Collisions between electrons and ions

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1. Introduction

The Fokker-Planck collision operator can be simplified significantly when we consider collisions between electrons and ions. The simplification is a result of the large mass difference. Assuming that the ion and electron temperatures are of the same order,

$$T_e \sim T_i, \tag{1.1}$$

we find that the thermal speed of the electrons v_{te} is much larger than the thermal speed of the ions v_{ti},

$$v_{te} \sim \sqrt{\frac{T_e}{m_e}} \gg v_{ti} \sim \sqrt{\frac{T_i}{m_i}} \sim \sqrt{\frac{m_e}{m_i}} v_{te}, \tag{1.2}$$

where m_e and m_i are the mass of electrons and ions, respectively. We will exploit this difference in characteristic velocities to study collisions between electrons and ions.

2. Simple estimates

Before attempting a rigorous derivation, we give some simple estimates. We consider electrons of charge $-e$ and mass m_e colliding with ions with charge Ze and mass m_i. Naively, to have a collision, the distance b_{ei} between the electron and the ion must be such that the kinetic energy of the electron is of the order of the potential energy due to the Coulomb force between the particles,

$$\frac{1}{2} m_e v_{te}^2 \sim T_e \sim \frac{Ze^2}{4\pi\epsilon_0 b_{ei}}. \tag{2.1}$$

Thus, only electrons at a distance

$$b_{ei} \sim \frac{Ze^2}{4\pi\epsilon_0 T_e} \tag{2.2}$$

of ions have significant collisions with those ions.

The characteristic collision frequency (the inverse of the time between collisions) can then be estimated using b_{ei}. An electron moves a mean free path λ_{ei} before it encounters an ion. Since the electron only notices ions at a distance b_{ei}, it samples a volume $\pi b_{ei}^2 \lambda_{ei}$ as it move a mean free path. In this volume, the probability of finding an ion is of order unity, so it must satisfy $n_i \pi b_{ei}^2 \lambda_{ei} \sim 1$. From this estimate we obtain the mean free path

$$\lambda_{ei} \sim \frac{1}{n_i \pi b_{ei}^2} \sim \frac{(4\pi\epsilon_0)^2 T_e^2}{Z^2 e^4 n_i}. \tag{2.3}$$

With the mean free path, we obtain the collision frequency

$$\nu_{ei} \sim \frac{v_{te}}{\lambda_{ei}} \sim \frac{Z^2 e^4 n_i}{(4\pi\epsilon_0)^2 m_e^{1/2} T_e^{3/2}}. \tag{2.4}$$

This estimate ignores the fact that weak collisions between particles separated by the Debye length λ_D dominate. To include the effect of these weak collisions, we only need to recall that we estimated the effect of weak collisions to be larger than the effect of collisions between particles at a distance b_{ei} by a Coulomb logarithm $\ln \Lambda_{ei} \gg 1$. A larger effect is roughly equivalent to more collisions, that is, it is equivalent to a larger collision frequency. Then, we need to multiply (2.4) by a factor of $\ln \Lambda_{ei}$ to obtain

$$\nu_{ei} \sim \frac{Z^2 e^4 n_i \ln \Lambda_{ei}}{(4\pi\epsilon_0)^2 m_e^{1/2} T_e^{3/2}}. \tag{2.5}$$

Similar estimates give us the typical collision frequency of electron-electron collisions and ion-ion collisions,

$$\nu_{ee} \sim \frac{e^4 n_e \ln \Lambda_{ee}}{(4\pi\epsilon_0)^2 m_e^{1/2} T_e^{3/2}} \tag{2.6}$$

and

$$\nu_{ii} \sim \frac{Z^4 e^4 n_i \ln \Lambda_{ii}}{(4\pi\epsilon_0)^2 m_i^{1/2} T_i^{3/2}}. \tag{2.7}$$

The estimate that led to (2.5) is not valid for the effect of ion-electron collisions on ions. It is true that electrons and ions collide often, but the effect of a single collision on an ion is small. Due to conservation of momentum, if the change of the electron velocity in a collision is $\Delta \mathbf{v}_e$, the change to the ion velocity is

$$\Delta \mathbf{v}_i = -\frac{m_e}{m_i} \Delta \mathbf{v}_e \sim \frac{m_e}{m_i} v_{te} \sim \sqrt{\frac{m_e}{m_i}} v_{ti} \ll v_{ti}. \tag{2.8}$$

Therefore, a single electron-ion collision only modifies the ion velocity by a small amount of the order of $(m_e/m_i)^{1/2} v_{ti} \ll v_{ti}$. This type of collision that only changes the velocity by a small amount can be thought of as a random walk in velocity space. To achieve a total change in the ion velocity of the order of v_{ti}, we need a large number N_c of electron-ion collisions,

$$N_c \sim \left(\frac{v_{ti}}{|\Delta \mathbf{v}_i|} \right)^2 \sim \frac{m_i}{m_e} \gg 1. \tag{2.9}$$

Then, the effective collision frequency of ion-electron collisions is $1/N_c$ smaller than the electron-ion collision frequency,

$$\nu_{ie} \sim \frac{\nu_{ei}}{N_c} \sim \frac{m_e}{m_i} \nu_{ei} \sim \frac{Z^2 e^4 n_i m_e^{1/2} \ln \Lambda_{ei}}{(4\pi\epsilon_0)^2 m_i T_e^{3/2}}. \tag{2.10}$$

Combining equations (2.5), (2.6), (2.7) and (2.10), and assuming $T_e \sim T_i$, $Z \sim 1$ and $n_e \sim n_i$, we obtain

$$\nu_{ee} \sim \nu_{ei} \gg \nu_{ii} \sim \sqrt{\frac{m_e}{m_i}} \nu_{ei} \gg \nu_{ie} \sim \frac{m_e}{m_i} \nu_{ei}. \tag{2.11}$$

The electrons collide with electrons as often as they collide with ions. Ions collide with ions much more rarely, and ions are affected by their collisions with the light electrons only after a time much longer than the time between collisions with other ions.

3. Electron-ion collision operator

We start with the effect on electrons of collisions with ions. We perform an expansion in $\sqrt{m_e/m_i} \ll 1$. We first consider the lowest order in $\sqrt{m_e/m_i} \ll 1$, and later we keep higher order terms.

3.1. *Electron-ion collision to lowest order in* $\sqrt{m_e/m_i} \ll 1$

The electron-ion Fokker-Planck collision operator is

$$C_{ei}[f_e, f_i] = \frac{\gamma_{ei}}{m_e} \nabla_v \cdot \left\{ \int \nabla_g \nabla_g g \cdot \left[\underbrace{\frac{f_i(\mathbf{v}')}{m_e} \nabla_v f_e(\mathbf{v})}_{\sim f_i f_e/\sqrt{m_e T_e}} - \underbrace{\frac{f_e(\mathbf{v})}{m_i} \nabla_{v'} f_i(\mathbf{v}')}_{\sim f_i f_e/\sqrt{m_i T_i}} \right] \mathrm{d}^3 v' \right\}. \tag{3.1}$$

To find the order of magnitude estimate, we have used $\nabla_v f_e \sim f_e/v_{te}$ and $\nabla_{v'} f_i \sim f_i/v_{ti}$. The term $m_i^{-1} f_e(\mathbf{v}) \nabla_{v'} f_i(\mathbf{v}')$ is then negligible. Moreover, we find

$$\mathbf{g} = \underbrace{\mathbf{v}}_{\sim v_{te}} - \underbrace{\mathbf{v}'}_{\sim v_{ti}} \simeq \mathbf{v}, \tag{3.2}$$

leading to

$$\nabla_g \nabla_g g \simeq \nabla_v \nabla_v v = \frac{v^2 \mathbf{I} - \mathbf{v}\mathbf{v}}{v^3}. \tag{3.3}$$

With this result, and using $\int f_i(\mathbf{v}')\, \mathrm{d}^3 v' = n_i$, equation (3.1) becomes to lowest order in $\sqrt{m_e/m_i} \ll 1$

$$C_{ei}[f_e, f_i] \simeq \mathcal{L}_{ei}[f_e] = \frac{\gamma_{ei} n_i}{m_e^2} \nabla_v \cdot \left(\frac{v^2 \mathbf{I} - \mathbf{v}\mathbf{v}}{v^3} \cdot \nabla_v f_e \right). \tag{3.4}$$

This approximate operator is known as Lorentz collision operator or pitch-angle scattering collision operator.

To understand the pitch-angle scattering operator, we rewrite it using the spherical coordinates $\{v, \alpha, \beta\}$ in velocity space, shown in figure 1. In the orthonormal basis $\{\hat{\mathbf{v}} = \mathbf{v}/v, \hat{\boldsymbol{\alpha}} = \nabla_v \alpha / |\nabla_v \alpha|, \hat{\boldsymbol{\beta}} = \nabla_v \beta / |\nabla_v \beta|\}$, the gradient and divergence with respect to the velocity of general functions f and $\boldsymbol{\Gamma}$ are

$$\nabla_v f = \nabla_v v \frac{\partial f}{\partial v} + \nabla_v \alpha \frac{\partial f}{\partial \alpha} + \nabla_v \beta \frac{\partial f}{\partial \beta} = \frac{\partial f}{\partial v} \hat{\mathbf{v}} + \frac{1}{v} \frac{\partial f}{\partial \alpha} \hat{\boldsymbol{\alpha}} + \frac{1}{v \sin\alpha} \frac{\partial f}{\partial \beta} \hat{\boldsymbol{\beta}} \tag{3.5}$$

and

$$\begin{aligned} \nabla_v \cdot \boldsymbol{\Gamma} &= \frac{1}{\mathcal{J}} \left[\frac{\partial}{\partial v} \left(\mathcal{J} \boldsymbol{\Gamma} \cdot \nabla_v v \right) + \frac{\partial}{\partial \alpha} \left(\mathcal{J} \boldsymbol{\Gamma} \cdot \nabla_v \alpha \right) + \frac{\partial}{\partial \beta} \left(\mathcal{J} \boldsymbol{\Gamma} \cdot \nabla_v \beta \right) \right] \\ &= \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 \boldsymbol{\Gamma} \cdot \hat{\mathbf{v}} \right) + \frac{1}{v \sin\alpha} \frac{\partial}{\partial \alpha} \left(\sin\alpha\, \boldsymbol{\Gamma} \cdot \hat{\boldsymbol{\alpha}} \right) + \frac{1}{v \sin\alpha} \frac{\partial}{\partial \beta} \left(\boldsymbol{\Gamma} \cdot \hat{\boldsymbol{\beta}} \right), \end{aligned} \tag{3.6}$$

where $\mathcal{J} = \det[\partial \mathbf{v} / \partial(v, \alpha, \beta)] = [\nabla_v v \cdot (\nabla_v \alpha \times \nabla_v \beta)]^{-1} = v^2 \sin\alpha$ is the determinant of the Jacobian of the transformation $\mathbf{v}(v, \alpha, \beta)$. Using (3.5) and (3.6), equation (3.4) becomes

$$\mathcal{L}_{ei}[f_e] = \frac{\gamma_{ei} n_i}{m_e^2 v^3} \left[\frac{1}{\sin\alpha} \frac{\partial}{\partial \alpha} \left(\sin\alpha \frac{\partial f_e}{\partial \alpha} \right) + \frac{1}{\sin^2\alpha} \frac{\partial^2 f_e}{\partial \beta^2} \right]. \tag{3.7}$$

The Lorentz operator diffuses the distribution function in α and β, but leaves its structure in v unchanged. The reason for this lack of diffusion in v is that electrons do not change the magnitude of its velocity when they collide with heavy ions. According to (2.8), the velocity of the ion barely changes in a collision with an electron. Then, the ion kinetic

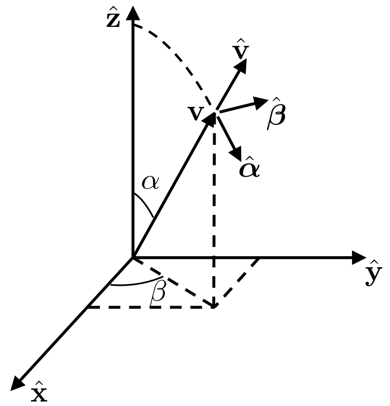

FIGURE 1. Spherical coordinates $\{v, \alpha, \beta\}$ in velocity space. The orthonormal basis $\{\hat{\mathbf{v}} = \mathbf{v}/v, \hat{\boldsymbol{\alpha}} = \nabla_v \alpha / |\nabla_v \alpha|, \hat{\boldsymbol{\beta}} = \nabla_v \beta / |\nabla_v \beta|\}$ is also sketched.

energy does not change, and since the total kinetic energy of both the electron and the ion is conserved, the kinetic energy of the electron is the same before and after the collision. Thus, only the direction of the electron velocity changes after a collision, leading to the diffusion in α and β seen in (3.7).

The Lorentz operator tends to make the electron distribution function isotropic, that is, it tends to give a function f_e that is only a function of v and not of α or β. To show this property, we prove that the Lorentz operator satisfies its own H-theorem. The entropy production due to the Lorentz operator is

$$\dot{\sigma}_{ei}^{\mathcal{L}} = -\int \ln f_e \, \mathcal{L}_{ei}[f_e]. \tag{3.8}$$

Using (3.4) and integrating by parts, we obtain

$$\begin{aligned} \dot{\sigma}_{ei}^{\mathcal{L}} &= \frac{\gamma_{ei} n_i}{m_e^2} \int f_e \, \nabla_v \ln f_e \cdot \frac{v^2 \mathbf{I} - \mathbf{v}\mathbf{v}}{v^3} \cdot \nabla_v \ln f_e \, \mathrm{d}^3 v \\ &= \frac{\gamma_{ei} n_i}{m_e^2} \int \frac{f_e}{v} \left| \nabla_v \ln f_e - \frac{\mathbf{v} \cdot \nabla_v \ln f_e}{v^2} \mathbf{v} \right|^2 \mathrm{d}^3 v \\ &= \frac{\gamma_{ei} n_i}{m_e^2} \int \frac{f_e}{v^3} \left[\left(\frac{\partial f_e}{\partial \alpha} \right)^2 + \frac{1}{\sin^2 \alpha} \left(\frac{\partial f_e}{\partial \beta} \right)^2 \right] \mathrm{d}^3 v \geqslant 0. \end{aligned} \tag{3.9}$$

Thus, the entropy grows until $\dot{\sigma}_{ei}^{\mathcal{L}} = 0$. The entropy production $\dot{\sigma}_{ei}^{\mathcal{L}}$ vanishes only when $\nabla_v \ln f_e$ is proportional to $\mathbf{v}$, that is, when $f_e(v)$ is only a function of the velocity magnitude. Interestingly, we did not need to add the entropy production of the ions due to electron-ion collisions to show that the entropy increases. The isotropization process is then independent of the ion distribution function because to this order in $\sqrt{m_e/m_i} \ll 1$, the ions seem just stationary particles compared to the fast electrons.

3.2. *Electron-ion collision to first order in* $\sqrt{m_e/m_i} \ll 1$

We have argued in (2.11) that the electron-ion collisions are much more frequent than other types of collisions. Thus, it is usual to have an electron distribution function that is isotropic to lowest order in $\sqrt{m_e/m_i} \ll 1$,

$$f_e(\mathbf{v}) = \underbrace{f_{e0}(v)}_{\text{isotropic}} + \underbrace{f_{e1}(\mathbf{v})}_{\sim\sqrt{\frac{m_e}{m_i}}f_{e0}} + \ldots \tag{3.10}$$

If this is the case, we need to continue the expansion of the electron-ion collision operator to next order in $\sqrt{m_e/m_i} \ll 1$. For $\nabla_g\nabla_g g$, instead of the lowest order approximation in (3.3), we keep the next order correction to find

$$\nabla_g\nabla_g g \equiv \mathbf{M}(\mathbf{g}) = \mathbf{M}(\mathbf{v}-\mathbf{v}') \simeq \mathbf{M}(\mathbf{v}) - \mathbf{v}'\cdot\nabla_v\mathbf{M}(\mathbf{v}) = \nabla_v\nabla_v v - \mathbf{v}'\cdot\nabla_v\nabla_v\nabla_v v. \tag{3.11}$$

Substituting this result and the expansion in (3.10) into (3.1), we obtain

$$C_{ei}[f_e,f_i] \simeq \frac{\gamma_{ei}}{m_e}\nabla_v\cdot\left\{\int\left[\frac{f_i(\mathbf{v}')}{m_e}\Big(\underbrace{\nabla_v\nabla_v v\cdot\nabla_v f_{e0}(v)}_{0\text{ due to isotropy}} + \nabla_v\nabla_v v\cdot\nabla_v f_{e1}(\mathbf{v})\right.\right.$$
$$\left.\left. -\mathbf{v}'\cdot\nabla_v\nabla_v\nabla_v v\cdot\nabla_v f_{e0}(v)\Big) - \frac{f_{e0}(v)}{m_i}\nabla_v\nabla_v v\cdot\nabla_{v'}f_i(\mathbf{v}')\right]\mathrm{d}^3v'\right\}. \tag{3.12}$$

Using

$$\nabla_v f_{e0}(v) = \frac{1}{v}\frac{\partial f_{e0}}{\partial v}\mathbf{v}, \tag{3.13}$$

and

$$\int f_i(\mathbf{v}')\,\mathrm{d}^3v' = n_i,\ \int f_i(\mathbf{v}')\mathbf{v}'\,\mathrm{d}^3v' = n_i\mathbf{u}_i,\ \int \nabla_{v'}f_i(\mathbf{v}')\,\mathrm{d}^3v' = 0, \tag{3.14}$$

the electron-ion collision operator in (3.12) can be rewritten as

$$C_{ei}[f_e,f_i] \simeq \frac{\gamma_{ei}n_i}{m_e^2}\nabla_v\cdot\left(\nabla_v\nabla_v v\cdot\nabla_v f_{e1} - \frac{1}{v}\frac{\partial f_{e0}}{\partial v}\mathbf{u}_i\cdot\nabla_v\nabla_v\nabla_v v\cdot\mathbf{v}\right). \tag{3.15}$$

One further useful manipulation is

$$\nabla_v\nabla_v\nabla_v v\cdot\mathbf{v} = \nabla_v(\underbrace{\nabla_v\nabla_v v\cdot\mathbf{v}}_{0}) - \underbrace{\nabla_v\mathbf{v}}_{\mathbf{I}}\cdot\nabla_v\nabla_v v = -\nabla_v\nabla_v v. \tag{3.16}$$

With this result equation (3.15) finally becomes

$$\begin{aligned} C_{ei}[f_e,f_i] &\simeq \frac{\gamma_{ei}n_i}{m_e^2}\nabla_v\cdot\left[\nabla_v\nabla_v v\cdot\left(\nabla_v f_{e1} + \frac{\mathbf{u}_i}{v}\frac{\partial f_{e0}}{\partial v}\right)\right] \\ &= \frac{\gamma_{ei}n_i}{m_e^2}\nabla_v\cdot\left[\nabla_v\nabla_v v\cdot\nabla_v\left(f_{e1} + \frac{\mathbf{v}\cdot\mathbf{u}_i}{v}\frac{\partial f_{e0}}{\partial v}\right)\right] = \mathcal{L}_{ei}\left[f_{e1} + \frac{\mathbf{v}\cdot\mathbf{u}_i}{v}\frac{\partial f_{e0}}{\partial v}\right]. \end{aligned} \tag{3.17}$$

The electron-electron collisions are usually as frequent as the electron-ion collisions (see (2.11)), and as a result, it is usually the case that the lowest order electron distribution function is not only isotropic, but also Maxwellian,

$$f_{e0}(v) = f_{Me}(v) \equiv n_e\left(\frac{m_e}{2\pi T_e}\right)^{3/2}\exp\left(-\frac{m_e v^2}{2T_e}\right). \tag{3.18}$$

If this is the case, equation (3.17) becomes

$$C_{ei}[f_e,f_i] \simeq \mathcal{L}_{ei}\left[f_{e1} - \frac{m_e\mathbf{v}\cdot\mathbf{u}_i}{T_e}f_{Me}\right]. \tag{3.19}$$

We have seen that the Lorentz operator tends to make the distribution function isotropic. Thus, the collision operator in (3.17) will give

$$f_{e1}(\mathbf{v}) = g_{e1}(v) - \frac{\mathbf{v}\cdot\mathbf{u}_i}{v}\frac{\partial f_{e0}}{\partial v}. \tag{3.20}$$

where $g_{e1}(v)$ is isotropic. When calculating the total distribution function $f_e \simeq f_{e0}(v) + f_{e1}(\mathbf{v})$, we can absorb the isotropic correction $g_{e1}(v)$ into the lowest order isotropic distribution function $f_{e0}(v)$, leading to

$$f_e(\mathbf{v}) \simeq f_{e0}(v) - \frac{\mathbf{v}\cdot\mathbf{u}_i}{v}\frac{\partial f_{e0}}{\partial v} \simeq f_{e0}(v) + (|\mathbf{v}-\mathbf{u}_i| - v)\frac{\partial f_{e0}}{\partial v} \simeq f_{e0}(|\mathbf{v}-\mathbf{u}_i|). \tag{3.21}$$

Then, the electron-ion collisions tend to give an electron distribution function that is isotropic around the average velocity of the ions $\mathbf{u}_i$.

We proceed to calculate the collisional friction force and the collisional energy exchange using (3.17).

3.2.1. *Electron-ion collisional friction force*

The collisional force on the electrons is

$$\mathbf{F}_{ei} = \int m_e \mathbf{v}\, C_{ei}[f_e, f_i]\, \mathrm{d}^3 v. \tag{3.22}$$

Substituting equation (3.17) into this expression, and integrating by parts, we find

$$\begin{aligned}\mathbf{F}_{ei} &= -\frac{\gamma_{ei} n_i}{m_e}\int \cancelto{\mathbf{I}}{\nabla_v \mathbf{v}}\cdot\nabla_v\nabla_v v\cdot\left(\nabla_v f_{e1} + \frac{\mathbf{u}_i}{v}\frac{\partial f_{e0}}{\partial v}\right)\mathrm{d}^3 v\\ &= -\frac{\gamma_{ei} n_i}{m_e}\int \nabla_v\nabla_v v\cdot\left(\nabla_v f_{e1} + \frac{\mathbf{u}_i}{v}\frac{\partial f_{e0}}{\partial v}\right)\mathrm{d}^3 v. \end{aligned} \tag{3.23}$$

Integrating by parts the first term in the integral, we find

$$\mathbf{F}_{ei} = \frac{\gamma_{ei} n_i}{m_e}\int\left(f_{e1}\,\nabla_v^2\nabla_v v - \frac{v^2\mathbf{I} - \mathbf{v}\mathbf{v}}{v^4}\cdot\mathbf{u}_i\frac{\partial f_{e0}}{\partial v}\right)\mathrm{d}^3 v. \tag{3.24}$$

To simplify the integral further, we use that $f_{e0}(v)$ is isotropic, and we take the integral in the spherical coordinates sketched in figure 1. Since $\mathbf{v} = v[\sin\alpha(\cos\beta\,\hat{\mathbf{x}} + \sin\beta\,\hat{\mathbf{y}}) + \cos\alpha\,\hat{\mathbf{z}}]$, the integral over the angles α and β gives

$$\frac{1}{4\pi}\int_0^{\pi}\mathrm{d}\alpha\int_0^{2\pi}\mathrm{d}\beta\,\sin\alpha\,\mathbf{v}\mathbf{v} = \frac{v^2}{3}(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}}) = \frac{v^2}{3}\mathbf{I}. \tag{3.25}$$

Then,

$$\int\frac{v^2\mathbf{I} - \mathbf{v}\mathbf{v}}{v^4}\frac{\partial f_{e0}}{\partial v}\,\mathrm{d}^3 v = \frac{8\pi}{3}\mathbf{I}\int_0^{\infty}\frac{\partial f_{e0}}{\partial v}\,\mathrm{d}v = -\frac{8\pi f_{e0}(0)}{3}\mathbf{I}. \tag{3.26}$$

Using this expression, and employing

$$\begin{aligned}\nabla_v^2\nabla_v v = \nabla_v[\nabla_v\cdot(\nabla_v v)] &= \nabla_v\left[\nabla_v\cdot\left(\frac{\mathbf{v}}{v}\right)\right] = \nabla_v\left(\frac{\nabla_v\cdot\mathbf{v}}{v} - \frac{\mathbf{v}\cdot\nabla_v v}{v^2}\right)\\ &= \nabla_v\left(\frac{3}{v} - \frac{\mathbf{v}\cdot\mathbf{v}}{v^3}\right) = \nabla_v\left(\frac{2}{v}\right) = -\frac{2\mathbf{v}}{v^3}, \end{aligned} \tag{3.27}$$

equation (3.24) becomes

$$\mathbf{F}_{ei} = \frac{\gamma_{ei} n_i}{m_e}\left(\frac{8\pi f_{e0}(0)}{3}\mathbf{u}_i - 2\int\frac{\mathbf{v}}{v^3}f_{e1}\,\mathrm{d}^3 v\right). \tag{3.28}$$

The friction force depends on the average ion velocity and on a moment of the correction to the electron distribution function f_{e1}. Note that only the value of f_{e0} at $v = 0$ enters in the expression, and that the integral over f_{e1} is weighed towards smaller v due to the factor v^{-3}. Low energy electrons determine the friction force because they are more likely to collide with ions.

Expression (3.28) becomes more transparent if we assume that the electron distribution function is a Maxwellian with average velocity $\mathbf{u}_e \sim \mathbf{u}_i \ll v_{te}$,

$$f_e(\mathbf{v}) = n_e \left(\frac{m_e}{2\pi T_e}\right)^{3/2} \exp\left(-\frac{m_e|\mathbf{v}-\mathbf{u}_e|^2}{2T_e}\right) = f_{Me}(|\mathbf{v}-\mathbf{u}_e|)$$
$$\simeq f_{Me}(v) - \frac{\mathbf{v}\cdot\mathbf{u}_e}{v}\frac{\partial f_{Me}}{\partial v} = \underbrace{f_{Me}(v)}_{f_{e0}(v)} + \underbrace{\frac{m_e \mathbf{v}\cdot\mathbf{u}_e}{T_e} f_{Me}(v)}_{f_{e1}(\mathbf{v})}, \tag{3.29}$$

where $f_{Me}(v)$ is the stationary Maxwellian defined in (3.18). In this simple case,

$$f_{e0}(0) = f_{Me}(0) = n_e \left(\frac{m_e}{2\pi T_e}\right)^{3/2}, \tag{3.30}$$

and using (3.25), we find that

$$\int \frac{\mathbf{v}}{v^3} f_{e1}\,\mathrm{d}^3v = \int \frac{m_e(\mathbf{v}\cdot\mathbf{u}_e)\mathbf{v}}{v^3 T_e} f_{Me}\,\mathrm{d}^3v$$
$$= \frac{2n_e\mathbf{u}_e}{3\sqrt{2\pi}}\left(\frac{m_e}{T_e}\right)^{5/2}\int_0^\infty \exp\left(-\frac{m_e v^2}{2T_e}\right) v\,\mathrm{d}v = \frac{2n_e}{3\sqrt{2\pi}}\left(\frac{m_e}{T_e}\right)^{3/2}\mathbf{u}_e. \tag{3.31}$$

With these results, equation (3.28) becomes

$$\mathbf{F}_{ei} = n_e m_e \nu_{ei}(\mathbf{u}_i - \mathbf{u}_e), \tag{3.32}$$

where the electron-ion collision frequency is defined to be

$$\nu_{ei} = \frac{4}{3\sqrt{2\pi}}\frac{\gamma_{ei} n_i}{m_e^{1/2} T_e^{3/2}} = \frac{4\sqrt{2\pi}}{3}\frac{Z^2 e^4 n_i \ln\Lambda_{ei}}{(4\pi\epsilon_0)^2 m_e^{1/2} T_e^{3/2}}. \tag{3.33}$$

3.2.2. *Electron-ion collisional energy exchange*

The collisional energy gained or lost by the electrons is

$$W_{ei} = \int \frac{1}{2} m_e v^2\, C_{ei}[f_e, f_i]\,\mathrm{d}^3v. \tag{3.34}$$

Substituting equation (3.17) into this expression, and integrating by parts, we find

$$W_{ei} = -\frac{\gamma_{ei} n_i}{m_e}\int \nabla_v\left(\frac{v^2}{2}\right)\cdot\nabla_v\nabla_v v\cdot\left(\nabla_v f_{e1} + \frac{\mathbf{u}_i}{v}\frac{\partial f_{e0}}{\partial v}\right)\mathrm{d}^3v$$
$$= -\frac{\gamma_{ei} n_i}{m_e}\int \cancelto{0}{\mathbf{v}\cdot\nabla_v\nabla_v v}\cdot\left(\nabla_v f_{e1} + \frac{\mathbf{u}_i}{v}\frac{\partial f_{e0}}{\partial v}\right)\mathrm{d}^3v = 0. \tag{3.35}$$

To this order in the expansion in $\sqrt{m_e/m_i} \ll 1$, there is no exchange of energy. The magnitude of the velocity of the electron barely changes in one collision, and as a result, the transfer of energy is minimal. To calculate the energy transfer, it is better to use the ion-electron collision operator than to expand the electron-ion collision operator to next order in $\sqrt{m_e/m_i} \ll 1$.

4. Ion-electron collision operator

We proceed to calculate the effect on ions of collisions with electrons. We perform an expansion in $\sqrt{m_e/m_i} \ll 1$. To simplify the problem, we assume that the electron distribution function is almost isotropic and hence it can be expanded as in (3.10).

The ion-electron Fokker-Planck collision operator is

$$C_{ie}[f_i, f_e] = \frac{\gamma_{ei}}{m_i} \nabla_v \cdot \left\{ \int \nabla_g \nabla_g g \cdot \left[\underbrace{\frac{f_e(\mathbf{v}')}{m_i} \nabla_v f_i(\mathbf{v})}_{\sim f_i f_e/\sqrt{m_i T_i}} - \underbrace{\frac{f_i(\mathbf{v})}{m_e} \nabla_{v'} f_e(\mathbf{v}')}_{\sim f_i f_e/\sqrt{m_e T_e}} \right] \mathrm{d}^3 v' \right\}. \tag{4.1}$$

The term $m_i^{-1} f_e(\mathbf{v}') \nabla_v f_i(\mathbf{v})$ is then small, and we can use the lowest order approximation $f_e(\mathbf{v}) \simeq f_{e0}(v)$ in it. We also have

$$\mathbf{g} = \underbrace{\mathbf{v}}_{\sim v_{ti}} - \underbrace{\mathbf{v}'}_{\sim v_{te}} \simeq -\mathbf{v}', \tag{4.2}$$

leading to

$$\nabla_g \nabla_g g \equiv \mathbf{M}(\mathbf{g}) = \mathbf{M}(\mathbf{v} - \mathbf{v}') \simeq \mathbf{M}(-\mathbf{v}') + \mathbf{v} \cdot \nabla_{v'} \mathbf{M}(-\mathbf{v}') = \nabla_{v'} \nabla_{v'} v' - \mathbf{v} \cdot \nabla_{v'} \nabla_{v'} \nabla_{v'} v'. \tag{4.3}$$

With these results, equation (4.1) becomes

$$C_{ie}[f_i, f_e] \simeq \frac{\gamma_{ei}}{m_i} \nabla_v \cdot \left\{ \int \left[\frac{f_{e0}(v')}{m_i} \nabla_{v'} \nabla_{v'} v' \cdot \nabla_v f_i(\mathbf{v}) - \frac{f_i(\mathbf{v})}{m_e} \Big(\underbrace{\nabla_{v'} \nabla_{v'} v' \cdot \nabla_{v'} f_{e0}(v')}_{\to 0} \right.\right.$$
$$\left.\left. + \nabla_{v'} \nabla_{v'} v' \cdot \nabla_{v'} f_{e1}(\mathbf{v}') - \mathbf{v} \cdot \nabla_{v'} \nabla_{v'} \nabla_{v'} v' \cdot \nabla_{v'} f_{e0}(v') \Big) \right] \mathrm{d}^3 v' \right\}. \tag{4.4}$$

Using (3.13) and (3.16), we find

$$\nabla_{v'} \nabla_{v'} \nabla_{v'} v' \cdot \nabla_{v'} f_{e0}(v') = -\frac{1}{v'} \frac{\partial f_{e0}(v')}{\partial v'} \nabla_{v'} \nabla_{v'} v'. \tag{4.5}$$

Employing (3.23), we obtain

$$-\frac{\gamma_{ei}}{m_i m_e} \int \nabla_{v'} \nabla_{v'} v' \cdot \nabla_{v'} f_{e1}(\mathbf{v}') \, \mathrm{d}^3 v' = \frac{\mathbf{F}_{ei}}{n_i m_i} + \frac{\gamma_{ei}}{m_i m_e} \mathbf{u}_i \cdot \int \frac{1}{v'} \frac{\partial f_{e0}(v')}{\partial v'} \nabla_{v'} \nabla_{v'} v' \, \mathrm{d}^3 v'. \tag{4.6}$$

With these results, equation (4.4) becomes

$$C_{ie}[f_i, f_e] \simeq \frac{\mathbf{F}_{ei}}{n_i m_i} \cdot \nabla_v f_i + \frac{\gamma_{ei}}{m_i} \nabla_v \cdot \left\{ \int \left[\frac{f_{e0}(v')}{m_i} \nabla_{v'} \nabla_{v'} v' \cdot \nabla_v f_i(\mathbf{v}) \right.\right.$$
$$\left.\left. - \frac{f_i(\mathbf{v})}{m_e v'} \frac{\partial f_{e0}(v')}{\partial v'} (\mathbf{v} - \mathbf{u}_i) \cdot \nabla_{v'} \nabla_{v'} v' \right] \mathrm{d}^3 v' \right\}. \tag{4.7}$$

We finish by taking the integrals in $\mathbf{v}'$. Using (3.25) and $\nabla_{v'} \nabla_{v'} v' = [(v')^2 \mathbf{I} - \mathbf{v}'\mathbf{v}']/(v')^3$, we find

$$\int f_{e0}(v') \nabla_{v'} \nabla_{v'} v' \, \mathrm{d}^3 v' = \frac{8\pi}{3} \mathbf{I} \int_0^\infty f_{e0}(v') v' \, \mathrm{d}v'. \tag{4.8}$$

Using this result and (3.26), equation (4.7) finally becomes

$$C_{ie}[f_i, f_e] \simeq \frac{\mathbf{F}_{ei}}{n_i m_i} \cdot \nabla_v f_i + \frac{8\pi\gamma_{ei} f_{e0}(0)}{3m_i m_e} \nabla_v \cdot \left[\frac{\nabla_v f_i}{m_i f_{e0}(0)} \int_0^\infty f_{e0}(v') m_e v' \, \mathrm{d}v' + (\mathbf{v} - \mathbf{u}_i) f_i \right]. \tag{4.9}$$

If the electron distribution function is a Maxwellian (see (3.18)), this operator simplifies to

$$C_{ie}[f_i, f_e] \simeq \frac{\mathbf{F}_{ei}}{n_i m_i} \cdot \nabla_v f_i + \frac{n_e m_e \nu_{ei}}{n_i m_i} \nabla_v \cdot \left[\frac{T_e}{m_i} \nabla_v f_i + (\mathbf{v} - \mathbf{u}_i) f_i \right], \tag{4.10}$$

where ν_{ei} is defined in (3.33).

We proceed to calculate the collisional friction force and the collisional energy exchange.

4.1. *Ion-electron collisional friction force*

The collisional force on the ions is

$$\mathbf{F}_{ie} = \int m_i \mathbf{v} \, C_{ie}[f_i, f_e] \, \mathrm{d}^3 v. \tag{4.11}$$

Substituting equation (4.9) into this expression, and integrating by parts, we find

$$\mathbf{F}_{ie} = -\frac{\mathbf{F}_{ei}}{n_i} \int f_i \, \mathrm{d}^3 v - \frac{8\pi\gamma_{ei} f_{e0}(0)}{3m_e} \int \left[\frac{\nabla_v f_i}{m_i f_{e0}(0)} \int_0^\infty f_{e0}(v') m_e v' \, \mathrm{d}v' + (\mathbf{v} - \mathbf{u}_i) f_i \right] \mathrm{d}^3 v. \tag{4.12}$$

Using (3.14), the collisional force becomes

$$\mathbf{F}_{ie} = -\mathbf{F}_{ei}, \tag{4.13}$$

as expected.

4.2. *Ion-electron collisional energy exchange*

The collisional energy gained or lost by the ions is

$$W_{ie} = \int \frac{1}{2} m_i v^2 \, C_{ie}[f_i, f_e] \, \mathrm{d}^3 v. \tag{4.14}$$

Substituting equation (4.9) into this expression, and integrating by parts, we find

$$\mathbf{F}_{ie} = -\frac{\mathbf{F}_{ei}}{n_i} \cdot \int f_i \mathbf{v} \, \mathrm{d}^3 v - \frac{8\pi\gamma_{ei} f_{e0}(0)}{3m_e} \int \left[\frac{\mathbf{v} \cdot \nabla_v f_i}{m_i f_{e0}(0)} \int_0^\infty f_{e0}(v') m_e v' \, \mathrm{d}v' + \mathbf{v} \cdot (\mathbf{v} - \mathbf{u}_i) f_i \right] \mathrm{d}^3 v. \tag{4.15}$$

Using that $\int f_i (\mathbf{v} - \mathbf{u}_i) \, \mathrm{d}^3 v = 0$, we can write

$$\int f_i \mathbf{v} \cdot (\mathbf{v} - \mathbf{u}_i) \, \mathrm{d}^3 v = \int f_i |\mathbf{v} - \mathbf{u}_i|^2 \mathrm{d}^3 v. \tag{4.16}$$

By integrating by parts, we obtain

$$\int \mathbf{v} \cdot \nabla_v f_i \, \mathrm{d}^3 v = -\int f_i (\nabla_v \cdot \mathbf{v}) \, \mathrm{d}^3 v = -3 \int f_i \, \mathrm{d}^3 v = -3 n_i. \tag{4.17}$$

With these results, equation (4.15) becomes

$$W_{ie} = -\mathbf{F}_{ei}\cdot\mathbf{u}_i + \frac{8\pi\gamma_{ei} n_i f_{e0}(0)}{m_i m_e}\left(\frac{1}{f_{e0}(0)}\int_0^\infty f_{e0}(v') m_e v'\,\mathrm{d}v' - \frac{1}{n_i}\int f_i \frac{m_i|\mathbf{v}-\mathbf{u}_i|^2}{3}\,\mathrm{d}^3v\right). \tag{4.18}$$

For an electron Maxwellian distribution function (see (3.18)) and an ion Maxwellian distribution function

$$f_i(\mathbf{v}) = f_{Mi}(\mathbf{v}) \equiv n_i\left(\frac{m_i}{2\pi T_i}\right)^{3/2}\exp\left(-\frac{m_i|\mathbf{v}-\mathbf{u}_i|^2}{2T_i}\right), \tag{4.19}$$

the collisional energy exchange becomes

$$W_{ie} = \underbrace{-\mathbf{F}_{ei}\cdot\mathbf{u}_i}_{\text{work done by friction force}} + \frac{3n_e m_e \nu_{ei}}{m_i}(T_e - T_i). \tag{4.20}$$

The first term in (4.20) is the work done by the collisional force $\mathbf{F}_{ie} = -\mathbf{F}_{ei}$ on the ions. The second term is a collisional energy exchange proportional to the temperature difference between electrons and ions. This term will tend to make the ion and electron temperatures equal, but at the slow rate

$$\frac{n_e m_e}{n_i m_i}\nu_{ei} \ll \nu_{ii} \ll \nu_{ee} \sim \nu_{ei}. \tag{4.21}$$

Then, the ions and electrons can have many collisions and their distribution functions become Maxwellians without their temperatures becoming equal. For this reason, it is possible to find plasmas with very different electron and ion temperatures.

Due to energy conservation, the electron energy gain or loss is

$$\begin{aligned} W_{ei} = -W_{ie} &= \mathbf{F}_{ei}\cdot\mathbf{u}_i - \frac{3n_e m_e \nu_{ei}}{m_i}(T_e - T_i) \\ &= \underbrace{\mathbf{F}_{ei}\cdot\mathbf{u}_e}_{\text{work done by friction force}} + \underbrace{\mathbf{F}_{ei}\cdot(\mathbf{u}_i - \mathbf{u}_e)}_{\text{Joule heating}} + \frac{3n_e m_e \nu_{ei}}{m_i}(T_i - T_e). \end{aligned} \tag{4.22}$$

This collisional energy gain has the work done by the friction force $\mathbf{F}_{ei}$ on the electrons, and the energy exchange due to the temperature difference, but in addition to these two terms, it contains Joule heating. This Joule heating term is due to the transfer of energy from the average electron flow to the electron temperature.