q}_e}_{\lesssim \frac{p_e v_{te}}{L}} - \underbrace{n_e T_e \nabla \cdot \mathbf{u}_e}_{\sim \frac{p_e v_{ti}}{L}} - \underbrace{\mathbf{\Pi}_e : \nabla \mathbf{u}_e}_{\lesssim \frac{p_e v_{ti}}{L}} + \underbrace{\mathbf{F}_{ei} \cdot (\mathbf{u}_i - \mathbf{u}_e) - \tilde{W}_{ie}}_{\sim \frac{p_e v_{ti}}{L}}. \tag{3.10}$$

Note that we have rewritten the collisional term $\tilde{W}_{ei}$ as

$$\tilde{W}_{ei} = \mathbf{F}_{ei} \cdot (\mathbf{u}_i - \mathbf{u}_e) - \tilde{W}_{ie}. \tag{3.11}$$

To obtain this formula, we use the fact that collisions conserve energy, leading to

$$W_{ei} = -W_{ie} = -\tilde{W}_{ie} - \mathbf{F}_{ie} \cdot \mathbf{u}_i = -\tilde{W}_{ie} + \mathbf{F}_{ei} \cdot \mathbf{u}_i, \tag{3.12}$$

where we have employed the definition of $\tilde{W}_{ss'}$ in (2.22), and the fact that collisions conserve momentum, $\mathbf{F}_{ie} = -\mathbf{F}_{ei}$. Then,

$$\tilde{W}_{ei} = W_{ei} - \mathbf{F}_{ei} \cdot \mathbf{u}_e = -\tilde{W}_{ie} + \mathbf{F}_{ei} \cdot (\mathbf{u}_i - \mathbf{u}_e), \tag{3.13}$$

which is equation (3.11). The term $\mathbf{F}_{ei} \cdot (\mathbf{u}_i - \mathbf{u}_e)$ is the Joule heating.

The dominant term in the momentum equation (3.9) is the term due to the electromagnetic force in the perpendicular direction, $-en_e(\mathbf{E}_\perp + \mathbf{u}_e \times \mathbf{B})$. Thus, to lowest order, the electron momentum equation becomes $\mathbf{E}_\perp + \mathbf{u}_e \times \mathbf{B} \simeq 0$, and the perpendicular electron average velocity is the $\mathbf{E} \times \mathbf{B}$ drift,

$$\mathbf{u}_{e\perp} \simeq \mathbf{v}_E \equiv \frac{1}{B} \mathbf{E} \times \hat{\mathbf{b}} \sim v_{ti}. \tag{3.14}$$

Note that if we had ordered the perpendicular electric field to be smaller than in (3.1), the perpendicular electron velocity would have been smaller than the ion thermal speed. The ion momentum equation is very similar to the electron momentum equation, giving $\mathbf{u}_{i\perp} \simeq \mathbf{v}_E \sim v_{ti}$. Since both $\mathbf{u}_{e\perp}$ and $\mathbf{u}_{i\perp}$ are equal to $\mathbf{v}_E$, their difference is indeed small as we assumed in (3.3).

We need to determine $\mathbf{\Pi}_e$, $\mathbf{q}_e$, $\mathbf{F}_{ei}$ and $\tilde{W}_{ie}$. To do so, we need to calculate the electron and ion distribution functions. From the ion distribution function, we will only need to know n_i, $\mathbf{u}_i$ and T_i. For the electron distribution function, we expand first in the small parameter

$$\frac{\rho_e}{L} \sim \sqrt{\frac{m_e}{m_i}} \sim \frac{\lambda_{ee}}{L} \sim \frac{\lambda_{ei}}{L} \ll 1. \tag{3.15}$$

In this expansion parameter, the electron distribution function is

$$f_e = f_{e0} + \underbrace{f_{e1}}_{\sim \frac{\rho_e}{L} f_{e0}} + \dots \tag{3.16}$$

We proceed to determine the electron distribution function order by order. We then calculate the terms that we need for the electron fluid equations.

3.1. *Electron Fokker-Planck equation*

It is convenient to write the Fokker-Planck kinetic equation (2.1) using the relative velocity $\mathbf{w} = \mathbf{v} - \mathbf{u}_e$. Using the chain rule for partial differentiation, we find

$$\left.\frac{\partial f_e}{\partial t}\right|_{\mathbf{r},\mathbf{v}} = \left.\frac{\partial f_e}{\partial t}\right|_{\mathbf{r},\mathbf{w}} + \left.\frac{\partial \mathbf{w}}{\partial t}\right|_{\mathbf{r},\mathbf{v}} \cdot \nabla_w f_e|_{t,\mathbf{r}} = \left.\frac{\partial f_e}{\partial t}\right|_{\mathbf{r},\mathbf{w}} - \frac{\partial \mathbf{u}_e}{\partial t} \cdot \nabla_w f_e|_{t,\mathbf{r}}, \tag{3.17}$$

$$\nabla f_e|_{t,\mathbf{v}} = \nabla f_e|_{t,\mathbf{w}} + \nabla \mathbf{w}|_{t,\mathbf{v}} \cdot \nabla_w f_e|_{t,\mathbf{r}} = \nabla f_e|_{t,\mathbf{w}} - \nabla \mathbf{u}_e \cdot \nabla_w f_e|_{t,\mathbf{r}} \tag{3.18}$$

and

$$\nabla_v f_e|_{t,\mathbf{r}} = \nabla_v \mathbf{w}|_{t,\mathbf{r}} \cdot \nabla_w f_e|_{t,\mathbf{r}} = \nabla_w f_e|_{t,\mathbf{r}}. \tag{3.19}$$

With these results, the Fokker-Planck equation (2.1) becomes

$$\begin{aligned} -\Omega_e(\mathbf{w} \times \hat{\mathbf{b}}) \cdot \nabla_w f_e - C_{ee}[f_e, f_e] - C_{ei}[f_e, f_i] = -\frac{\partial f_e}{\partial t} - (\mathbf{w} + \mathbf{u}_e) \cdot \nabla f_e \\ + \left[\frac{e}{m_e}(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + \frac{\partial \mathbf{u}_e}{\partial t} + (\mathbf{w} + \mathbf{u}_e) \cdot \nabla \mathbf{u}_e \right] \cdot \nabla_w f_e, \end{aligned} \tag{3.20}$$

where $\Omega_e = eB/m_e$ is the electron gyrofrequency, and $\hat{\mathbf{b}} = \mathbf{B}/B$ is the unit vector in the direction of $\mathbf{B}$. Using (3.9) for $\partial \mathbf{u}_e/\partial t$, equation (3.20) can be rewritten as

$$-\underbrace{\Omega_e(\mathbf{w}\times\hat{\mathbf{b}})\cdot\nabla_w f_e}_{\sim\Omega_e f_e \sim \frac{L}{\rho_e}\frac{v_{te}}{L}f_e \gg \frac{v_{te}}{L}f_e} \underbrace{-C_{ee}[f_e,f_e] - C_{ei}[f_e,f_i]}_{\sim\nu_{ee}f_e\sim\frac{L}{\lambda_{ee}}\frac{v_{te}}{L}f_e\gg\frac{v_{te}}{L}f_e} = \underbrace{-\frac{\partial f_e}{\partial t} - \mathbf{u}_e\cdot\nabla f_e}_{\sim\frac{v_{ti}}{L}f_e\sim\sqrt{\frac{m_e}{m_i}}\frac{v_{te}}{L}f_e\ll\frac{v_{te}}{L}f_e} - \underbrace{\mathbf{w}\cdot\nabla f_e}_{\sim\frac{v_{te}}{L}f_e}$$
$$+\underbrace{\frac{\mathbf{F}_{ei} - \nabla p_e - \nabla\cdot\mathbf{\Pi}_e}{n_e m_e}\cdot\nabla_w f_e}_{\sim\frac{v_{te}}{L}f_e} + \underbrace{\mathbf{w}\cdot\nabla\mathbf{u}_e\cdot\nabla_w f_e}_{\sim\frac{v_{ti}}{L}f_e\sim\sqrt{\frac{m_e}{m_i}}\frac{v_{te}}{L}f_e\ll\frac{v_{te}}{L}f_e}. \tag{3.21}$$

Due to our choice of coordinate, $\mathbf{w} = \mathbf{v} - \mathbf{u}_e$, the electron distribution function has to satisfy the condition

$$\int f_e \mathbf{w}\,\mathrm{d}^3w = 0. \tag{3.22}$$

Thus, equation (3.21) must be solved in conjunction with condition (3.22).

3.2. *Zeroth order electron distribution function*

Substituting the expansion (3.16) into the Fokker-Planck equation (3.21) and the condition (3.22), we obtain the lowest order equation

$$-\Omega_e(\mathbf{w}\times\hat{\mathbf{b}})\cdot\nabla_w f_{e0} - C_{ee}[f_{e0},f_{e0}] - C_{ei}[f_{e0},f_i] = 0 \tag{3.23}$$

with the lowest order condition

$$\int f_{e0}\mathbf{w}\,\mathrm{d}^3w = 0. \tag{3.24}$$

Since we are expanding in $\sqrt{m_e/m_i} \ll 1$, we can use the approximation that we deduced for the electron-ion collision operator: $C_{ei}[f_{e0},f_i] \simeq \mathcal{L}_{ei}[f_{e0}]$, where $\mathcal{L}_{ei}[f_{e0}]$ is the Lorentz collision operator. Thus, to lowest order, the equation for f_{e0} is

$$-\Omega_e(\mathbf{w}\times\hat{\mathbf{b}})\cdot\nabla_w f_{e0} - C_{ee}[f_{e0},f_{e0}] - \mathcal{L}_{ei}[f_{e0}] = 0. \tag{3.25}$$

The solution to equation (3.25) is a Maxwellian. To show it, we use the H-theorem. Multiplying (3.25) by $-\ln f_{e0}$, we find

$$\nabla_w\cdot[\Omega_e(\mathbf{w}\times\hat{\mathbf{b}})(f_{e0}\ln f_{e0} - f_{e0})] = -\ln f_{e0}\,C_{ee}[f_{e0},f_{e0}] - \ln f_{e0}\,\mathcal{L}_{ei}[f_{e0}]. \tag{3.26}$$

Integrating over velocity space, we obtain

$$-\int \ln f_{e0}\,C_{ee}[f_{e0},f_{e0}]\,\mathrm{d}^3w - \int \ln f_{e0}\,\mathcal{L}_{ei}[f_{e0}]\,\mathrm{d}^3w = 0. \tag{3.27}$$

Both of these integrals are positive, and their sum is equal to zero only if both integrals vanish. The integral over C_{ee} only vanishes if f_{e0} is a Maxwellian, and the integral over $\mathcal{L}_{ei}$ only vanishes if f_{e0} is isotropic. Thus, f_{e0} is an isotropic Maxwellian

$$f_{e0} = f_{Me} \equiv n_e\left(\frac{m_e}{2\pi T_e}\right)^{3/2}\exp\left(-\frac{m_e w^2}{2T_e}\right). \tag{3.28}$$

Note that this function satisfies $(\mathbf{w}\times\hat{\mathbf{b}})\cdot\nabla_w f_{Me} = 0$, and it is hence the only possible solution to equation (3.25). It also satisfies condition (3.24).

We define the density n_e and the temperature T_e in f_{Me} such that

$$\int f_{Me}\,\mathrm{d}^3w = \int f_e\,\mathrm{d}^3w = n_e, \quad \int\frac{1}{3}m_e w^2 f_{Me}\,\mathrm{d}^3w = \int\frac{1}{3}m_e w^2 f_e\,\mathrm{d}^3w = n_eT_e. \tag{3.29}$$

Then, the higher order corrections must satisfy

$$\int f_{e1}\,\mathrm{d}^3w = 0, \quad \int \frac{1}{3} m_e w^2\, f_{e1}\,\mathrm{d}^3w = 0. \tag{3.30}$$

3.3. *First order electron distribution function*

To next order in $\rho_e/L \sim \sqrt{m_e/m_i} \sim \lambda_{ee}/L \sim \lambda_{ei}/L \ll 1$, equation (3.21) becomes

$$-\Omega_e(\mathbf{w}\times\hat{\mathbf{b}})\cdot\nabla_w f_{e1} - C_{ee}^{(\ell)}[f_{e1}] - \mathcal{L}_{ei}\left[f_{e1} - \frac{m_e\mathbf{w}\cdot(\mathbf{u}_i-\mathbf{u}_e)}{T_e} f_{Me}\right] = -\mathbf{w}\cdot\nabla f_{Me} + \frac{\mathbf{F}_{ei} - \nabla p_e - \nabla\cdot\mathbf{\Pi}_e}{n_e m_e}\cdot\nabla_w f_{Me}, \tag{3.31}$$

and condition (3.22) becomes

$$\int f_{e1}\mathbf{w}\,\mathrm{d}^3w = 0. \tag{3.32}$$

Note that we have use the approximation that we derived for the electron-ion collision operator, valid for $\sqrt{m_e/m_i} \ll 1$ (here we are using the relative velocity $\mathbf{w} = \mathbf{v} - \mathbf{u}_e$, and hence in the electron-ion collision operator, we need to use the average ion velocity in the frame moving with $\mathbf{u}_e$, $\mathbf{u}_i - \mathbf{u}_e$).

Equation (3.31) can be simplified using the following approximations:

- In the expansion in $\sqrt{m_e/m_i} \ll 1$, the friction force is given by

$$\mathbf{F}_{ei} = -\frac{2\gamma_{ei}n_i}{m_e}\int f_{e1}(\mathbf{w}')\frac{\mathbf{w}'}{(w')^3}\,\mathrm{d}^3w' + n_e m_e \nu_{ei}(\mathbf{u}_i - \mathbf{u}_e). \tag{3.33}$$

- The piece of the electron-ion collision operator proportional to $\mathbf{u}_i - \mathbf{u}_e$ can be written as

$$\mathcal{L}_{ei}\left[\frac{m_e\mathbf{w}\cdot(\mathbf{u}_i-\mathbf{u}_e)}{T_e}f_{Me}\right] = -\frac{3\sqrt{\pi}}{4}\left(\frac{2T_e}{m_e w^2}\right)^{3/2}\frac{m_e\nu_{ei}(\mathbf{u}_i-\mathbf{u}_e)\cdot\mathbf{w}}{T_e}f_{Me}, \tag{3.34}$$

where we have used that $\gamma_{ei} = (3\sqrt{2\pi}/4)(m_e^{1/2}T_e^{3/2}/n_i)\nu_{ei}$.

- The viscosity is negligible because $f_e \simeq f_{Me}$ and hence

$$\mathbf{\Pi}_e \simeq \int f_{Me} m_e\left(\mathbf{w}\mathbf{w} - \frac{w^2}{3}\mathbf{I}\right) = 0. \tag{3.35}$$

With all these results, and using

$$\nabla f_{Me} = \left[\frac{\nabla p_e}{p_e} + \left(\frac{m_e w^2}{2T_e} - \frac{5}{2}\right)\frac{\nabla T_e}{T_e}\right] f_{Me}, \quad \nabla_w f_{Me} = -\frac{m_e\mathbf{w}}{T_e} f_{Me}, \tag{3.36}$$

equation (3.31) becomes

$$\begin{aligned} -\Omega_e(\mathbf{w}\times\hat{\mathbf{b}})\cdot\nabla_w f_{e1} - C_{ee}^{(\ell)}[f_{e1}] - \mathcal{L}_{ei}[f_{e1}] - \frac{2\gamma_{ei}n_i f_{Me}}{m_e p_e}\mathbf{w}\cdot\int f_{e1}(\mathbf{w}')\frac{\mathbf{w}'}{(w')^3}\,\mathrm{d}^3w' \\ = -\left[1 - \frac{3\sqrt{\pi}}{4}\left(\frac{2T_e}{m_e w^2}\right)^{3/2}\right]\frac{m_e\nu_{ei}(\mathbf{u}_i-\mathbf{u}_e)\cdot\mathbf{w}}{T_e}f_{Me} \\ - \left(\frac{m_e w^2}{2T_e} - \frac{5}{2}\right)\mathbf{w}\cdot\nabla\ln T_e\, f_{Me}. \end{aligned} \tag{3.37}$$

To solve equation (3.37) with condition (3.32), we first change to a set of convenient coordinates. We then split the distribution function into two components.

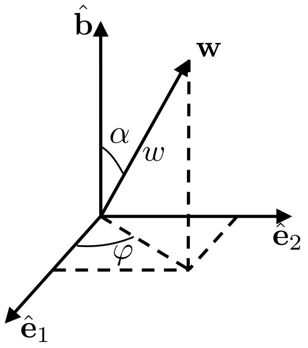

FIGURE 1. Spherical coordinates $\{w, \alpha, \varphi\}$.

3.3.1. *Change of coordinates*

To solve equation (3.37), we will use the spherical coordinates $\{w, \alpha, \varphi\}$, where $w = |\mathbf{w}|$ is the magnitude of the velocity, $\alpha = \arccos(\mathbf{w} \cdot \hat{\mathbf{b}}/w)$ is the angle between the velocity and the magnetic field $\mathbf{B}$, and $\varphi = \arctan(\mathbf{w} \cdot \hat{\mathbf{e}}_2/\mathbf{w} \cdot \hat{\mathbf{e}}_1)$ is the gyrophase (the angle between the perpendicular velocity and a vector $\hat{\mathbf{e}}_1$ perpendicular to the magnetic field $\mathbf{B}$). See figure 1 for a sketch of these spherical coordinates. The vectors $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{b}}\}$ form an orthonormal, right handed basis. In these coordinates,

$$\mathbf{w} = w[\cos\alpha\,\hat{\mathbf{b}} + \sin\alpha(\cos\varphi\,\hat{\mathbf{e}}_1 + \sin\varphi\,\hat{\mathbf{e}}_2)]. \tag{3.38}$$

The gradients with respect to velocity of these coordinates are

$$\nabla_w w = \frac{\mathbf{w}}{w}, \quad \nabla_w \alpha = \frac{1}{w\sin\alpha}\left(\cos\alpha\frac{\mathbf{w}}{w} - \hat{\mathbf{b}}\right), \quad \nabla_w \varphi = -\frac{1}{w^2\sin^2\alpha}\mathbf{w}\times\hat{\mathbf{b}}, \tag{3.39}$$

and hence

$$(\mathbf{w}\times\hat{\mathbf{b}})\cdot\nabla_w f_{e1} = (\mathbf{w}\times\hat{\mathbf{b}})\cdot\left(\nabla_w w\frac{\partial f_{e1}}{\partial w} + \nabla_w\alpha\frac{\partial f_{e1}}{\partial\alpha} + \nabla_w\varphi\frac{\partial f_{e1}}{\partial\varphi}\right) = -\frac{\partial f_{e1}}{\partial\varphi}. \tag{3.40}$$

Then, equation (3.37) becomes

$$\begin{aligned}\Omega_e\frac{\partial f_{e1}}{\partial\varphi} - C_{ee}^{(\ell)}[f_{e1}] - \mathcal{L}_{ei}[f_{e1}] - \frac{2\gamma_{ei}n_i f_{Me}}{m_e p_e}\mathbf{w}\cdot\int f_{e1}(\mathbf{w}')\frac{\mathbf{w}'}{(w')^3}\,\mathrm{d}^3w' \\ = -\left[1 - \frac{3\sqrt{\pi}}{4}\left(\frac{2T_e}{m_e w^2}\right)^{3/2}\right]\frac{m_e\nu_{ei}(\mathbf{u}_i - \mathbf{u}_e)\cdot\mathbf{w}}{T_e}f_{Me} \\ -\left(\frac{m_e w^2}{2T_e} - \frac{5}{2}\right)\mathbf{w}\cdot\nabla\ln T_e\,f_{Me}.\end{aligned} \tag{3.41}$$

To solve equation (3.41) with condition (3.32), it will be convenient to split f_{e1} into its gyrophase independent piece, $\langle f_{e1}\rangle_\varphi$, where

$$\langle f\rangle_\varphi = \frac{1}{2\pi}\int_0^{2\pi} f(w, \alpha, \varphi)\,\mathrm{d}\varphi \tag{3.42}$$

is the gyroaverage, and its gyrophase dependent piece, $\tilde{f}_{e1} = f_{e1} - \langle f_{e1}\rangle_\varphi$.

3.3.2. *Gyrophase independent piece of f_{e1}*

The equation for the gyrophase independent piece is obtained by gyroaveraging equation (3.41). Using the isotropy of the linearized collision operator, we obtain $\langle C_{ee}^{(\ell)}[f_{e1}]\rangle_\varphi =$

$C_{ee}^{(\ell)}[\langle f_{e1}\rangle_\varphi]$. The gyroaverage of the Lorentz collision operator is

$$\langle \mathcal{L}_{ei}[f_{e1}]\rangle_\varphi = \frac{\gamma_{ei} n_i}{m_e^2 w^3} \left\langle \frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha \frac{\partial f_{e1}}{\partial \alpha}\right) + \frac{1}{\sin^2\alpha}\frac{\partial^2 f_{e1}}{\partial\varphi^2}\right\rangle_\varphi$$
$$= \frac{\gamma_{ei} n_i}{m_e^2 w^3}\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha\frac{\partial\langle f_{e1}\rangle_\varphi}{\partial\alpha}\right) = \mathcal{L}_{ei}[\langle f_{e1}\rangle_\varphi]. \tag{3.43}$$

We finish by noting that using the form of $\mathbf{w}$ in (3.38), $\langle\mathbf{w}\rangle_\varphi = w\cos\alpha\,\hat{\mathbf{b}}$, and hence

$$\langle\mathbf{w}\rangle_\varphi\cdot\int f_{e1}(\mathbf{w}')\frac{\mathbf{w}'}{(w')^3}\,\mathrm{d}^3w' = w\cos\alpha\int f_{e1}(\mathbf{w}')\frac{w'\cos\alpha'}{(w')^3}\,\mathrm{d}^3w'$$
$$= w\cos\alpha\int\langle f_{e1}\rangle_\varphi(\mathbf{w}')\frac{w'\cos\alpha'}{(w')^3}\,\mathrm{d}^3w'. \tag{3.44}$$

With all these results, the gyroaverage of (3.41) becomes

$$C_{ee}^{(\ell)}[\langle f_{e1}\rangle_\varphi] + \mathcal{L}_{ei}[\langle f_{e1}\rangle_\varphi] + \frac{2\gamma_{ei}n_i f_{Me}}{m_e p_e} w\cos\alpha\int\langle f_{e1}\rangle_\varphi(\mathbf{w}')\frac{w'\cos\alpha'}{(w')^3}\,\mathrm{d}^3w'$$
$$= \left[1 - \frac{3\sqrt{\pi}}{4}\left(\frac{2T_e}{m_e w^2}\right)^{3/2}\right]\frac{m_e\nu_{ei}(u_{i\parallel}-u_{e\parallel})w\cos\alpha}{T_e} f_{Me}$$
$$+ \left(\frac{m_e w^2}{2T_e} - \frac{5}{2}\right) w\cos\alpha\,\hat{\mathbf{b}}\cdot\nabla\ln T_e\, f_{Me}. \tag{3.45}$$

This equation has to be solved in conjunction with condition (3.32). Taking the parallel component of (3.32), we obtain

$$\int f_{e1} w\cos\alpha\,\mathrm{d}^3w = \int\langle f_{e1}\rangle_\varphi w\cos\alpha\,\mathrm{d}^3w = 0. \tag{3.46}$$

To solve (3.45), it is typical to use a technique similar to the one that we used for the Spitzer-Härm problem. We describe the calculation in Appendix A. The solution $\langle f_{e1}\rangle_\varphi$ is assumed to be a summation of modified Laguerre polynomials $L_p^{(3/2)}(x)$,

$$\langle f_{e1}\rangle_\varphi = \sum_{p=1}^\infty a_p L_p^{(3/2)}(x) f_{Me}(w) w\cos\alpha, \tag{3.47}$$

where $x = m_e w^2/2T_e$, and a_p are coefficients that we need to determine. Note that unlike in the Spitzer-Härm problem, the polynomial $L_0^{(3/2)}(x)$ is not included in the summation because the function has to satisfy condition (3.46). Truncating the series after the first two terms, one obtains

$$\langle f_{e1}\rangle_\varphi \simeq \left[\left(\frac{1.265}{\nu_{ee}}\hat{\mathbf{b}}\cdot\nabla\ln T_e + 0.284\,\frac{m_e(u_{i\parallel}-u_{e\parallel})}{T_e}\right)L_1^{(3/2)}(x)\right.$$
$$\left. + \left(-\frac{0.633}{\nu_{ee}}\,\hat{\mathbf{b}}\cdot\nabla\ln T_e + 0.032\,\frac{m_e(u_{i\parallel}-u_{e\parallel})}{T_e}\right)L_2^{(3/2)}(x)\right] f_{Me}(w)\,w\cos\alpha, \tag{3.48}$$

where the electron-electron collision frequency is defined by Braginskii to be

$$\nu_{ee} = \frac{4\sqrt{2\pi}}{3}\frac{e^4 n_e\ln\Lambda_{ee}}{(4\pi\epsilon_0)^2 m_e^{1/2} T_e^{3/2}}. \tag{3.49}$$

For singly charge ions, $\nu_{ei}\simeq\nu_{ee}$, and we can use both frequencies indistinctly. We will use ν_{ei} instead of ν_{ee} in the terms that would not exist without electron-ion collisions.

3.3.3. *Gyrophase dependent piece of f_{e1}*

The equation for the gyrophase dependent piece $\tilde{f}_{e1}$ is obtained by subtracting (3.45) from (3.41) to find

$$\underbrace{\Omega_e \frac{\partial \tilde{f}_{e1}}{\partial \varphi}}_{\sim \Omega_e \tilde{f}_{e1}} \underbrace{- C_{ee}^{(\ell)}[\tilde{f}_{e1}] - \mathcal{L}_{ei}[\tilde{f}_{e1}] - \frac{2\gamma_{ei} n_i f_{Me}}{m_e p_e} \mathbf{w}_\perp \cdot \int \tilde{f}_{e1}(\mathbf{w}') \frac{\mathbf{w}'_\perp}{(w')^3} \, \mathrm{d}^3 w'}_{\sim \nu_{ee} \tilde{f}_{e1}}$$
$$= \underbrace{- \left[1 - \frac{3\sqrt{\pi}}{4} \left(\frac{2T_e}{m_e w^2} \right)^{3/2} \right] \frac{m_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e) \cdot \mathbf{w}_\perp}{T_e} f_{Me}}_{\sim \frac{m_e \nu_{ei} v_{te}}{T_e} \frac{\rho_i}{L} v_{ti} f_{Me} \sim \frac{\nu_{ee}}{\Omega_e} \frac{v_{te}}{L} f_{Me}}$$
$$\underbrace{- \left(\frac{m_e w^2}{2T_e} - \frac{5}{2} \right) \mathbf{w}_\perp \cdot \nabla \ln T_e \, f_{Me}}_{\sim \frac{v_{te}}{L} f_{Me}}, \tag{3.50}$$

where

$$\mathbf{w}_\perp = w \sin\alpha (\cos\varphi \, \hat{\mathbf{e}}_1 + \sin\varphi \, \hat{\mathbf{e}}_2). \tag{3.51}$$

Equation (3.50) has to be solved in conjunction with condition (3.32). Taking the perpendicular component of (3.32), we obtain

$$\int f_{e1} \mathbf{w}_\perp \, \mathrm{d}^3 w = \int \tilde{f}_{e1} \mathbf{w}_\perp \, \mathrm{d}^3 w = 0. \tag{3.52}$$

Equation (3.50) can be solved numerically, but it is more interesting to take the subsidiary limit $\nu_{ee}/\Omega_e \ll 1$ (as we announced in (1.10)). In this subsidiary expansion, the gyrophase dependent piece becomes

$$\tilde{f}_{e1} = \tilde{f}_{e1}^{(0)} + \underbrace{\tilde{f}_{e1}^{(1)}}_{\sim \frac{\nu_{ee}}{\Omega_e} \tilde{f}_{e1}^{(0)} \ll \tilde{f}_{e1}^{(0)}} + \ldots \tag{3.53}$$

Using this expansion in (3.50), we find to lowest order

$$\Omega_e \frac{\partial \tilde{f}_{e1}^{(0)}}{\partial \varphi} = - \left(\frac{m_e w^2}{2T_e} - \frac{5}{2} \right) \mathbf{w}_\perp \cdot \nabla \ln T_e \, f_{Me}, \tag{3.54}$$

and to next order

$$\Omega_e \frac{\partial \tilde{f}_{e1}^{(1)}}{\partial \varphi} = C_{ee}^{(\ell)}[\tilde{f}_{e1}^{(0)}] + \mathcal{L}_{ei}[\tilde{f}_{e1}^{(0)}] + \frac{2\gamma_{ei} n_i f_{Me}}{m_e p_e} \mathbf{w}_\perp \cdot \int \tilde{f}_{e1}^{(0)}(\mathbf{w}') \frac{\mathbf{w}'_\perp}{(w')^3} \, \mathrm{d}^3 w'$$
$$- \left[1 - \frac{3\sqrt{\pi}}{4} \left(\frac{2T_e}{m_e w^2} \right)^{3/2} \right] \frac{m_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e) \cdot \mathbf{w}_\perp}{T_e} f_{Me}. \tag{3.55}$$

We can integrate equation (3.54), finding

$$\tilde{f}_{e1}^{(0)} = - \frac{1}{\Omega_e} \left(\frac{m_e w^2}{2T_e} - \frac{5}{2} \right) (\mathbf{w}_\perp \times \hat{\mathbf{b}}) \cdot \nabla \ln T_e \, f_{Me} = \frac{L_1^{(3/2)}(x)}{\Omega_e} (\mathbf{w}_\perp \times \hat{\mathbf{b}}) \cdot \nabla \ln T_e \, f_{Me}, \tag{3.56}$$

where we have used $L_1^{(3/2)}(x) = 5/2 - x$ and

$$\mathbf{w}_\perp \times \hat{\mathbf{b}} = w \sin\alpha (\sin\varphi \, \hat{\mathbf{e}}_1 - \cos\varphi \, \hat{\mathbf{e}}_2). \tag{3.57}$$

We could integrate (3.55), but as we will see, it is not necessary to integrate it to find what we need for the electron fluid equations.

3.4. *Electron heat flux, electron viscosity, friction force and collisional energy exchange*

Once the electron distribution function is known, we can calculate the electron heat flux, the electron viscosity and the collisional terms.

3.4.1. *Electron heat flux*

The electron heat flux is given by

$$\begin{aligned}\mathbf{q}_e &\simeq \underbrace{\cancel{\int \frac{1}{2} m_e w^2 \mathbf{w}\, f_{Me}\, \mathrm{d}^3 w}}_{0} + \int \frac{1}{2} m_e w^2 \mathbf{w}\, f_{e1}\, \mathrm{d}^3 w \\ &= T_e \int \left(\frac{m_e w^2}{2T_e} - \frac{5}{2} \right) \mathbf{w}\, f_{e1}\, \mathrm{d}^3 w = -T_e \int L_1^{(3/2)}(x)\, \mathbf{w}\, f_{e1}\, \mathrm{d}^3 w, \end{aligned} \tag{3.58}$$

where we have used the no flow condition (3.32) to write the heat flux as an integral of $(m_e w^2/2 - 5T_e/2)\mathbf{w}$. It is a widespread form of writing the heat flux because it can then be easily written as an integral over the modified Laguerre polynomial $L_1^{(3/2)}(x) = 5/2 - x$.

The first order correction that appears in (3.58) is $f_{e1} \simeq \langle f_{e1} \rangle_\varphi + \tilde{f}_{e1}^{(0)} + \tilde{f}_{e1}^{(1)}$. Using this decomposition, we split the heat flux into three pieces,

$$\mathbf{q}_e = q_{e\|} \hat{\mathbf{b}} + \mathbf{q}_{e\times} + \mathbf{q}_{e\perp}, \tag{3.59}$$

where the parallel heat flux is

$$\boxed{q_{e\|} = -T_e \int L_1^{(3/2)}(x)\, w \cos\alpha\, \langle f_{e1} \rangle_\varphi\, \mathrm{d}^3 w = -\frac{3.16 p_e}{m_e \nu_{ee}} \hat{\mathbf{b}} \cdot \nabla T_e - 0.71 p_e (u_{i\|} - u_{e\|}),} \tag{3.60}$$

the diamagnetic heat flux is

$$\boxed{\mathbf{q}_{e\times} = -T_e \int L_1^{(3/2)}(x)\, \mathbf{w}_\perp\, \tilde{f}_{e1}^{(0)}\, \mathrm{d}^3 w = -\frac{5}{2} \frac{p_e}{m_e \Omega_e} \hat{\mathbf{b}} \times \nabla T_e,} \tag{3.61}$$

and the perpendicular heat flux is given by the integral

$$\mathbf{q}_{e\perp} = -T_e \int L_1^{(3/2)}(x)\, \mathbf{w}_\perp\, \tilde{f}_{e1}^{(1)}\, \mathrm{d}^3 w. \tag{3.62}$$

To take the integral in (3.62), we use that $\mathbf{w}_\perp = \partial(\mathbf{w}_\perp \times \hat{\mathbf{b}})/\partial\varphi$, and we integrate by parts in the gyrophase φ to find

$$\mathbf{q}_{e\perp} = T_e \int L_1^{(3/2)}(x)\, (\mathbf{w}_\perp \times \hat{\mathbf{b}})\, \frac{\partial \tilde{f}_{e1}^{(1)}}{\partial \varphi}\, \mathrm{d}^3 w. \tag{3.63}$$

Using (3.55), this equation becomes

$$\begin{aligned}\mathbf{q}_{e\perp} = &\frac{T_e}{\Omega_e} \int L_1^{(3/2)}(x)\, (\mathbf{w}_\perp \times \hat{\mathbf{b}}) \left(C_{ee}^{(\ell)}[\tilde{f}_{e1}^{(0)}] + \mathcal{L}_{ei}[\tilde{f}_{e1}^{(0)}] \right) \mathrm{d}^3 w \\ &- \frac{T_e}{\Omega_e} \int L_1^{(3/2)}(x)\, (\mathbf{w}_\perp \times \hat{\mathbf{b}}) \left[1 - \frac{3\sqrt{\pi}}{4} \left(\frac{2T_e}{m_e w^2} \right)^{3/2} \right] \frac{m_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e) \cdot \mathbf{w}_\perp}{T_e} f_{Me} \mathrm{d}^3 w. \end{aligned} \tag{3.64}$$

The integrals in this equation finally give

$$\boxed{\mathbf{q}_{e\perp} = -\left(\sqrt{2} + \frac{13}{4}\right)\frac{p_e\nu_{ee}}{m_e\Omega_e^2}\nabla_\perp T_e - \frac{3}{2}\frac{p_e\nu_{ei}}{\Omega_e}\hat{\mathbf{b}}\times(\mathbf{u}_i - \mathbf{u}_e).} \tag{3.65}$$

3.4.2. *Electron viscosity*

The electron viscosity is zero to the order that we have calculated the distribution function because the integral $\int_0^\pi \mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\varphi \, \sin\alpha \, \mathbf{w}[\mathbf{ww} - (w^2/3)\mathbf{I}]$ over the angles α and φ vanishes. We would need a piece of the distribution function composed of spherical harmonics with $l = 2$ to find a non-zero viscosity. Hence, we can take the electron viscosity to be zero,

$$\boxed{\mathbf{\Pi}_e \simeq \cancelto{0}{\int f_{Me}\, m_e\left(\mathbf{ww} - \frac{w^2}{3}\mathbf{I}\right)\mathrm{d}^3w} + \int f_{e1}\, m_e\left(\mathbf{ww} - \frac{w^2}{3}\mathbf{I}\right)\mathrm{d}^3w = 0.} \tag{3.66}$$

3.4.3. *Friction force*

The friction force, defined in (3.33), can be split into two pieces

$$\mathbf{F}_{ei} = F_{ei,\|}\hat{\mathbf{b}} + \mathbf{F}_{ei,\perp}, \tag{3.67}$$

where the parallel friction force is

$$\boxed{\begin{aligned} F_{ei,\|} &= -\frac{2\gamma_{ei}n_i}{m_e}\int \langle f_{e1}\rangle_\varphi \frac{\cos\alpha}{w^2}\,\mathrm{d}^3w + n_e m_e \nu_{ei}(u_{i\|} - u_{e\|}) \\ &= 0.51 n_e m_e \nu_{ei}(u_{i\|} - u_{e\|}) - 0.71 n_e \hat{\mathbf{b}}\cdot\nabla T_e \end{aligned}} \tag{3.68}$$

and the perpendicular friction force is

$$\boxed{\begin{aligned} \mathbf{F}_{ei,\perp} &= -\frac{2\gamma_{ei}n_i}{m_e}\int \tilde{f}_{e1}\frac{\mathbf{w}_\perp}{w^3}\,\mathrm{d}^3w + n_e m_e\nu_{ei}(\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}) \\ &\simeq -\frac{2\gamma_{ei}n_i}{m_e}\int \tilde{f}_{e1}^{(0)}\frac{\mathbf{w}_\perp}{w^3}\,\mathrm{d}^3w + n_e m_e\nu_{ei}(\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}) \\ &= n_e m_e \nu_{ei}(\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}) - \frac{3}{2}\frac{n_e\nu_{ei}}{\Omega_e}\hat{\mathbf{b}}\times\nabla T_e. \end{aligned}} \tag{3.69}$$

3.4.4. *Collisional energy exchange*

To lowest order, the electron distribution function is a Maxwellian. Using the same method that we used to show that the lowest order electron distribution function is a Maxwellian, we can show that the ion distribution function is also a Maxwellian. The electron and ion temperature will be different in general, and as a result, it is sufficient to calculate the collisional energy exchange between two Maxwellians because the first order corrections f_{e1} and f_{i1} will only give a small correction to the large Maxwellian contribution. We have already calculated the energy exchange between two Maxwellians, and it is given by

$$\boxed{\tilde{W}_{ie} = \frac{3n_e m_e\nu_{ei}}{m_i}(T_e - T_i).} \tag{3.70}$$

3.5. *Discussion*

Since we have assumed that the smallest parameter in our expansion is $\rho_e/L \ll 1$, electrons are magnetized, and they gyrate several times around the magnetic field before

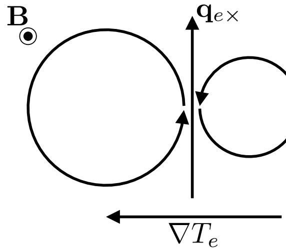

FIGURE 2. Sketch of a plasma with a gradient of electron temperature, ∇T_e, and a magnetic field $\mathbf{B}$ pointing towards the reader. Due to ∇T_e, the gyro-orbits on the left have on average higher velocity than the gyro-orbits on the right, leading to a net energy flow in the direction $-\hat{\mathbf{b}} \times \nabla T_e$.

they collide with other electrons or ions (recall that $\nu_{ee}/\Omega_e \sim \nu_{ei}/\Omega_e \ll 1$). As a result, electrons can move freely along field lines, but they barely move across them. This disparity in the magnitude of the net parallel and perpendicular motion leads to very different heat fluxes and friction forces in the parallel and perpendicular directions. The terms proportional to ∇T_e in equations (3.60), (3.61) and (3.65) are of order

$$q_{e\|} \sim \sqrt{\frac{m_i}{m_e}} \frac{\lambda_{ee}}{L} p_e v_{ti} \gg \mathbf{q}_{e\times} \sim \frac{\rho_i}{L} p_e v_{ti} \gg \mathbf{q}_{e\perp} \sim \sqrt{\frac{m_e}{m_i}} \frac{\nu_{ii}}{\Omega_i} \frac{\rho_i}{L} p_e v_{ti}. \tag{3.71}$$

Heat diffuses rapidly along magnetic field lines, it moves in the direction $-\hat{\mathbf{b}} \times \nabla T_e$ at a slower rate, and diffuses even more slowly along $\nabla_\perp T_e$. The difference in size between $q_{e\|}$ and $\mathbf{q}_{e\perp}$ can be explained using a random walk argument. Particles that move a distance Δl in a time Δt and then change direction randomly before moving a distance Δl in a time Δt again behave diffusively on average with a diffusive coefficient $D \sim (\Delta l)^2/\Delta t$. In a collisional plasma, collisions are the randomizing events, and hence $\Delta t \sim \nu_{ee}^{-1}$. The different size of the parallel and perpendicular heat fluxes is due to the different distances that the particles move during the interval between collisions: along a magnetic field line, they move $\Delta l \sim \lambda_{ee}$, giving $D_\| \sim \nu_{ee} \lambda_{ee}^2 \sim v_{te}^2/\nu_{ee}$, whereas across a magnetic field line, particles are only displaced a distance $\Delta l \sim \rho_e$, leading to $D_\perp \sim \rho_e^2 \nu_{ee} \sim v_{te}^2 \nu_{ee}/\Omega_e^2$. We then obtain $q_{e\|} \sim -n_e D_\| \hat{\mathbf{b}} \cdot \nabla T_e$ and $\mathbf{q}_{e\perp} \sim -n_e D_\perp \nabla_\perp T_e$. The diamagnetic heat flux is a result of the ∇B and curvature drifts and of the finite size of the gyromotion. Figure 2 shows how a gradient of electron temperature gives a heat flux in the direction $-\hat{\mathbf{b}} \times \nabla T_e$.

Another interesting feature of the electron heat flux is that it contains terms proportional to $\mathbf{u}_i - \mathbf{u}_e$. The terms $-0.71 p_e (u_{i\|} - u_{e\|})$ and $-(3/2)(p_e \nu_{ei}/\Omega_e) \hat{\mathbf{b}} \times (\mathbf{u}_i - \mathbf{u}_e)$ arise from the fact that slow electrons are more likely to collide with ions, and as a result they tend to acquire the ion average velocity, whereas the fast electrons will not collide as much with ions. This difference in velocity between slow and fast electrons gives the heat fluxes proportional to $\mathbf{u}_i - \mathbf{u}_e$. The terms proportional to ∇T_e in the friction force have a similar origin. The terms $-0.71 n_e \hat{\mathbf{b}} \cdot \nabla T_e$ and $-(3/2)(n_e \nu_{ei}/\Omega_e) \hat{\mathbf{b}} \times \nabla T_e$ are due to the fact that in the presence of a temperature gradient, the energy of a particle is correlated with its direction: particles coming from the higher temperature region will have more energy than particles coming from the opposite direction. This difference in energy leads to a net friction force because particles coming from the low temperature region will collide more often and will loss more momentum than particles going in the opposite direction.

Finally, note that the parallel electron heat flux $q_{e\|}$ and the parallel friction force $F_{ei,\|}$ are order unity contributions to the electron energy equation (3.10) and the parallel component of the electron momentum equation (3.9), whereas the perpendicular components of the heat flux and the friction force are usually small in the same equations.

4. Ion equations

Equations (2.4), (2.13) and (2.21) for ions are

$$\underbrace{\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i)}_{\sim \frac{n_i v_{ti}}{L}} = 0, \tag{4.1}$$

$$\underbrace{n_i m_i \left(\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i \right)}_{\sim \frac{p_i}{L}} = -\underbrace{\nabla p_i}_{\sim \frac{p_i}{L}} - \underbrace{\nabla \cdot \mathbf{\Pi}_i}_{\lesssim \frac{\lambda_{ii}}{L} \frac{p_i}{L} \ll \frac{p_i}{L}} + \underbrace{e n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B})}_{\sim e n_i v_{ti} B \sim \frac{L}{\rho_i} \frac{p_i}{L} \gg \frac{p_i}{L}} - \underbrace{\mathbf{F}_{ei}}_{\sim \frac{p_i}{L}}, \tag{4.2}$$

$$\underbrace{\frac{3}{2} n_i \left(\frac{\partial T_i}{\partial t} + \mathbf{u}_i \cdot \nabla T_i \right)}_{\sim \frac{p_i v_{ti}}{L}} = -\underbrace{\nabla \cdot \mathbf{q}_i}_{\lesssim \frac{\lambda_{ii}}{L} \frac{p_i v_{ti}}{L} \ll \frac{p_i v_{ti}}{L}} - \underbrace{n_i T_i \nabla \cdot \mathbf{u}_i}_{\sim \frac{p_i v_{ti}}{L}} - \underbrace{\mathbf{\Pi}_i : \nabla \mathbf{u}_i}_{\lesssim \frac{\lambda_{ii}}{L} \frac{p_i v_{ti}}{L} \ll \frac{p_i v_{ti}}{L}} + \underbrace{\tilde{W}_{ie}}_{\sim \frac{p_i v_{ti}}{L}}. \tag{4.3}$$

There two important considerations about these ion equations:

• Instead of equation (4.2), it is common to add equations (3.9) and (4.2) and to use quasineutrality $n_e = n_i$ to obtain a conservation equation for the total momentum,

$$\underbrace{n_e m_i \left(\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i \right)}_{\sim \frac{p_i}{L}} \simeq -\underbrace{\nabla (p_i + p_e)}_{\sim \frac{p_i}{L}} - \underbrace{\nabla \cdot \mathbf{\Pi}_i}_{\sim \frac{\lambda_{ii}}{L} \frac{p_i}{L} \ll \frac{p_i}{L}} + \underbrace{e n_e (\mathbf{u}_i - \mathbf{u}_e) \times \mathbf{B}}_{\sim e n_e \frac{\rho_i}{L} v_{ti} B \sim \frac{p_i}{L}}. \tag{4.4}$$

Here we have used quasineutrality, $n_i = n_e$, to cancel the electric field, and we have neglected the electron inertia $n_e m_e (\partial \mathbf{u}_e / \partial t + \mathbf{u}_e \cdot \nabla \mathbf{u}_e)$ and the electron viscosity $\mathbf{\Pi}_e$. Equation (4.4) has the advantage that the large electromagnetic force terms of equations (3.9) and (4.2) cancel. Note as well that this equation proves that the ordering for the perpendicular velocity difference in (3.3) is correct.

• Since $n_e = n_i$ due to quasineutrality, it is common to replace the ion continuity equation (4.1) by the difference of equations (3.8) and (4.1),

$$\nabla \cdot [n_e (\mathbf{u}_i - \mathbf{u}_e)] = 0. \tag{4.5}$$

This is the current conservation equation.

The ion heat flux and ion viscosity are calculated following a similar procedure to the one we followed to calculate the electron heat flux, electron viscosity and the collisional terms. The ion distribution function is also close to a Maxwellian, and the small correction to the Maxwellian gives the heat flux and the viscosity. For the ions, the collisions with electrons are negligible, and only the collisions between ions matter. The final ion heat flux is given by

$$\mathbf{q}_i = q_{i\|} \hat{\mathbf{b}} + \mathbf{q}_{i\times} + \mathbf{q}_{i\perp}, \tag{4.6}$$

where the parallel, diamagnetic and perpendicular components of the heat flux are analogous to the corresponding quantities for electrons, and they are given by

$$\boxed{q_{i\|} = -\frac{3.9 p_i}{m_i \nu_{ii}} \hat{\mathbf{b}} \cdot \nabla T_i, \quad \mathbf{q}_{i\times} = \frac{5}{2} \frac{p_i}{m_i \Omega_i} \hat{\mathbf{b}} \times \nabla T_i, \quad \mathbf{q}_{i\perp} = -\frac{2 p_i \nu_{ii}}{m_i \Omega_i^2} \nabla_\perp T_i.} \tag{4.7}$$

The ion-ion collision frequency was defined by Braginskii as

$$\nu_{ii} = \frac{4\sqrt{\pi}}{3}\frac{e^4 n_i \ln\Lambda_{ii}}{(4\pi\epsilon_0)^2 m_i^{1/2} T_i^{3/2}}. \tag{4.8}$$

Note that there is a difference of $\sqrt{2}$ between Braginskii's definitions of ν_{ii} and ν_{ee} (see (3.49)). The ion viscosity is also split into three different terms

$$\mathbf{\Pi}_i = \Pi_{i\parallel}\left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{\mathbf{I}}{3}\right) + \mathbf{\Pi}_{i\times} + \mathbf{\Pi}_{i\perp}. \tag{4.9}$$

Here the parallel viscosity is

$$\boxed{\Pi_{i\parallel} = -\frac{0.96 p_i}{\nu_{ii}}\left(3\hat{\mathbf{b}}\cdot\nabla\mathbf{u}_i\cdot\hat{\mathbf{b}} - \nabla\cdot\mathbf{u}_i\right);} \tag{4.10}$$

the gyroviscosity is

$$\boxed{\mathbf{\Pi}_{i\times} = \frac{p_i}{4\Omega_i}\left[\hat{\mathbf{b}}\times\left(\nabla\mathbf{u}_i + (\nabla\mathbf{u}_i)^{\mathrm{T}}\right)\cdot(\mathbf{I}+3\hat{\mathbf{b}}\hat{\mathbf{b}}) - (\mathbf{I}+3\hat{\mathbf{b}}\hat{\mathbf{b}})\cdot\left(\nabla\mathbf{u}_i + (\nabla\mathbf{u}_i)^{\mathrm{T}}\right)\times\hat{\mathbf{b}}\right],} \tag{4.11}$$

where $\mathbf{M}^{\mathrm{T}}$ is the transpose of matrix $\mathbf{M}$ and $\hat{\mathbf{b}}\times\mathbf{M}$ is, in Einstein's repeated index notation, $(\hat{\mathbf{b}}\times\mathbf{M})_{ij} = \epsilon_{ikl}\hat{b}_k M_{lj}$; and the perpendicular viscosity is

$$\boxed{\begin{aligned}\mathbf{\Pi}_{i\perp} = -\frac{3p_i\nu_{ii}}{10\Omega_i^2}&\Bigg\{(\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}})\cdot\left(\nabla\mathbf{u}_i + (\nabla\mathbf{u}_i)^{\mathrm{T}}\right)\cdot(\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}})\\ &+\left(\hat{\mathbf{b}}\cdot\nabla\mathbf{u}_i\cdot\hat{\mathbf{b}} - \nabla\cdot\mathbf{u}_i\right)(\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}})\\ &+4\Big[(\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}})\cdot\left(\nabla\mathbf{u}_i + (\nabla\mathbf{u}_i)^{\mathrm{T}}\right)\cdot\hat{\mathbf{b}}\hat{\mathbf{b}} + \hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\left(\nabla\mathbf{u}_i + (\nabla\mathbf{u}_i)^{\mathrm{T}}\right)\cdot(\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}})\Big]\Bigg\}.\end{aligned}} \tag{4.12}$$

Comparing the ion heat flux with the electron heat flux (see (3.71)), we find

$$\begin{aligned}q_{e\parallel} \sim \sqrt{\frac{m_i}{m_e}}\frac{\lambda_{ee}}{L}p_e v_{ti} \gg q_{i\parallel} \sim \frac{\lambda_{ii}}{L}p_i v_{ti} \gg \mathbf{q}_{i\times} \sim \mathbf{q}_{e\times} \sim \frac{\rho_i}{L}p_i v_{ti}\\ \gg \mathbf{q}_{i\perp} \sim \frac{\nu_{ii}}{\Omega_i}\frac{\rho_i}{L}p_i v_{ti} \gg \mathbf{q}_{e\perp} \sim \sqrt{\frac{m_e}{m_i}}\frac{\nu_{ii}}{\Omega_i}\frac{\rho_i}{L}p_e v_{ti}.\end{aligned} \tag{4.13}$$

Thus, the electrons are much more efficient at transporting energy along magnetic field lines than the ions due to their large thermal speed, but they are very slow in the perpendicular direction due to their small gyroradii. The ion heat flux is usually a small contribution to the ion energy equation (4.3).

The different pieces of the viscosity are of order

$$\Pi_{i\parallel} \sim \frac{\lambda_{ii}}{L}\frac{|\mathbf{u}_i|}{v_{ti}}p_i \gg \mathbf{\Pi}_{i\times} \sim \frac{\rho_i}{L}\frac{|\mathbf{u}_i|}{v_{ti}}p_i \gg \mathbf{\Pi}_{i\perp} \sim \frac{\nu_{ii}}{\Omega_i}\frac{|\mathbf{u}_i|}{v_{ti}}\frac{\rho_i}{L}p_i, \tag{4.14}$$

where we have indicated that the viscosity is proportional to the size of $\nabla\mathbf{u}_i$ and hence to the size of $\mathbf{u}_i$. For $|\mathbf{u}_i| \sim v_{ti}$, the viscosity is small compared to the other terms in the total momentum equation (4.4). For sufficiently small ion velocity, the parallel viscosity can become comparable to the convective term $n_e m_i \mathbf{u}_i\cdot\nabla\mathbf{u}_i \sim (|\mathbf{u}_i|^2/v_{ti}^2)(p_i/L)$. Indeed, for

$$\frac{\rho_i}{L} \ll \frac{|\mathbf{u}_i|}{v_{ti}} \sim \frac{\lambda_{ii}}{L} \ll 1, \tag{4.15}$$

the parallel viscosity is as important as the convective term. Braginskii's expansion is valid in this limit. However, if the ion flow becomes as small as $|\mathbf{u}_i| \sim (\rho_i/L)v_{ti}$, Braginskii's viscosity is not sufficiently accurate and one needs to keep higher order terms (Mikhailovskii & Tsypin 1971; Catto & Simakov 2004).

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Appendix A. Solving for the electron gyrophase independent piece of the distribution function $\langle f_{e1}\rangle_\varphi$

Equation (3.45) can be solved using a variational principle.

Due to condition (3.46), we search for solutions in the vector space GINF of Gyrophase Independent distribution functions h_e with No Flow, i.e. $\int h_e w \cos\alpha \, \mathrm{d}^3 w = 0$. The linearized electron-electron collision operator $C_{ee}^{(\ell)}[h_e]$ and the modified Lorentz collision operator

$$\mathcal{L}_{ei}^{\mathrm{mod}}[h_e] = \mathcal{L}_{ei}[h_e] + \frac{2\gamma_{ei} n_i f_{Me}}{m_e p_e} w\cos\alpha \int h_e(\mathbf{w}') \frac{w'\cos\alpha'}{(w')^3}\,\mathrm{d}^3 w' \tag{A 1}$$

convert functions of the space GINF into functions of GINF since

- when applied to a gyrophase independent function, they give another gyrophase independent function, and
- $\int C_{ee}^{(\ell)}[h_e] w\cos\alpha\,\mathrm{d}^3 w = 0$ and

$$\begin{aligned}
\int \mathcal{L}_{ei}^{\mathrm{mod}}[h_e] w\cos\alpha\,\mathrm{d}^3 w &= \frac{\gamma_{ei} n_i}{m_e^2} \int \nabla_w \cdot \left(\nabla_w \nabla_w w \cdot \nabla_w h_e\right) \mathbf{w}\cdot\hat{\mathbf{b}}\,\mathrm{d}^3 w \\
&+ \frac{2\gamma_{ei} n_i}{m_e p_e} \int f_{Me} w^2 \cos^2\alpha\,\mathrm{d}^3 w \int h_e(\mathbf{w}')\frac{\mathbf{w}'\cdot\hat{\mathbf{b}}}{(w')^3}\,\mathrm{d}^3 w' \\
&= -\frac{\gamma_{ei} n_i}{m_e^2} \int \hat{\mathbf{b}}\cdot\nabla_w\nabla_w w \cdot \nabla_w h_e\,\mathrm{d}^3 w + \frac{2\gamma_{ei} n_i}{m_e^2}\int h_e(\mathbf{w}')\frac{\mathbf{w}'\cdot\hat{\mathbf{b}}}{(w')^3}\,\mathrm{d}^3 w' \\
&= \frac{\gamma_{ei} n_i}{m_e^2} \int h_e\,\hat{\mathbf{b}}\cdot\nabla_w\nabla_w^2 w\,\mathrm{d}^3 w + \frac{2\gamma_{ei} n_i}{m_e^2}\int h_e(\mathbf{w}')\frac{\mathbf{w}'\cdot\hat{\mathbf{b}}}{(w')^3}\,\mathrm{d}^3 w' = 0,
\end{aligned} \tag{A 2}$$

where we have integrated by parts several times, and we have used $\nabla_w \nabla_w^2 w = -2\mathbf{w}/w^3$.

We define the scalar product

$$\langle k_e, h_e\rangle = \int \frac{1}{f_{Me}} k_e h_e\,\mathrm{d}^3 w \tag{A 3}$$

in the vector space GINF. The operators $C_{ee}^{(\ell)}[h_e]$ and $\mathcal{L}_{ei}^{\mathrm{mod}}[h_e]$ are self-adjoint in GINF with this scalar product. In the case of $\mathcal{L}_{ei}^{\mathrm{mod}}[h_e]$, we use

$$\langle k_e, w\cos\alpha f_{Me}\rangle = \int k_e w\cos\alpha\,\mathrm{d}^3 w = 0 \tag{A 4}$$

to write

$$\langle k_e, \mathcal{L}_{ei}^{\mathrm{mod}}[h_e]\rangle = \langle k_e, \mathcal{L}_{ei}[h_e]\rangle. \tag{A 5}$$

The Lorentz collision operator is clearly self-adjoint.

Using the self-adjointness of $C_{ee}^{(\ell)}[h_e]$ and $\mathcal{L}_{ei}^{\mathrm{mod}}[h_e]$, it is easy to show that the function $\langle f_{e1}\rangle_\varphi$, solution to (3.45), is the mininum of the functional

$$\begin{aligned}
\Sigma[h_e] = &-\left\langle h_e, C_{ee}^{(\ell)}[h_e]\right\rangle - \langle h_e, \mathcal{L}_{ei}[h_e]\rangle \\
&+ 2\left\langle h_e, \left(\frac{m_e w^2}{2T_e} - \frac{5}{2}\right) w\cos\alpha\,\hat{\mathbf{b}}\cdot\nabla\ln T_e\, f_{Me}\right\rangle \\
&+ 2\left\langle h_e, \left[1 - \frac{3\sqrt{\pi}}{4}\left(\frac{2T_e}{m_e w^2}\right)^{3/2}\right] \frac{m_e \nu_{ei} w\cos\alpha(u_{i\|} - u_{e\|})}{T_e} f_{Me}\right\rangle.
\end{aligned} \tag{A 6}$$

Following the solution of the Spitzer-Härm problem, we choose to write $\langle f_{e1}\rangle_\varphi$ in the form given in (3.47) to obtain

$$\Sigma = \frac{p_e}{m_e}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}(\nu_{ee}K_{pq}^{ee}+\nu_{ei}K_{pq}^{ei})a_p a_q - \frac{5n_e\hat{\mathbf{b}}\cdot\nabla T_e}{m_e}a_1 - 2n_e\nu_{ei}(u_{i\|}-u_{e\|})\sum_{p=1}^{\infty}C_p a_p, \quad \text{(A 7)}$$

where K_{pq}^{ee} and K_{pq}^{ei} were calculated for the Spitzer-Härm problem, and

$$\begin{aligned} C_p &= \frac{2}{n_e}\left\langle x^{1/2}L_p^{(3/2)}(x)f_{Me}(w)\cos\alpha, x^{1/2}\left(\frac{3\sqrt{\pi}}{4x^{3/2}}-1\right)f_{Me}(w)\cos\alpha\right\rangle \\ &= \frac{2}{n_e}\left\langle x^{1/2}L_p^{(3/2)}(x)f_{Me}(w)\cos\alpha, \frac{3\sqrt{\pi}}{4x}f_{Me}(w)\cos\alpha\right\rangle \Rightarrow \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 3/2 \\ 15/8 \\ 35/16 \\ \vdots \end{pmatrix}. \end{aligned} \quad \text{(A 8)}$$

The coefficients a_p are then determined by finding the stationary values of Σ.