The linearized Fokker-Planck collision operator and the Spitzer-Härm problem

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1. Introduction

In many systems of interest, collisions are sufficiently frequent that the distribution function is close to a Maxwellian. When this happens, we can assume the distribution functions to be

$$f_s = f_{Ms} + h_s, \tag{1.1}$$

where f_{Ms} are Maxwellians that make the collision operators of interest vanish (that is, they will have the same average velocity and the same temperature), and $h_s \ll f_{Ms}$ are corrections to the Maxwellians. The corrections h_s are driven by electric and magnetic fields and gradients in density, temperature and flows.

The linearized collision operator is the lowest order result of applying the collision operator to a distribution function of the form shown in (1.1). We proceed to study the properties of the linearized collision operator.

2. Landau form of the linearized Fokker-Planck collision operator

When the masses of species s and s' are comparable, the distribution functions that solve the collision operators for collisions between species s and s' are Maxwellians with the same average velocity $\mathbf{u}$ and temperature T,

$$\begin{aligned} f_{Ms} &= n_s \left(\frac{m_s}{2\pi T} \right)^{3/2} \exp\left(- \frac{m_s |\mathbf{v} - \mathbf{u}|^2}{2T} \right), \\ f_{Ms'} &= n_{s'} \left(\frac{m_{s'}}{2\pi T} \right)^{3/2} \exp\left(- \frac{m_{s'} |\mathbf{v} - \mathbf{u}|^2}{2T} \right). \end{aligned} \tag{2.1}$$

Species with very different masses, e.g., electrons and ions, can have different temperatures. The linearized collision operator for electrons and ions can be deduced from the approximate collision operators given in the notes about collisions between electrons and ions. We will not consider electron-ion collisions again in these notes until we solve the Spitzer-Härm problem.

Using (1.1) and (2.1), and taking into account that the Fokker-Planck collision operator is bilinear, we obtain

$$\begin{aligned} C_{ss'}[f_s, f_{s'}] = C_{ss'}[f_{Ms} + h_s, f_{Ms'} + h_{s'}] &= \cancelto{0}{C_{ss'}[f_{Ms}, f_{Ms'}]} + C_{ss'}[h_s, f_{Ms'}] \\ &+ C_{ss'}[f_{Ms}, h_{s'}] + \cancelto{\text{quadratic} \Rightarrow \text{small}}{C_{ss'}[h_s, h_{s'}]}. \end{aligned} \tag{2.2}$$

Considering the Fokker-Planck collision operator bilinear in f_s and $f_{s'}$ is an approximation because all the distribution functions enter in the Coulomb logarithm $\ln \Lambda_{ss'}$ via densities and temperatures. Since we have neglected order unity corrections to the definition of $\ln \Lambda_{ss'} \gg 1$, keeping very small corrections to the densities and temperatures due to h_s would not be consistent. Thus, as long as $\ln \Lambda_{ss'} \gg 1$, we can ignore the perturbations to the Coulomb logarithm.

Equation (2.2) gives the linearized collision operator,

$$C_{ss'}[f_s, f_{s'}] \simeq C_{ss'}^{(\ell)}[h_s; h_{s'}] \equiv C_{ss'}[h_s, f_{Ms'}] + C_{ss'}[f_{Ms}, h_{s'}]. \tag{2.3}$$

The semi-colon in the linearized collision operator indicates that the arguments are h_s and $h_{s'}$, but the operator is not bilinear – it is composed of two pieces: one linear in h_s and the other linear in $h_{s'}$.

Using the Landau form of the Fokker-Planck collision operator,

$$C_{ss'}[f_s, f_{s'}] = \frac{\gamma_{ss'}}{m_s} \nabla_v \cdot \left\{ \int \nabla_g \nabla_g g \cdot \left[\frac{f_{s'}(\mathbf{v}')}{m_s} \nabla_v f_s(\mathbf{v}) - \frac{f_s(\mathbf{v})}{m_{s'}} \nabla_{v'} f_{s'}(\mathbf{v}') \right] \mathrm{d}^3 v' \right\}, \tag{2.4}$$

the first term in equation (2.3) becomes

$$C_{ss'}[h_s, f_{Ms'}] = \frac{\gamma_{ss'}}{m_s} \nabla_v \cdot \left\{ \int \nabla_g \nabla_g g \cdot \left[\frac{f_{Ms'}(\mathbf{v}')}{m_s} \nabla_v h_s(\mathbf{v}) - \frac{h_s(\mathbf{v})}{m_{s'}} \nabla_{v'} f_{Ms'}(\mathbf{v}') \right] \mathrm{d}^3 v' \right\}. \tag{2.5}$$

Realizing that $\nabla_g \nabla_g g \cdot \mathbf{g} = 0$ and hence $\nabla_g \nabla_g g \cdot \mathbf{v} = \nabla_g \nabla_g g \cdot \mathbf{v}'$, we can write

$$\begin{aligned} -\frac{1}{m_{s'}} \nabla_g \nabla_g g \cdot \nabla_{v'} f_{Ms'}(\mathbf{v}') = \frac{f_{Ms'}(\mathbf{v}')}{T} \nabla_g \nabla_g g \cdot (\mathbf{v}' - \mathbf{u}) = \frac{f_{Ms'}(\mathbf{v}')}{T} \nabla_g \nabla_g g \cdot (\mathbf{v} - \mathbf{u}) \\ = \frac{f_{Ms}(\mathbf{v}) f_{Ms'}(\mathbf{v}')}{m_s} \nabla_g \nabla_g g \cdot \nabla_v \left(\frac{1}{f_{Ms}(\mathbf{v})} \right). \end{aligned} \tag{2.6}$$

With this result, equation (2.5) becomes

$$C_{ss'}[h_s, f_{Ms'}] = \frac{\gamma_{ss'}}{m_s^2} \nabla_v \cdot \left[f_{Ms}(\mathbf{v}) \int f_{Ms'}(\mathbf{v}') \nabla_g \nabla_g g \cdot \nabla_v \left(\frac{h_s(\mathbf{v})}{f_{Ms}(\mathbf{v})} \right) \mathrm{d}^3 v' \right]. \tag{2.7}$$

Using a similar manipulation, we can rewrite the second term in equation (2.3) as

$$C_{ss'}[f_{Ms}, h_{s'}] = -\frac{\gamma_{ss'}}{m_s m_{s'}} \nabla_v \cdot \left[f_{Ms}(\mathbf{v}) \int f_{Ms'}(\mathbf{v}') \nabla_g \nabla_g g \cdot \nabla_{v'} \left(\frac{h_{s'}(\mathbf{v}')}{f_{Ms'}(\mathbf{v}')} \right) \mathrm{d}^3 v' \right]. \tag{2.8}$$

Substituting equations (2.7) and (2.8) into equation (2.3), we find

$$\begin{aligned} C_{ss'}^{(\ell)}[h_s; h_{s'}] = \frac{\gamma_{ss'}}{m_s} \nabla_v \cdot \Bigg\{ f_{Ms}(\mathbf{v}) \int f_{Ms'}(\mathbf{v}') \nabla_g \nabla_g g \cdot \Bigg[\frac{1}{m_s} \nabla_v \left(\frac{h_s(\mathbf{v})}{f_{Ms}(\mathbf{v})} \right) \\ - \frac{1}{m_{s'}} \nabla_{v'} \left(\frac{h_{s'}(\mathbf{v}')}{f_{Ms'}(\mathbf{v}')} \right) \Bigg] \mathrm{d}^3 v' \Bigg\}. \end{aligned} \tag{2.9}$$

With this Landau form of the collision operator, we can easily prove its conservation properties and an H-theorem.

2.1. *Conservation properties*

Following the same procedure that we used for the full Fokker-Planck collision operator (integration by parts and exchange of dummy integration variables $\mathbf{v}$ and $\mathbf{v}'$), one can prove conservation of particles,

$$\int C^{(\ell)}_{ss'}[h_s; h_{s'}]\,\mathrm{d}^3v = 0; \tag{2.10}$$

conservation of momentum,

$$\delta\mathbf{F}_{ss'} + \delta\mathbf{F}_{s's} = 0, \tag{2.11}$$

where

$$\delta\mathbf{F}_{ss'} = \int m_s\mathbf{v}\,C^{(\ell)}_{ss'}[h_s; h_{s'}]\,\mathrm{d}^3v \tag{2.12}$$

is the perturbed collisional friction force on species s due to collisions with species s'; and conservation of energy,

$$\delta W_{ss'} + \delta W_{s's} = 0, \tag{2.13}$$

where

$$\delta W_{ss'} = \int \frac{1}{2}m_s v^2\,C^{(\ell)}_{ss'}[h_s; h_{s'}]\,\mathrm{d}^3v \tag{2.14}$$

is the perturbed collisional energy gain or loss of species s due to collisions with species s'.

2.2. *H-theorem and solutions to the linearized collision operator*

The entropy production of the linearized collision operator can be deduced from the entropy production of the full Fokker-Planck collision operator. We start from

$$\begin{aligned}\dot{\sigma}_{ss'} &= -\int \ln(f_{Ms} + h_s)\,C_{ss'}[f_{Ms} + h_s, f_{Ms'} + h_{s'}]\,\mathrm{d}^3v \\ &= -\int \ln(f_{Ms} + h_s)\left(C^{(\ell)}_{ss'}[h_s; h_{s'}] + C_{ss'}[h_s, h_{s'}]\right)\mathrm{d}^3v.\end{aligned} \tag{2.15}$$

Using

$$\ln(f_{Ms} + h_s) = \ln f_{Ms} + \ln\left(1 + \frac{h_s}{f_{Ms}}\right) = \ln f_{Ms} + \frac{h_s}{f_{Ms}} + O\left(\frac{h_s^2}{f_{Ms}^2}\right) \tag{2.16}$$

and

$$\ln f_{Ms} = \ln\left[n_s\left(\frac{m_s}{2\pi T}\right)^{3/2}\right] - \frac{m_s u^2}{2T} + \frac{m_s\mathbf{v}\cdot\mathbf{u}}{T} - \frac{m_s v^2}{2T}, \tag{2.17}$$

and neglecting terms that are cubic or higher order in h_s and $h_{s'}$, equation (2.15) becomes

$$\begin{aligned}\dot{\sigma}_{ss'} \simeq \frac{1}{T}\int\left(\frac{1}{2}m_s v^2 - m_s\mathbf{v}\cdot\mathbf{u}\right)\left(C^{(\ell)}_{ss'}[h_s; h_{s'}] + C_{ss'}[h_s, h_{s'}]\right)\mathrm{d}^3v \\ - \int \frac{h_s}{f_{Ms}}C^{(\ell)}_{ss'}[h_s; h_{s'}]\,\mathrm{d}^3v.\end{aligned} \tag{2.18}$$

Adding the entropy productions $\dot{\sigma}_{ss'}$ and $\dot{\sigma}_{s's}$, and using the conservation of momentum and energy of the linearized and full collision operators, several terms linear and quadratic in h_s cancel,

$$\begin{aligned}&\frac{1}{T}\int\left(\frac{1}{2}m_s v^2 - m_s\mathbf{v}\cdot\mathbf{u}\right)\left(C^{(\ell)}_{ss'}[h_s; h_{s'}] + C_{ss'}[h_s, h_{s'}]\right)\mathrm{d}^3v \\ &\quad + \frac{1}{T}\int\left(\frac{1}{2}m_{s'} v^2 - m_{s'}\mathbf{v}\cdot\mathbf{u}\right)\left(C^{(\ell)}_{s's}[h_{s'}; h_s] + C_{s's}[h_{s'}, h_s]\right)\mathrm{d}^3v = 0,\end{aligned} \tag{2.19}$$

and we find

$$\dot{\sigma}_{ss'} + \dot{\sigma}_{s's} \simeq \delta\dot{\sigma}_{ss'} + \delta\dot{\sigma}_{s's}, \tag{2.20}$$

where the perturbed entropy productions

$$\delta\dot{\sigma}_{ss'} = -\int \frac{h_s}{f_{Ms}} C^{(\ell)}_{ss'}[h_s; h_{s'}]\,\mathrm{d}^3v \tag{2.21}$$

are only a piece of the total entropy production in (2.18).

From the H-theorem of the full Fokker-Planck collision operator, it is evident that $\delta\dot{\sigma}_{ss'} + \delta\dot{\sigma}_{s's} \geqslant 0$. This property can also be shown using (2.9) and (2.21), integrating by parts and exchanging the dummy integration variables $\mathbf{v}$ and $\mathbf{v}'$ to write

$$\delta\dot{\sigma}_{ss'} + \delta\dot{\sigma}_{s's} = \gamma_{ss'} \int \mathrm{d}^3v \int \mathrm{d}^3v'\, f_{Ms}(\mathbf{v}) f_{Ms'}(\mathbf{v}')\, \mathbf{a}\cdot\nabla_g\nabla_g g\cdot\mathbf{a} \geqslant 0, \tag{2.22}$$

where

$$\mathbf{a} = \frac{1}{m_s}\nabla_v\left(\frac{h_s(\mathbf{v})}{f_{Ms}(\mathbf{v})}\right) - \frac{1}{m_{s'}}\nabla_{v'}\left(\frac{h_{s'}(\mathbf{v}')}{f_{Ms'}(\mathbf{v}')}\right). \tag{2.23}$$

According to (2.22), $\delta\dot{\sigma}_{ss'} + \delta\dot{\sigma}_{s's} = 0$ only when $\mathbf{a} \propto \mathbf{g}$. Following the same procedure that we used for the H-theorem, we can show that $\mathbf{a} \propto \mathbf{g}$ implies

$$\frac{1}{m_s}\nabla_v\left(\frac{h_s(\mathbf{v})}{f_{Ms}(\mathbf{v})}\right) = k\mathbf{v} + \mathbf{c} = \frac{1}{m_{s'}}\nabla_v\left(\frac{h_{s'}(\mathbf{v})}{f_{Ms'}(\mathbf{v})}\right), \tag{2.24}$$

where k and $\mathbf{c}$ are constants. We rename the constants k and $\mathbf{c}$

$$k = \frac{\delta T}{T^2} \text{ and } \mathbf{c} = \frac{\delta\mathbf{u}}{T} - \frac{\delta T\,\mathbf{u}}{T^2}. \tag{2.25}$$

We will see that δT and $\delta\mathbf{u}$ are perturbations to the temperature and the average velocity. Using (2.25) and integrating (2.24), we obtain

$$h_s(\mathbf{v}) = \left[\delta n_s + \frac{m_s\delta\mathbf{u}\cdot(\mathbf{v}-\mathbf{u})}{T} + \frac{\delta T}{T}\left(\frac{m_s|\mathbf{v}-\mathbf{u}|^2}{2T} - \frac{3}{2}\right)\right] f_{Ms}(\mathbf{v}),$$
$$h_{s'}(\mathbf{v}) = \left[\delta n_{s'} + \frac{m_{s'}\delta\mathbf{u}\cdot(\mathbf{v}-\mathbf{u})}{T} + \frac{\delta T}{T}\left(\frac{m_{s'}|\mathbf{v}-\mathbf{u}|^2}{2T} - \frac{3}{2}\right)\right] f_{Ms'}(\mathbf{v}). \tag{2.26}$$

The functions in (2.26) are the only solutions to the linearized Fokker-Planck collision operator because they make the entropy production vanish. They can also be understood as perturbations to the background Maxwellians since

$$(n_s + \delta n_s)\left(\frac{m_s}{2\pi(T+\delta T)}\right)^{3/2} \exp\left(-\frac{m_s|\mathbf{v}-\mathbf{u}-\delta\mathbf{u}|^2}{2(T+\delta T)}\right) \simeq f_{Ms}(\mathbf{v}) + h_s(\mathbf{v}),$$
$$(n_{s'} + \delta n_{s'})\left(\frac{m_{s'}}{2\pi(T+\delta T)}\right)^{3/2} \exp\left(-\frac{m_{s'}|\mathbf{v}-\mathbf{u}-\delta\mathbf{u}|^2}{2(T+\delta T)}\right) \simeq f_{Ms'}(\mathbf{v}) + h_{s'}(\mathbf{v}). \tag{2.27}$$

The perturbation to the velocity and the temperature, $\delta\mathbf{u}$ and δT, are the same for both species.

For like particle collision operators, the entropy production is always positive,

$$\delta\dot{\sigma}_{ss} = -\int \frac{h_s}{f_{Ms}} C^{(\ell)}_{ss}[h_s]\,\mathrm{d}^3v \geqslant 0, \tag{2.28}$$

and it only vanishes for

$$h_s(\mathbf{v}) = \left[\delta n_s + \frac{m_s\delta\mathbf{u}_s\cdot(\mathbf{v}-\mathbf{u}_s)}{T_s} + \frac{\delta T_s}{T_s}\left(\frac{m_s|\mathbf{v}-\mathbf{u}_s|^2}{2T_s} - \frac{3}{2}\right)\right] f_{Ms}(\mathbf{v}). \tag{2.29}$$

The perturbations to the average velocity and the temperature, $\delta\mathbf{u}_s$ and δT_s, do not have to be equal to those of any other species.

3. Isotropy of the linearized collision operator

If we use the relative velocities $\mathbf{w} = \mathbf{v} - \mathbf{u}$ and $\mathbf{w}' = \mathbf{v}' - \mathbf{u}$ in equation (2.9), we find

$$C_{ss'}^{(\ell)}[h_s; h_{s'}] = \frac{\gamma_{ss'}}{m_s} \nabla_w \cdot \left\{ f_{Ms}(w) \int f_{Ms'}(w') \nabla_g \nabla_g g \cdot \left[\frac{1}{m_s} \nabla_w \left(\frac{h_s(\mathbf{w})}{f_{Ms}(w)} \right) \right.\right.$$
$$\left.\left. - \frac{1}{m_{s'}} \nabla_{w'} \left(\frac{h_{s'}(\mathbf{w}')}{f_{Ms'}(w')} \right) \right] \mathrm{d}^3 w' \right\}, \tag{3.1}$$

where $\mathbf{g} = \mathbf{w} - \mathbf{w}'$, and the Maxwellians

$$f_{Ms}(w) = n_s \left(\frac{m_s}{2\pi T} \right)^{3/2} \exp\left(-\frac{m_s w^2}{2T} \right),$$
$$f_{Ms'}(w) = n_{s'} \left(\frac{m_{s'}}{2\pi T} \right)^{3/2} \exp\left(-\frac{m_{s'} w^2}{2T} \right), \tag{3.2}$$

only depend on the magnitude of $\mathbf{w}$, $w = |\mathbf{w}|$. The linearized collision operator in (3.1) does not depend on a particular direction, that is, it is isotropic.

The isotropy of the linearized collision operator can be formally expressed using rotations of the basis of the velocity space. For a given orthonormal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, a rotation gives a new orthonormal basis $\{\hat{\mathbf{e}}_1^R, \hat{\mathbf{e}}_2^R, \hat{\mathbf{e}}_3^R\}$ characterized by

$$\hat{\mathbf{e}}_i^R = \Theta_{ji} \hat{\mathbf{e}}_j, \tag{3.3}$$

where we are using Einstein's convention for repeated indices. The matrix $\boldsymbol{\Theta}$ with components $\Theta_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j^R$ is a rotation matrix, that is, it is an orthogonal matrix,

$$\boldsymbol{\Theta} \cdot \boldsymbol{\Theta}^T = \mathbf{I} \Rightarrow \Theta_{ik} \Theta_{jk} = \delta_{ij}, \tag{3.4}$$

and its determinant is

$$\det(\boldsymbol{\Theta}) = \epsilon_{ijk} \Theta_{1i} \Theta_{2j} \Theta_{3k} = 1, \tag{3.5}$$

where ϵ_{ijk} is the Levi-Civita tensor. Using (3.3), we find that the relation between the coordinates (w_1^b, w_2^b, w_3^b) before the rotation, $\mathbf{w} = w_i^b \hat{\mathbf{e}}_i$, and the coordinates (w_1^a, w_2^a, w_3^a) after the rotation, $\mathbf{w} = w_i^a \hat{\mathbf{e}}_i^R$, is

$$w_i^b = \Theta_{ij} w_j^a. \tag{3.6}$$

The rotation of the basis has not changed the vector $\mathbf{w}$, or the physics of the problem, and hence the equations have to be invariant under this rotation. Due to the isotropy of the collision operator, we only need to consider the rotation in the arguments of the operator, h_s and $h_{s'}$, and not in the operator itself. If the function h_s in the basis before the rotation $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ is $h_s^b(w_1^b, w_2^b, w_3^b)$, the function $h_s^a(w_1^a, w_2^a, w_3^a)$ after the rotation is simply

$$h_s^a(w_1^a, w_2^a, w_3^a) = h_s^b(w_1^b, w_2^b, w_3^b) = h_s^b(\Theta_{1i} w_i^a, \Theta_{2i} w_i^a, \Theta_{3i} w_i^a), \tag{3.7}$$

where we have used (3.6). This expression for $h_s^a(w_1^a, w_2^a, w_3^a)$ can be written in vector form as

$$h_s^a(\mathbf{w}^a) = h_s^b(\boldsymbol{\Theta} \cdot \mathbf{w}^a). \tag{3.8}$$

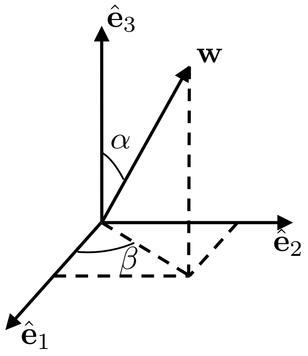

FIGURE 1. Spherical coordinates $\{w, \alpha, \beta\}$.

The isotropy of the linearized collision operator is the fact that

$$C_{ss'}^{(\ell)}\Big[h_s(\boldsymbol{\Theta}\cdot\mathbf{w}); h_{s'}(\boldsymbol{\Theta}\cdot\mathbf{w})\Big](\mathbf{w}) = C_{ss'}^{(\ell)}\Big[h_s(\mathbf{w}); h_{s'}(\mathbf{w})\Big](\boldsymbol{\Theta}\cdot\mathbf{w}) \tag{3.9}$$

for any matrix $\boldsymbol{\Theta}$ that satisfies (3.4) and (3.5) (see Appendix A for a direct proof). Property (3.9) is non-trivial. If the linearized collision operator had depended on a vector different from $\mathbf{w}$, such as a background magnetic field $\mathbf{B}$ or an average velocity $\mathbf{u}$, it would not have been sufficient to rotate the arguments h_s and $h_{s'}$; we would have also had to rotate the background magnetic field $\mathbf{B}$ and the average velocity $\mathbf{u}$.

Some useful properties can be deduced from (3.9). These properties are best understood in the spherical coordinates $\{w, \alpha, \beta\}$ shown in figure 1.

(*a*) **Angular averages.** It is common to take averages over angles around an axis (recall, for example, gyroraverages). This is equivalent to averaging over the angle β. This average can be made using a particular rotation matrix $\boldsymbol{\Theta}$. In the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, the matrix is

$$\boldsymbol{\Theta}(\gamma) = \cos\gamma\,\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \sin\gamma(\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_1\hat{\mathbf{e}}_2) + \cos\gamma\,\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3 = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.10}$$

Under this transformation, $\mathbf{w} = w[\sin\alpha(\cos\beta\,\hat{\mathbf{e}}_1 + \sin\beta\,\hat{\mathbf{e}}_2) + \cos\alpha\,\hat{\mathbf{e}}_3]$ becomes

$$\boldsymbol{\Theta}(\gamma)\cdot\mathbf{w} = w[\sin\alpha(\cos(\beta+\gamma)\,\hat{\mathbf{e}}_1 + \sin(\beta+\gamma)\,\hat{\mathbf{e}}_2) + \cos\alpha\,\hat{\mathbf{e}}_3], \tag{3.11}$$

i.e. the rotation of the basis is equivalent to adding γ to β. Then, we can write an average over β of a function $f(\mathbf{w})$ as

$$\langle f\rangle_\beta = \frac{1}{2\pi}\int_0^{2\pi} f(w,\alpha,\beta)\,\mathrm{d}\beta = \frac{1}{2\pi}\int_0^{2\pi} f(w,\alpha,\beta+\gamma)\,\mathrm{d}\gamma = \frac{1}{2\pi}\int_0^{2\pi} f(\boldsymbol{\Theta}(\gamma)\cdot\mathbf{w})\,\mathrm{d}\gamma, \tag{3.12}$$

that is, the average over β is equivalent to averaging over the rotation of the basis described by $\boldsymbol{\Theta}(\gamma)$. Then, averaging over the linearized collision operator and using equation (3.9), we obtain

$$\begin{aligned}\langle C_{ss'}^{(\ell)}[h_s; h_{s'}]\rangle_\beta &= \frac{1}{2\pi}\int_0^{2\pi} C_{ss'}^{(\ell)}[h_s(\mathbf{w}); h_{s'}(\mathbf{w})](\boldsymbol{\Theta}(\gamma)\cdot\mathbf{w})\,\mathrm{d}\gamma \\ &= \frac{1}{2\pi}\int_0^{2\pi} C_{ss'}^{(\ell)}[h_s(\boldsymbol{\Theta}(\gamma)\cdot\mathbf{w}); h_{s'}(\boldsymbol{\Theta}(\gamma)\cdot\mathbf{w})](\mathbf{w})\,\mathrm{d}\gamma.\end{aligned} \tag{3.13}$$

Since the collision operator does not operate on γ, we can apply the integral over γ

directly over the distribution functions $h_s(\mathbf{\Theta}(\gamma)\cdot\mathbf{w})$ and $h_{s'}(\mathbf{\Theta}(\gamma)\cdot\mathbf{w})$, leading to

$$\langle C_{ss'}^{(\ell)}[h_s;h_{s'}]\rangle_\beta = C_{ss'}^{(\ell)}[\langle h_s\rangle_\beta;\langle h_{s'}\rangle_\beta]. \tag{3.14}$$

(*b*) **Spherical harmonics.** The collision operator is diagonal in the basis of spherical harmonics. The spherical harmonics $Y_l^m(\alpha,\beta)$ are eigenfunctions of the differential operators

$$\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\sin\alpha\frac{\partial}{\partial\alpha}+\frac{1}{\sin^2\alpha}\frac{\partial^2}{\partial\beta^2}, \quad -\mathrm{i}\frac{\partial}{\partial\beta}. \tag{3.15}$$

Indeed,

$$\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha\frac{\partial Y_l^m}{\partial\alpha}\right)+\frac{1}{\sin^2\alpha}\frac{\partial^2 Y_l^m}{\partial\beta^2} = -l(l+1)Y_l^m \tag{3.16}$$

with $l=0,1,2,\ldots$, and

$$-\mathrm{i}\frac{\partial Y_l^m}{\partial\beta} = mY_l^m \tag{3.17}$$

with $m=-l,-l+1,\ldots,-1,0,1,\ldots,l-1,l$. The operators in equation (3.15) are Hermitian, and consequently the spherical harmonics are orthogonal to each other, that is,

$$\int_0^\pi \mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\beta\,\sin\alpha\left[Y_{l'}^{m'}(\alpha,\beta)\right]^* Y_l^m(\alpha,\beta) = 0 \tag{3.18}$$

for $l\neq l'$ and $m\neq m'$. Conventionally, the spherical harmonics are normalized such that

$$\int_0^\pi \mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\beta\,\sin\alpha\left[Y_{l'}^{m'}(\alpha,\beta)\right]^* Y_l^m(\alpha,\beta) = \delta_{ll'}\delta_{mm'}. \tag{3.19}$$

The first few spherical harmonics are

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}, \tag{3.20}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\alpha, \qquad Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\alpha\exp(\pm\mathrm{i}\beta), \tag{3.21}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\alpha-1), \qquad Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\alpha\cos\alpha\exp(\pm\mathrm{i}\beta),$$

$$Y_2^{\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\alpha\exp(\pm2\mathrm{i}\beta), \tag{3.22}$$

To prove that the collision operator is diagonal in the basis of spherical harmonics, we take an infinitesimal rotation,

$$\mathbf{\Theta}\cdot\mathbf{w} \simeq \mathbf{w}+\mathbf{\Omega}\times\mathbf{w}, \tag{3.23}$$

where $\mathbf{\Omega}$ is assumed to be small, $|\mathbf{\Omega}|\ll 1$. Under this rotation, a function $f(\mathbf{w})$ becomes

$$f(\mathbf{\Theta}\cdot\mathbf{w}) \simeq f(\mathbf{w}) + (\mathbf{\Omega}\times\mathbf{w})\cdot\nabla_w f(\mathbf{w}) = f + \mathrm{i}\mathbf{\Omega}\cdot\boldsymbol{\mathcal{M}}f, \tag{3.24}$$

where we have defined the differential operator

$$\boldsymbol{\mathcal{M}} = -\mathrm{i}\mathbf{w}\times\nabla_w. \tag{3.25}$$

This operator is the same as the angular momentum operator of quantum mechanics, $-\mathrm{i}\hbar\mathbf{r}\times\nabla$. Using the same techniques as in quantum mechanics (see, for example, Sakurai 1993; Binney & Skinner 2013), one can derive that the spherical harmonics $Y_l^m(\alpha,\beta)$ are the eigenfunctions of the operators $\hat{\mathbf{e}}_3 \cdot \boldsymbol{\mathcal{M}}$ and $-\boldsymbol{\mathcal{M}} \cdot \boldsymbol{\mathcal{M}}$. For us, it is sufficient to note that

$$-\boldsymbol{\mathcal{M}} \cdot \boldsymbol{\mathcal{M}} = \frac{1}{\sin\alpha} \frac{\partial}{\partial\alpha} \sin\alpha \frac{\partial}{\partial\alpha} + \frac{1}{\sin^2\alpha} \frac{\partial^2}{\partial\beta^2}, \tag{3.26}$$

$$\hat{\mathbf{e}}_3 \cdot \boldsymbol{\mathcal{M}} = -\mathrm{i}\frac{\partial}{\partial\beta}. \tag{3.27}$$

Using (3.24) in (3.9), and keeping terms up to first order in $\boldsymbol{\Omega}$, we obtain

$$\boldsymbol{\mathcal{M}} C_{ss'}^{(\ell)}[h_s; h_{s'}] = C_{ss'}^{(\ell)}[\boldsymbol{\mathcal{M}} h_s; \boldsymbol{\mathcal{M}} h_{s'}], \tag{3.28}$$

that is, the operators $C_{ss'}^{(\ell)}$ and $\boldsymbol{\mathcal{M}}$ commute. This implies that they must have a common basis of eigenfunctions, which is equivalent to saying that

$$C_{ss'}^{(\ell)}[H_s(w) Y_l^m(\alpha,\beta); H_{s'}(w) Y_l^m(\alpha,\beta)] = F(w) Y_l^m(\alpha,\beta). \tag{3.29}$$

The relation between the functions $H_s(w)$, $H_{s'}(w)$ and $F(w)$ is not trivial. To show equation (3.29), we apply $-\boldsymbol{\mathcal{M}} \cdot \boldsymbol{\mathcal{M}}$ and $\hat{\mathbf{e}}_3 \cdot \boldsymbol{\mathcal{M}}$ to $C_{ss'}^{(\ell)}[H_s(w) Y_l^m(\alpha,\beta); H_{s'}(w) Y_l^m(\alpha,\beta)]$. Equation (3.28) implies that the result of applying these operators to the linearized collision operator is equivalent to applying these operators to the arguments of the linearized collision operator. Hence, using the spherical harmonic properties (3.16) and (3.17), we obtain

$$\begin{aligned} &- \boldsymbol{\mathcal{M}} \cdot \boldsymbol{\mathcal{M}} C_{ss'}^{(\ell)}[H_s(w) Y_l^m(\alpha,\beta); H_{s'}(w) Y_l^m(\alpha,\beta)] \\ &\qquad = -l(l+1) C_{ss'}^{(\ell)}[H_s(w) Y_l^m(\alpha,\beta); H_{s'}(w) Y_l^m(\alpha,\beta)], \end{aligned} \tag{3.30}$$

$$\begin{aligned} &\hat{\mathbf{e}}_3 \cdot \boldsymbol{\mathcal{M}} C_{ss'}^{(\ell)}[H_s(w) Y_l^m(\alpha,\beta); H_{s'}(w) Y_l^m(\alpha,\beta)] \\ &\qquad = m C_{ss'}^{(\ell)}[H_s(w) Y_l^m(\alpha,\beta); H_{s'}(w) Y_l^m(\alpha,\beta)], \end{aligned} \tag{3.31}$$

i.e. $C_{ss'}^{(\ell)}[H_s(w) Y_l^m(\alpha,\beta); H_{s'}(w) Y_l^m(\alpha,\beta)]$ has to be proportional to the spherical harmonic Y_l^m, proving equation (3.29).

4. Self-adjointness of the linearized collision operator

For collisions between species s and s', we consider the 2D vectors

$$\mathbf{h}_{ss'}(\mathbf{w}) = \begin{pmatrix} h_s(\mathbf{w}) \\ h_{s'}(\mathbf{w}) \end{pmatrix}. \tag{4.1}$$

For vectors of these form, we can define the scalar product

$$\langle \mathbf{k}_{ss'}, \mathbf{h}_{ss'} \rangle = \int \frac{1}{f_{Ms}(w)} k_s(\mathbf{w}) h_s(\mathbf{w})\, \mathrm{d}^3 w + \int \frac{1}{f_{Ms'}(w)} k_{s'}(\mathbf{w}) h_{s'}(\mathbf{w})\, \mathrm{d}^3 w. \tag{4.2}$$

The collision operator for the 2D vectors in (4.1) is

$$\mathbf{C}_{ss'}^{(\ell)}[\mathbf{h}_{ss'}] = \begin{pmatrix} C_{ss'}^{(\ell)}[h_s; h_{s'}] \\ C_{s's}^{(\ell)}[h_{s'}; h_s] \end{pmatrix}. \tag{4.3}$$

This collision operator is self-adjoint, that is,

$$\langle \mathbf{k}_{ss'}, \mathbf{C}_{ss'}^{(\ell)}[\mathbf{h}_{ss'}] \rangle = \langle \mathbf{C}_{ss'}^{(\ell)}[\mathbf{k}_{s'}], \mathbf{h}_{ss'} \rangle. \tag{4.4}$$

To prove this expression, we use the definition of the scalar product in (4.2) to write

$$\langle \mathbf{k}_{ss'}, \mathbf{C}^{(\ell)}_{ss'}[\mathbf{h}_{ss'}]\rangle = \int \frac{k_s}{f_{Ms}} C^{(\ell)}_{ss'}[h_s; h_{s'}]\, \mathrm{d}^3 w + \int \frac{k_{s'}}{f_{Ms'}} C^{(\ell)}_{s's}[h_{s'}; h_s]\, \mathrm{d}^3 w. \tag{4.5}$$

Using (3.1), integrating by parts, and exchanging the dummy integration variables $\mathbf{w}$ and $\mathbf{w}'$, equation (4.5) becomes

$$\langle \mathbf{k}_{ss'}, \mathbf{C}^{(\ell)}_{ss'}[\mathbf{h}_{ss'}]\rangle = -\gamma_{ss'} \int \mathrm{d}^3 w \int \mathrm{d}^3 w'\, f_{Ms}(w) f_{Ms'}(w') \left[\frac{1}{m_s} \nabla_w \left(\frac{k_s(\mathbf{w})}{f_{Ms}(w)} \right) \right. \\ \left. - \frac{1}{m_{s'}} \nabla_{w'} \left(\frac{k_{s'}(\mathbf{w}')}{f_{Ms'}(w')} \right) \right] \cdot \nabla_g \nabla_g g \cdot \left[\frac{1}{m_s} \nabla_w \left(\frac{h_s(\mathbf{w})}{f_{Ms}(w)} \right) - \frac{1}{m_{s'}} \nabla_{w'} \left(\frac{h_{s'}(\mathbf{w}')}{f_{Ms'}(w')} \right) \right]. \tag{4.6}$$

Since we can swap k_s with h_s and $k_{s'}$ with $h_{s'}$ on the right side of this expression, we have proved (4.4).

For like particle collisions, we do not need to consider the 2D vector space in (4.1). We only consider the perturbed distribution functions $h_s(\mathbf{w})$. The scalar product is simply

$$\langle k_s, h_s\rangle = \int \frac{1}{f_{Ms}(w)} k_s(\mathbf{w}) h_s(\mathbf{w})\, \mathrm{d}^3 w, \tag{4.7}$$

The self-adjointness of the like particle collision operator is

$$\langle k_s, C^{(\ell)}_{ss}[h_s]\rangle = \langle C^{(\ell)}_{ss}[k_s], h_s\rangle. \tag{4.8}$$

The proof of self-adjointness for the like-particle collision operator is slightly different from collisions between different species. Using equation (3.1) and integrating by parts, we obtain

$$\langle k_s, C^{(\ell)}_{ss}[h_s]\rangle = -\frac{\gamma_{ss}}{m_s^2} \int \mathrm{d}^3 w \int \mathrm{d}^3 w'\, f_{Ms}(w) f_{Ms}(w') \left[\nabla_w \left(\frac{h_s(\mathbf{w})}{f_{Ms}(w)} \right) \right. \\ \left. - \nabla_{w'} \left(\frac{h_s(\mathbf{w}')}{f_{Ms}(w')} \right) \right] \cdot \nabla_g \nabla_g g \cdot \nabla_w \left(\frac{k_s(\mathbf{w})}{f_{Ms}(w)} \right). \tag{4.9}$$

This integral can be split into two equal halves, and in the second half, we can exchange the dummy integration variables to obtain the symmetrized form

$$\langle k_s, C^{(\ell)}_{ss}[h_s]\rangle = -\frac{\gamma_{ss}}{2m_s^2} \int \mathrm{d}^3 w \int \mathrm{d}^3 w'\, f_{Ms}(w) f_{Ms}(w') \left[\nabla_w \left(\frac{k_s(\mathbf{w})}{f_{Ms}(w)} \right) \right. \\ \left. - \nabla_{w'} \left(\frac{k_s(\mathbf{w}')}{f_{Ms}(w')} \right) \right] \cdot \nabla_g \nabla_g g \cdot \left[\nabla_w \left(\frac{h_s(\mathbf{w})}{f_{Ms}(w)} \right) - \nabla_{w'} \left(\frac{h_s(\mathbf{w}')}{f_{Ms}(w')} \right) \right]. \tag{4.10}$$

The symmetry of this expression proves (4.8).

5. The Spitzer-Härm problem

The Spitzer-Härm problem is the calculation of the response of a collisional, uniform, steady state quasineutral plasma to an applied electric field in the absence of a magnetic field. Since the electrons are the lightest, most mobile species, the problem reduces to the response of the electrons to this electric field. In particular, we are interested in the electron flow $n_e \mathbf{u}_e = \int f_e \mathbf{v} \, \mathrm{d}^3 v$ that gives the plasma current,

$$\mathbf{J} = \sum_i Z_i e n_i \mathbf{u}_i - e n_e \mathbf{u}_e. \tag{5.1}$$

We assume that there is only one ion species with charge Ze and mass $m_i \gg m_e$. We also assume that the electric field is sufficiently small (we will see how small later) that the electron distribution function can be split into a Maxwellian and a small correction, as shown in (1.1). Without loss of generality, we assume that the velocity $\mathbf{u}_e$ in the electron Maxwellian is zero, $\mathbf{u}_e = 0$,

$$f_{Me}(v) = n_e \left(\frac{m_e}{2\pi T_e} \right)^{3/2} \exp \left(- \frac{m_e v^2}{2T_e} \right), \tag{5.2}$$

and hence the electron distribution function is

$$f_e(\mathbf{v}) = f_{Me}(v) + h_e(\mathbf{v}). \tag{5.3}$$

The kinetic equation for electrons is

$$\underbrace{\frac{\partial f_e}{\partial t}}_{\text{steady state}} + \mathbf{v} \cdot \underbrace{\nabla f_e}_{\text{uniform}} - \frac{e}{m_e} \mathbf{E} \cdot \nabla_v f_e = C_{ee}[f_e, f_e] + C_{ei}[f_e, f_i]. \tag{5.4}$$

Using the expansion of f_e in (5.3), and the expansion in $\sqrt{m_e/m_i} \ll 1$ that we performed in the notes about electron-ion collisions, the electron-ion collision operator becomes

$$C_{ei}[f_e, f_i] \simeq \mathcal{L}_{ei} \left[h_e - \frac{m_e \mathbf{v} \cdot \mathbf{u}_i}{T_e} f_{Me} \right]. \tag{5.5}$$

With this result, equation (5.4) can be rewritten as

$$C_{ee}^{(\ell)}[h_e] + \mathcal{L}_{ei} \left[h_e - \frac{m_e \mathbf{v} \cdot \mathbf{u}_i}{T_e} f_{Me} \right] = - \frac{e}{m_e} \mathbf{E} \cdot \nabla_v f_{Me} = \frac{e \mathbf{E} \cdot \mathbf{v}}{T_e} f_{Me}. \tag{5.6}$$

When we investigated the entropy production of linearized like-collision operators, we deduced that these operators vanish when applied to functions of the form $(m_s \mathbf{v} \cdot \delta \mathbf{u}_s / T_s) f_{Ms}$. Thus, we find

$$C_{ee}^{(\ell)} \left[h_e - \frac{m_e \mathbf{v} \cdot \mathbf{u}_i}{T_e} f_{Me} \right] = C_{ee}^{(\ell)}[h_e] - \underbrace{C_{ee}^{(\ell)} \left[\frac{m_e \mathbf{v} \cdot \mathbf{u}_i}{T_e} f_{Me} \right]}_{0} = C_{ee}^{(\ell)}[h_e]. \tag{5.7}$$

Using this result, equation (5.6) becomes

$$C_{ee}^{(\ell)} \left[h_e - \frac{m_e \mathbf{v} \cdot \mathbf{u}_i}{T_e} f_{Me} \right] + \mathcal{L}_{ei} \left[h_e - \frac{m_e \mathbf{v} \cdot \mathbf{u}_i}{T_e} f_{Me} \right] = \frac{e \mathbf{E} \cdot \mathbf{v}}{T_e} f_{Me}. \tag{5.8}$$

The solution to this equation has two different terms,

$$h_e = \frac{m_e \mathbf{v} \cdot \mathbf{u}_i}{T_e} f_{Me} + f_{e,\mathrm{SH}}, \tag{5.9}$$

where the Spitzer-Härm piece $f_{e,\mathrm{SH}}$ satisfies the equation

$$C_{ee}^{(\ell)}[f_{e,\mathrm{SH}}] + \mathcal{L}_{ei}[f_{e,\mathrm{SH}}] = \frac{e \mathbf{E} \cdot \mathbf{v}}{T_e} f_{Me}. \tag{5.10}$$

If we solve this equation to obtain $f_{e,\mathrm{SH}}$, we can find the electron flow $n_e \mathbf{u}_e = \int f_e \mathbf{v} \, \mathrm{d}^3 v \simeq$

$n_e\mathbf{u}_i + \int f_{e,\mathrm{SH}}\mathbf{v}\,\mathrm{d}^3v$. Then, the plasma current becomes

$$\mathbf{J} = en_e(\mathbf{u}_i - \mathbf{u}_e) \simeq -e\int f_{e,\mathrm{SH}}\mathbf{v}\,\mathrm{d}^3v. \tag{5.11}$$

We proceed to obtain the current in (5.11) using a variational principle. We first present the variational principle, and we then propose a form of the solution. Finally, we obtain an approximate solution for the current $\mathbf{J}$.

5.1. *Variational principle*

Instead of calculating $f_{e,\mathrm{SH}}(\mathbf{v})$ with great accuracy, we will use a variational principle to achieve a much less ambitious objective: to calculate only the current in (5.11) with great accuracy.

Using the scalar product in (4.7), we can show that equation (5.10) has a variational principle. The solution to equation (5.10) is the minimum of the quadratic functional

$$\Sigma[k_e] = -\langle k_e, C_{ee}^{(\ell)}[k_e]\rangle - \langle k_e, \mathcal{L}_{ei}[k_e]\rangle + 2\left\langle k_e, \frac{e\mathbf{E}\cdot\mathbf{v}}{T_e}f_{Me}\right\rangle. \tag{5.12}$$

We proceed to show that equation (5.10) can be derived by setting the lowest order variation of $\Sigma[k_e]$ to zero. Taking the value of Σ for $k_{e,\min} + \delta k_e$ with $\delta k_e \ll k_{e,\min}$, and neglecting terms quadratic in δk_e, we find

$$\begin{aligned}\Sigma[k_{e,\min} + \delta k_e] - \Sigma[k_{e,\min}] \simeq{}& -\langle \delta k_e, C_{ee}^{(\ell)}[k_{e,\min}]\rangle - \langle k_{e,\min}, C_{ee}^{(\ell)}[\delta k_e]\rangle \\ & -\langle \delta k_e, \mathcal{L}_{ei}[k_{e,\min}]\rangle - \langle k_{e,\min}, \mathcal{L}_{ei}[\delta k_e]\rangle + 2\left\langle \delta k_e, \frac{e\mathbf{E}\cdot\mathbf{v}}{T_e}f_{Me}\right\rangle.\end{aligned} \tag{5.13}$$

Using the self-adjointness of $C_{ee}^{(\ell)}$, we find $\langle k_{e,\min}, C_{ee}^{(\ell)}[\delta k_e]\rangle = \langle \delta k_e, C_{ee}^{(\ell)}[k_{e,\min}]\rangle$. The Lorentz operator is also self-adjoint, giving $\langle k_{e,\min}, \mathcal{L}_{ei}[\delta k_e]\rangle = \langle \delta k_e, \mathcal{L}_{ei}[k_{e,\min}]\rangle$. Thus, equation (5.13) becomes

$$\Sigma[k_{e,\min} + \delta k_e] - \Sigma[k_{e,\min}] \simeq -2\left\langle \delta k_e, C_{ee}^{(\ell)}[k_{e,\min}] + \mathcal{L}_{ei}[k_{e,\min}] - \frac{e\mathbf{E}\cdot\mathbf{v}}{T_e}f_{Me}\right\rangle. \tag{5.14}$$

We request that Σ be stationary for any perturbation δk_e. Then, $k_{e,\min}$ must satisfy equation (5.10) and we have

$$k_{e,\min} = f_{e,\mathrm{SH}}. \tag{5.15}$$

Since the variation linear in δk_e vanishes, we need to keep quadratic terms, leading to

$$\begin{aligned}\Sigma[f_{e,\mathrm{SH}} + \delta k_e] - \Sigma[f_{e,\mathrm{SH}}] &= -\langle \delta k_e, C_{ee}^{(\ell)}[\delta k_e]\rangle - \langle \delta k_e, \mathcal{L}_{ei}[\delta k_e]\rangle \\ &= -\int \frac{\delta k_e}{f_{Me}}C_{ee}^{(\ell)}[\delta k_e]\,\mathrm{d}^3v - \int \frac{\delta k_e}{f_{Me}}\mathcal{L}_{ei}[\delta k_e]\,\mathrm{d}^3v \geqslant 0.\end{aligned} \tag{5.16}$$

Note that the quadratic variation is the entropy production due to the perturbation δk_e and it is hence positive. Thus, $f_{e,\mathrm{SH}}$ is indeed the minimum of the functional defined in (5.12).

If we substitute equation (5.10) into (5.12), we find that the minimum of Σ is

$$\Sigma_{\min} = \left\langle f_{e,\mathrm{SH}}, \frac{e\mathbf{E}\cdot\mathbf{v}}{T_e}f_{Me}\right\rangle = \frac{e\mathbf{E}}{T_e}\cdot\int f_{e,\mathrm{SH}}\mathbf{v}\,\mathrm{d}^3v = -\frac{\mathbf{E}\cdot\mathbf{J}}{T_e}. \tag{5.17}$$

Thus, we can obtain the component of the current parallel to $\mathbf{E}$ using the minimum of Σ. Due to the symmetry of the problem, we expect $\mathbf{J}$ to be parallel to the applied electric field. Then, the current can be written using a conductivity σ,

$$\mathbf{J} = -e \int f_{e,\mathrm{SH}} \mathbf{v} \, \mathrm{d}^3 v = \sigma \mathbf{E}. \tag{5.18}$$

The conductivity σ can be obtained from the minimum of Σ in (5.17),

$$\sigma = - \frac{T_e \Sigma_{\min}}{E^2}. \tag{5.19}$$

Calculating σ with a variational principle is advantageous. There is a relatively large region around the solution $f_{e,\mathrm{SH}}$ for which the value of Σ is close to the value $\Sigma_{\min}$ (recall that the minimum is a stationary point). Thus, we can use a bad approximation to $f_{e,\mathrm{SH}}$ to get an accurate value of $\Sigma_{\min}$ and hence of σ.

5.2. *Form of the solution* $f_{e,\mathrm{SH}}$

To propose a form for the solution to (5.10), we use the spherical coordinates $\{v, \alpha, \beta\}$ in figure 1. We align the basis vector $\hat{\mathbf{e}}_3$ with $\mathbf{E}$. Then, equation (5.10) becomes

$$C_{ee}^{(\ell)}[f_{e,\mathrm{SH}}] + \mathcal{L}_{ei}[f_{e,\mathrm{SH}}] = \frac{eEv\cos\alpha}{T_e} f_{Me}. \tag{5.20}$$

Since $\cos\alpha \propto Y_1^0(\alpha, \beta)$, the solution must be of the form

$$f_{e,\mathrm{SH}}(\mathbf{v}) = \frac{eEv\cos\alpha}{T_e} F_{e,\mathrm{SH}}(v) f_{Me}(v) = \frac{e\mathbf{E}\cdot\mathbf{v}}{T_e} F_{e,\mathrm{SH}}(v) f_{Me}(v). \tag{5.21}$$

Integrating this distribution function over velocity space, we find that the current $\mathbf{J} = -e \int f_{e,\mathrm{SH}} \mathbf{v} \, \mathrm{d}^3 v$ is parallel to the electric field, as we predicted in (5.18).

For the function $F_{e,\mathrm{SH}}(v)$, instead of the magnitude v, we use the normalized coordinate

$$x = \frac{m_e v^2}{2T_e}. \tag{5.22}$$

In this normalized coordinate, we describe the function $F_{e,\mathrm{SH}}(v)$ as a series of generalized Laguerre polynomials $L_p^{(\gamma)}(x)$ (also known as Sonine polynomials). These are orthogonal polynomials that satisfy the orthogonality condition

$$\int_0^\infty x^\gamma \exp(-x) L_p^{(\gamma)}(x) L_q^{(\gamma)}(x) \, \mathrm{d}x = \frac{\Gamma(p + \gamma + 1)}{p!} \delta_{pq}, \tag{5.23}$$

where $\Gamma(\nu) = \int_0^\infty x^{\nu - 1} \exp(-x) \, \mathrm{d}x$ is Euler's gamma function. These polynomials have the generating function

$$S_\gamma(\xi, x) \equiv \frac{1}{(1 - \xi)^{\gamma + 1}} \exp\left(- \frac{x\xi}{1 - \xi} \right) = \sum_{p=0}^\infty \xi^p L_p^{(\gamma)}(x). \tag{5.24}$$

The fact that they have this generating function will be useful. The first three polynomials are

$$\begin{aligned} L_0^{(\gamma)}(x) &= 1, \\ L_1^{(\gamma)}(x) &= \gamma + 1 - x, \\ L_2^{(\gamma)}(x) &= \frac{1}{2}[(\gamma + 1)(\gamma + 2) - 2(\gamma + 2)x + x^2]. \end{aligned} \tag{5.25}$$

For the particular case that we are considering, we will use the polynomials with

$\gamma = 3/2$,

$$F_{e,\mathrm{SH}}(v) = \sum_{m=0}^{\infty} a_p L_p^{(3/2)}(x). \tag{5.26}$$

The choice of $\gamma = 3/2$ is consistent with the fact that we are only considering spherical harmonics with $l = 1$. For example, the last term in (5.12) simplifies for $\gamma = 3/2$. Using (5.21), (5.23) and (5.26), we obtain.

$$\begin{aligned}
2\left\langle f_{e,\mathrm{SH}}, \frac{e\mathbf{E}\cdot\mathbf{v}}{T_e} f_{Me}\right\rangle &= \frac{2e\mathbf{E}}{T_e}\cdot\int f_{e,\mathrm{SH}}\mathbf{v}\,\mathrm{d}^3v = \frac{2e^2E^2}{T_e^2}\int v^2\cos^2\alpha F_{e,\mathrm{SH}}(v) f_{Me}(v)\,\mathrm{d}^3v \\
&= \frac{2e^2n_eE^2}{T_e^2}\left(\frac{m_e}{2\pi T_e}\right)^{3/2}\int_0^\infty v^4 F_{e,\mathrm{SH}}(v)\exp\left(-\frac{m_ev^2}{2T_e}\right)\mathrm{d}v\int_0^\pi \cos^2\alpha\sin\alpha\,\mathrm{d}\alpha\int_0^{2\pi}\mathrm{d}\beta \\
&= \frac{8e^2n_eE^2}{3m_eT_e\sqrt{\pi}}\int_0^\infty x^{3/2}\underbrace{L_0^{(3/2)}(x)}_{=1}F_{e,\mathrm{SH}}(v)\exp(-x)\,\mathrm{d}x = \frac{8e^2n_eE^2a_0}{3m_eT_e\sqrt{\pi}}\Gamma(5/2) \\
&= \frac{2e^2n_eE^2}{m_eT_e}a_0.
\end{aligned} \tag{5.27}$$

5.3. *Final form of the variational principle*

Using (5.21) and the decomposition in (5.26), the functional Σ in (5.12) becomes

$$\Sigma[f_{e,\mathrm{SH}}] = \frac{e^2n_eE^2}{m_eT_e}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left(\nu_{ee}K_{pq}^{ee} + \nu_{ei}K_{pq}^{ei}\right)a_pa_q + \frac{2e^2n_eE^2}{m_eT_e}a_0. \tag{5.28}$$

where

$$K_{pq}^{ee} = -\frac{2}{n_e\nu_{ee}}\left\langle x^{1/2}L_p^{(3/2)}(x)f_{Me}(v)\cos\alpha, C_{ee}^{(\ell)}\left[x^{1/2}L_q^{(3/2)}(x)f_{Me}(v)\cos\alpha\right]\right\rangle \tag{5.29}$$

and

$$K_{pq}^{ei} = -\frac{2}{n_e\nu_{ei}}\left\langle x^{1/2}L_p^{(3/2)}(x)f_{Me}(v)\cos\alpha, \mathcal{L}_{ei}\left[x^{1/2}L_q^{(3/2)}(x)f_{Me}(v)\cos\alpha\right]\right\rangle. \tag{5.30}$$

We use the conventional Braginskii definitions of the collision frequencies,

$$\nu_{ee} = \frac{4\sqrt{2\pi}}{3}\frac{e^4n_e\ln\Lambda_{ee}}{(4\pi\epsilon_0)^2m_e^{1/2}T_e^{3/2}},\ \nu_{ei} = \frac{4\sqrt{2\pi}}{3}\frac{Z^2e^4n_i\ln\Lambda_{ei}}{(4\pi\epsilon_0)^2m_e^{1/2}T_e^{3/2}}. \tag{5.31}$$

We still need to calculate the numerical coefficients K_{pq}^{ee} and K_{pq}^{ei}. To do this calculation, it is usually easier to use the generating function in (5.24). We calculate the functions

$$G^{ee}(\xi,\eta) = -\frac{2}{n_e\nu_{ee}}\left\langle x^{1/2}S_{3/2}(\xi,x)f_{Me}(v)\cos\alpha, C_{ee}^{(\ell)}\left[x^{1/2}S_{3/2}(\eta,x)f_{Me}(v)\cos\alpha\right]\right\rangle \tag{5.32}$$

and

$$G^{ei}(\xi,\eta) = -\frac{2}{n_e\nu_{ei}}\left\langle x^{1/2}S_{3/2}(\xi,x)f_{Me}(v)\cos\alpha, \mathcal{L}_{ei}\left[x^{1/2}S_{3/2}(\eta,x)f_{Me}(v)\cos\alpha\right]\right\rangle \tag{5.33}$$

that are relatively easy to evaluate. We then Taylor expand these functions to obtain the coefficients of interest,

$$G^{ee}(\xi,\eta) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}K_{pq}^{ee}\xi^p\eta^q,\ G^{ei}(\xi,\eta) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}K_{pq}^{ei}\xi^p\eta^q. \tag{5.34}$$

In Appendix B, we calculate $G^{ee}(\xi,\eta)$ and $G^{ei}(\xi,\eta)$,

$$G^{ee}(\xi,\eta) = \frac{\xi\eta}{(1-\xi\eta)^2(2-\xi-\eta)^{5/2}}(8-4\xi-4\eta-\xi\eta+2\xi^2\eta+2\xi\eta^2-3\xi^2\eta^2) \tag{5.35}$$

and

$$G^{ei}(\xi,\eta) = \frac{1}{(1-\xi\eta)(1-\xi)^{3/2}(1-\eta)^{3/2}}. \tag{5.36}$$

The first few coefficients of the Taylor expansion of these generating functions are

$$\begin{pmatrix} K^{ee}_{00} & K^{ee}_{01} & K^{ee}_{02} & K^{ee}_{03} & \cdots \\ K^{ee}_{10} & K^{ee}_{11} & K^{ee}_{12} & K^{ee}_{13} & \cdots \\ K^{ee}_{20} & K^{ee}_{21} & K^{ee}_{22} & K^{ee}_{23} & \cdots \\ K^{ee}_{30} & K^{ee}_{31} & K^{ee}_{32} & K^{ee}_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3/4 & 15/32 & \cdots \\ 0 & 3/4 & 45/16 & 309/128 & \cdots \\ 0 & 15/32 & 309/128 & 5657/1024 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5.37}$$

and

$$\begin{pmatrix} K^{ei}_{00} & K^{ei}_{01} & K^{ei}_{02} & K^{ei}_{03} & \cdots \\ K^{ei}_{10} & K^{ei}_{11} & K^{ei}_{12} & K^{ei}_{13} & \cdots \\ K^{ei}_{20} & K^{ei}_{21} & K^{ei}_{22} & K^{ei}_{23} & \cdots \\ K^{ei}_{30} & K^{ei}_{31} & K^{ei}_{32} & K^{ei}_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 3/2 & 15/8 & 35/16 & \cdots \\ 3/2 & 13/4 & 69/16 & 165/32 & \cdots \\ 15/8 & 69/16 & 433/64 & 1077/128 & \cdots \\ 35/16 & 165/32 & 1077/128 & 2957/256 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.38}$$

5.4. *Spitzer-Härm conductivity*

For the truncated solution

$$F_{e,\mathrm{SH}}(v) = a_0 + a_1 L_1^{(3/2)}(x) + a_2 L_2^{(3/2)}(x), \tag{5.39}$$

the functional Σ becomes

$$\Sigma(a_0,a_1,a_2) = \frac{2e^2 n_e E^2}{m_e T_e} a_0 + \frac{e^2 n_e E^2}{m_e T_e}\begin{pmatrix} a_0 & a_1 & a_2 \end{pmatrix}\begin{pmatrix} \nu_{ei} & \frac{3\nu_{ei}}{2} & \frac{15\nu_{ei}}{8} \\ \frac{3\nu_{ei}}{2} & \frac{13\nu_{ei}}{4}+\sqrt{2}\nu_{ee} & \frac{69\nu_{ei}}{16}+\frac{3\sqrt{2}\nu_{ee}}{4} \\ \frac{15\nu_{ei}}{8} & \frac{69\nu_{ei}}{16}+\frac{3\sqrt{2}\nu_{ee}}{4} & \frac{433\nu_{ei}}{64}+\frac{45\sqrt{2}\nu_{ee}}{16} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}. \tag{5.40}$$

Minimizing Σ with respect to a_0, a_1 and a_2, we find the equations

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial\Sigma/\partial a_0 \\ \partial\Sigma/\partial a_1 \\ \partial\Sigma/\partial a_2 \end{pmatrix} = \frac{2e^2 n_e E^2}{m_e T_e}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \nu_{ei} & \frac{3\nu_{ei}}{2} & \frac{15\nu_{ei}}{8} \\ \frac{3\nu_{ei}}{2} & \frac{13\nu_{ei}}{4}+\sqrt{2}\nu_{ee} & \frac{69\nu_{ei}}{16}+\frac{3\sqrt{2}\nu_{ee}}{4} \\ \frac{15\nu_{ei}}{8} & \frac{69\nu_{ei}}{16}+\frac{3\sqrt{2}\nu_{ee}}{4} & \frac{433\nu_{ei}}{64}+\frac{45\sqrt{2}\nu_{ee}}{16} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}\right]. \tag{5.41}$$

For ions with charge $Ze = e$, quasineutrality implies $n_i = n_e$, and the Coulomb logarithm satisfies $\ln \Lambda_{ee} \simeq \ln \Lambda_{ei}$. Hence, $\nu_{ee} \simeq \nu_{ei}$, and we can invert the matrix to find

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -1.95/\nu_{ei} \\ 0.55/\nu_{ei} \\ 0.06/\nu_{ei} \end{pmatrix}. \tag{5.42}$$

Then, the minimum of Σ is

$$\Sigma_{\min} = -\frac{1.95 e^2 n_e E^2}{m_e T_e \nu_{ei}}, \tag{5.43}$$

and using (5.19), we find the Spitzer-Härm conductivity

$$\sigma = \frac{1.95 e^2 n_e}{m_e \nu_{ei}}. \tag{5.44}$$

The plasma current is

$$\mathbf{J} = e n_e (\mathbf{u}_i - \mathbf{u}_e) = \sigma \mathbf{E} = \frac{1.95 e^2 n_e}{m_e \nu_{ei}} \mathbf{E}. \tag{5.45}$$

This equation can be rewritten as

$$e n_e \mathbf{E} = \mathbf{F}_{ei} = 0.51 n_e m_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e). \tag{5.46}$$

The electric field force is balancing the friction force between ions and electrons. Note that the friction force is half the value of the force that we calculated by assuming that the electron and the ion distribution functions are pure Maxwellians, $\mathbf{F}_{ei} = n_e m_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e)$. The reason for this difference is that the electric field tends to accelerate more the more energetic particles because the energetic particles collide much less often, and these energetic particles can carry current more efficiently than the slow particles. The Spitzer-Härm solution has a distribution function with a tail at high energies.

Finally, the Spitzer-Härm solution is of the order of

$$f_{e,\mathrm{SH}} \sim \frac{eEv_{te}}{T_e \nu_{ei}} f_{Me} \sim \frac{eE}{\nu_{ei} m_e v_{te}} f_{Me}. \tag{5.47}$$

We have assumed that $f_{e,\mathrm{SH}} \ll f_{Me}$. For this assumption to be true, the electric field must satisfy

$$\frac{eE}{\nu_{ei}} \ll m_e v_{te}. \tag{5.48}$$

Electrons gain momentum between collisions and after each collision with an ion, they lose their momentum to the ion. Thus, we need to impose that the momentum gained by an electron due to the electric field in the time interval between collisions must be smaller than the typical electron momentum to make sure that the Maxwellian distribution function is not distorted.

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Appendix A. Proof of equation (3.9)

Using equation (3.1), the left side of (3.9) becomes

$$\begin{aligned} & C^{(\ell)}_{ss'}\Big[h_s(\boldsymbol{\Theta}\cdot\mathbf{w}); h_{s'}(\boldsymbol{\Theta}\cdot\mathbf{w})\Big](\mathbf{w}) \\ &= \frac{\gamma_{ss'}}{m_s}\nabla_w\cdot\left\{ f_{Ms}(w)\int f_{Ms'}(w')\nabla_g\nabla_g g\cdot\left[\frac{1}{m_s}\nabla_w\left(\frac{h_s(\boldsymbol{\Theta}\cdot\mathbf{w})}{f_{Ms}(w)}\right)\right.\right. \\ &\left.\left. - \frac{1}{m_{s'}}\nabla_{w'}\left(\frac{h_{s'}(\boldsymbol{\Theta}\cdot\mathbf{w}')}{f_{Ms'}(w')}\right)\right]\mathrm{d}^3w'\right\}. \qquad \text{(A 1)} \end{aligned}$$

We change variables to $\mathbf{w}_R = \boldsymbol{\Theta}\cdot\mathbf{w}$ and $\mathbf{w}'_R = \boldsymbol{\Theta}\cdot\mathbf{w}'$. Then, $\mathbf{g}_R = \mathbf{w}_R - \mathbf{w}'_R = \boldsymbol{\Theta}\cdot(\mathbf{w}-\mathbf{w}') = \boldsymbol{\Theta}\cdot\mathbf{g}$. Note as well that according to (3.4), $w = w_R$, $w' = w'_R$ and $g = g_R$. Using these results and Einstein's repeated index convention,

$$\begin{aligned} & \nabla_w\cdot\left[f_{Ms}(w)\nabla_g\nabla_g g\cdot\nabla_w\left(\frac{h_s(\boldsymbol{\Theta}\cdot\mathbf{w})}{f_{Ms}(w)}\right)\right] = \frac{\partial}{\partial w_i}\left[f_{Ms}(w)\frac{\partial^2 g}{\partial g_i\partial g_j}\frac{\partial}{\partial w_j}\left(\frac{h_s(\boldsymbol{\Theta}\cdot\mathbf{w})}{f_{Ms}(w)}\right)\right] \\ &= \frac{\partial w_{R,k}}{\partial w_i}\frac{\partial}{\partial w_{R,k}}\left[f_{Ms}(w_R)\frac{\partial g_{R,l}}{\partial g_i}\frac{\partial}{\partial g_{R,l}}\left(\frac{\partial g_{R,m}}{\partial g_j}\frac{\partial g_R}{\partial g_{R,m}}\right)\frac{\partial w_{R,p}}{\partial w_j}\frac{\partial}{\partial w_{R,p}}\left(\frac{h_s(\mathbf{w}_R)}{f_{Ms}(w_R)}\right)\right] \\ &= \Theta_{ki}\frac{\partial}{\partial w_{R,k}}\left[f_{Ms}(w_R)\Theta_{li}\Theta_{mj}\frac{\partial^2 g_R}{\partial g_{R,l}\partial g_{R,m}}\Theta_{pj}\frac{\partial}{\partial w_{R,p}}\left(\frac{h_s(\mathbf{w}_R)}{f_{Ms}(w_R)}\right)\right]. \qquad \text{(A 2)} \end{aligned}$$

Equation (3.4) implies that $\Theta_{ik}\Theta_{jk} = \delta_{ij}$, and hence equation (A 2) becomes

$$\begin{aligned} & \nabla_w\cdot\left[f_{Ms}(w)\nabla_g\nabla_g g\cdot\nabla_w\left(\frac{h_s(\boldsymbol{\Theta}\cdot\mathbf{w})}{f_{Ms}(w)}\right)\right] \\ &= \frac{\partial}{\partial w_{R,k}}\left[f_{Ms}(w_R)\frac{\partial^2 g_R}{\partial g_{R,k}\partial g_{R,m}}\frac{\partial}{\partial w_{R,m}}\left(\frac{h_s(\mathbf{w}_R)}{f_{Ms}(w_R)}\right)\right] \\ &= \nabla_{w_R}\cdot\left[f_{Ms}(w_R)\nabla_{g_R}\nabla_{g_R}g_R\cdot\nabla_{w_R}\left(\frac{h_s(\mathbf{w}_R)}{f_{Ms}(w_R)}\right)\right]. \qquad \text{(A 3)} \end{aligned}$$

A similar manipulation gives

$$\begin{aligned} & \nabla_w\cdot\left[f_{Ms}(w)\nabla_g\nabla_g g\cdot\nabla_{w'}\left(\frac{h_{s'}(\boldsymbol{\Theta}\cdot\mathbf{w}')}{f_{Ms'}(w')}\right)\right] \\ &= \nabla_{w_R}\cdot\left[f_{Ms}(w_R)\nabla_{g_R}\nabla_{g_R}g_R\cdot\nabla_{w'_R}\left(\frac{h_{s'}(\mathbf{w}'_R)}{f_{Ms'}(w'_R)}\right)\right]. \qquad \text{(A 4)} \end{aligned}$$

Finally, equation (3.5) implies that

$$\mathrm{d}^3w' = |\det(\boldsymbol{\Theta})|^{-1}\,\mathrm{d}^3w'_R = \mathrm{d}^3w'_R. \qquad \text{(A 5)}$$

Substituting (A 3), (A 4) and (A 5) into (A 1), we obtain

$$
\begin{aligned}
&C_{ss'}^{(\ell)}\Big[h_s(\boldsymbol{\Theta}\cdot\mathbf{w});h_{s'}(\boldsymbol{\Theta}\cdot\mathbf{w})\Big](\mathbf{w}) \\
&\quad=\frac{\gamma_{ss'}}{m_s}\nabla_{w_R}\cdot\left\{f_{Ms}(w_R)\int f_{Ms'}(w_R')\nabla_{g_R}\nabla_{g_R}g_R\cdot\left[\frac{1}{m_s}\nabla_{w_R}\left(\frac{h_s(\mathbf{w}_R)}{f_{Ms}(w_R)}\right)\right.\right. \\
&\quad\left.\left.-\frac{1}{m_{s'}}\nabla_{w_R'}\left(\frac{h_{s'}(\mathbf{w}_R')}{f_{Ms'}(w_R')}\right)\right]\mathrm{d}^3w_R'\right\}=C_{ss'}^{(\ell)}\Big[h_s(\mathbf{w});h_{s'}(\mathbf{w})\Big](\mathbf{w}_R) \\
&\quad=C_{ss'}^{(\ell)}\Big[h_s(\mathbf{w});h_{s'}(\mathbf{w})\Big](\boldsymbol{\Theta}\cdot\mathbf{w}),
\end{aligned}
\tag{A 6}
$$

proving (3.9).

Appendix B. Calculation of the generating functions $G^{ee}(\xi,\eta)$ and $G^{ei}(\xi,\eta)$

In this appendix, we calculate the generating functions $G^{ee}(\xi,\eta)$ and $G^{ei}(\xi,\eta)$, defined in (5.32) and (5.33).

B.1. *Generating functions* $G^{ee}(\xi,\eta)$

To calculate $G^{ee}(\xi,\eta)$, defined in (5.32), we use equation (4.10). Thus, we need to evaluate $\nabla_v[x^{1/2}S_{3/2}(\xi,x)\cos\alpha]$. We write this gradient as

$$
\nabla_v[x^{1/2}S_{3/2}(\xi,x)\cos\alpha]=\sqrt{\frac{m_e}{2T_e}}\nabla_v[S_{3/2}(\xi,x)\mathbf{v}\cdot\hat{\mathbf{e}}_3]=\sqrt{\frac{m_e}{2T_e}}S_{3/2}(\xi,x)\mathbf{S}(\xi,\mathbf{v}),
\tag{B 1}
$$

where

$$
\mathbf{S}(\xi,\mathbf{v})=\hat{\mathbf{e}}_3-\frac{\xi}{1-\xi}\frac{m_e(\mathbf{v}\cdot\hat{\mathbf{e}}_3)\mathbf{v}}{T_e}.
\tag{B 2}
$$

Using this expression for $\nabla_v[x^{1/2}S_{3/2}(\xi,x)\cos\alpha]$, equation (4.10) and the definition of ν_{ee} in (5.31), equation (5.32) becomes

$$
\begin{aligned}
G^{ee}(\xi,\eta)=&\frac{3\sqrt{\pi}}{8}\sqrt{\frac{2T_e}{m_e}}\int\mathrm{d}^3v\int\mathrm{d}^3v'\,f_{Me}(v)f_{Me}(v') \\
&\times[S_{3/2}(\xi,x)S_{3/2}(\eta,x)\mathbf{S}(\xi,\mathbf{v})\cdot\nabla_g\nabla_g g\cdot\mathbf{S}(\eta,\mathbf{v}) \\
&-S_{3/2}(\xi,x)S_{3/2}(\eta,x')\mathbf{S}(\xi,\mathbf{v})\cdot\nabla_g\nabla_g g\cdot\mathbf{S}(\eta,\mathbf{v}') \\
&-S_{3/2}(\xi,x')S_{3/2}(\eta,x)\mathbf{S}(\xi,\mathbf{v}')\cdot\nabla_g\nabla_g g\cdot\mathbf{S}(\eta,\mathbf{v}) \\
&+S_{3/2}(\xi,x')S_{3/2}(\eta,x')\mathbf{S}(\xi,\mathbf{v}')\cdot\nabla_g\nabla_g g\cdot\mathbf{S}(\eta,\mathbf{v}')],
\end{aligned}
\tag{B 3}
$$

where $x'=m_ev'^2/2T_e$. Exchanging the dummy integration variables $\mathbf{v}$ and $\mathbf{v}'$ in the last two terms inside the square brackets, we reduce the calculation of $G^{ee}(\xi,\eta)$ to evaluating two integrals, that is,

$$
G^{ee}(\xi,\eta)=\frac{1}{(1-\xi)^{5/2}(1-\eta)^{5/2}}\left(I_1^{ee}+I_2^{ee}\right),
\tag{B 4}
$$

where

$$
\begin{aligned}
I_1^{ee}=\frac{3}{4\pi^{5/2}}\left(\frac{m_e}{2T_e}\right)^{5/2}\int\mathrm{d}^3v\int\mathrm{d}^3v'\,\exp\left(-\frac{1-\xi\eta}{(1-\xi)(1-\eta)}\frac{m_ev^2}{2T_e}-\frac{m_e(v')^2}{2T_e}\right) \\
\times\mathbf{S}(\xi,\mathbf{v})\cdot\nabla_g\nabla_g g\cdot\mathbf{S}(\eta,\mathbf{v})
\end{aligned}
\tag{B 5}
$$

and

$$I_2^{ee} = -\frac{3}{4\pi^{5/2}} \left(\frac{m_e}{2T_e} \right)^{5/2} \int \mathrm{d}^3 v \int \mathrm{d}^3 v' \, \exp\left(- \frac{1}{1-\xi} \frac{m_e v^2}{2T_e} - \frac{1}{1-\eta} \frac{m_e (v')^2}{2T_e} \right) \times \mathbf{S}(\xi, \mathbf{v}) \cdot \nabla_g \nabla_g g \cdot \mathbf{S}(\eta, \mathbf{v}'). \quad \text{(B 6)}$$

Integral (B 5) can be written as

$$I_1^{ee} = \frac{3}{4\pi^{5/2}} \left(\frac{m_e}{2T_e} \right)^{5/2} \int \mathrm{d}^3 v \int \mathrm{d}^3 v' \, \exp\left(- \frac{m_1 v^2}{2T_e} - \frac{m_e (v')^2}{2T_e} \right) \times \mathbf{S}(\xi, \mathbf{v}) \cdot \nabla_g \nabla_g g \cdot \mathbf{S}(\eta, \mathbf{v}), \quad \text{(B 7)}$$

with

$$m_1 = \frac{1 - \xi\eta}{(1-\xi)(1-\eta)} m_e. \quad \text{(B 8)}$$

To calculate the integral (B 7), we change the integration variables to $\mathbf{U}$ and $\mathbf{g}$, defined by

$$\mathbf{v} = \mathbf{U} + \frac{m_e}{m_1 + m_e} \mathbf{g} \quad \text{(B 9)}$$

and

$$\mathbf{v}' = \mathbf{U} - \frac{m_1}{m_1 + m_e} \mathbf{g}. \quad \text{(B 10)}$$

Using $\mathbf{g} \cdot \nabla_g \nabla_g g = 0$, the change to the variables $\mathbf{U}$ and $\mathbf{g}$ leaves

$$I_1^{ee} = \frac{3}{4\pi^{5/2}} \left(\frac{m_e}{2T_e} \right)^{5/2} \int \mathrm{d}^3 U \int \mathrm{d}^3 g \, \exp\left(- \frac{(m_1 + m_e) U^2}{2T_e} - \frac{m_1 m_e}{m_1 + m_e} \frac{g^2}{2T_e} \right) \times \mathbf{S}_1(\xi, \mathbf{U}, \mathbf{g}) \cdot \left(\frac{\mathbf{I}}{g} - \frac{\mathbf{g}\mathbf{g}}{g^3} \right) \cdot \mathbf{S}_1(\eta, \mathbf{U}, \mathbf{g}), \quad \text{(B 11)}$$

with

$$\mathbf{S}_1(\xi, \mathbf{U}, \mathbf{g}) = \hat{\mathbf{e}}_3 - \frac{\xi}{1-\xi} \frac{m_e (\mathbf{U} \cdot \hat{\mathbf{e}}_3) \mathbf{U}}{T_e} - \frac{\xi}{1-\xi} \frac{m_e^2}{m_1 + m_e} \frac{(\mathbf{g} \cdot \hat{\mathbf{e}}_3) \mathbf{U}}{T_e}. \quad \text{(B 12)}$$

This integral can be calculated analytically because it is composed of moments of Maxwellians in $\mathbf{U}$ and $\mathbf{g}$. Calculating first the moments in $\mathbf{g}$ and using

$$\int \exp\left(- \frac{m_1 m_e}{m_1 + m_e} \frac{g^2}{2T_e} \right) \left(\frac{\mathbf{I}}{g} - \frac{\mathbf{g}\mathbf{g}}{g^3} \right) \mathrm{d}^3 g = \frac{8\pi (m_1 + m_e) T_e}{3 m_1 m_e} \mathbf{I},$$

$$\int \exp\left(- \frac{m_1 m_e}{m_1 + m_e} \frac{g^2}{2T_e} \right) \left(\frac{\mathbf{I}}{g} - \frac{\mathbf{g}\mathbf{g}}{g^3} \right) (\mathbf{g} \cdot \hat{\mathbf{e}}_3) \, \mathrm{d}^3 g = 0,$$

$$\int \exp\left(- \frac{m_1 m_e}{m_1 + m_e} \frac{g^2}{2T_e} \right) \left(\frac{\mathbf{I}}{g} - \frac{\mathbf{g}\mathbf{g}}{g^3} \right) (\mathbf{g} \cdot \hat{\mathbf{e}}_3)^2 \, \mathrm{d}^3 g = \frac{32\pi (m_1 + m_e)^2 T_e^2}{15 m_1^2 m_e^2} \left(\mathbf{I} - \frac{1}{2} \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3 \right), \quad \text{(B 13)}$$

we are left with the following moments in $\mathbf{U}$:

$$\begin{aligned} \int \exp\left(-\frac{(m_1+m_e)U^2}{2T_e}\right) \mathrm{d}^3U &= \pi^{3/2}\left(\frac{2T_e}{m_1+m_e}\right)^{3/2}, \\ \int \exp\left(-\frac{(m_1+m_e)U^2}{2T_e}\right) U^2\,\mathrm{d}^3U &= \frac{3\pi^{3/2}}{2}\left(\frac{2T_e}{m_1+m_e}\right)^{5/2}, \\ \int \exp\left(-\frac{(m_1+m_e)U^2}{2T_e}\right) (\mathbf{U}\cdot\hat{\mathbf{e}}_3)^2\,\mathrm{d}^3U &= \frac{\pi^{3/2}}{2}\left(\frac{2T_e}{m_1+m_e}\right)^{5/2}, \\ \int \exp\left(-\frac{(m_1+m_e)U^2}{2T_e}\right) U^2(\mathbf{U}\cdot\hat{\mathbf{e}}_3)^2\,\mathrm{d}^3U &= \frac{5\pi^{3/2}}{4}\left(\frac{2T_e}{m_1+m_e}\right)^{7/2}. \end{aligned} \tag{B 14}$$

With these integrals, the final result for I_1^{ee} is

$$\begin{aligned} I_1^{ee} =& \frac{m_1+m_e}{m_1}\left(\frac{m_e}{m_1+m_e}\right)^{3/2}\left[1-\frac{\xi+\eta-2\xi\eta}{(1-\xi)(1-\eta)}\frac{m_e}{m_1+m_e}\right. \\ &\left.+\frac{5\xi\eta}{(1-\xi)(1-\eta)}\left(\frac{m_e}{m_1+m_e}\right)^2+\frac{2\xi\eta}{(1-\xi)(1-\eta)}\frac{m_e^3}{m_1(m_1+m_e)^2}\right]. \end{aligned} \tag{B 15}$$

Using (B 8), we find

$$I_1^{ee} = \frac{(1-\xi)^{5/2}(1-\eta)^{5/2}}{(1-\xi\eta)^2(2-\xi-\eta)^{5/2}}(4-2\xi-2\eta+3\xi\eta-3\xi^2\eta^2). \tag{B 16}$$

Integral (B 6) can be obtained using a similar method. We write it as

$$\begin{aligned} I_2^{ee} = -\frac{3}{4\pi^{5/2}}\left(\frac{m_e}{2T_e}\right)^{5/2}\int \mathrm{d}^3v \int \mathrm{d}^3v'\,\exp\left(-\frac{m_3v^2}{2T_e}-\frac{m_4(v')^2}{2T_e}\right) \\ \times\mathbf{S}(\xi,\mathbf{v})\cdot\nabla_g\nabla_g g\cdot\mathbf{S}(\eta,\mathbf{v}'), \end{aligned} \tag{B 17}$$

with

$$m_3 = \frac{1}{1-\xi}m_e \tag{B 18}$$

and

$$m_4 = \frac{1}{1-\eta}m_e. \tag{B 19}$$

To calculate the integral (B 17), we change the integration variables to $\mathbf{U}$ and $\mathbf{g}$, defined by

$$\mathbf{v} = \mathbf{U} + \frac{m_4}{m_3+m_4}\mathbf{g} \tag{B 20}$$

and

$$\mathbf{v}' = \mathbf{U} - \frac{m_3}{m_3+m_4}\mathbf{g}. \tag{B 21}$$

Using $\mathbf{g}\cdot\nabla_g\nabla_g g = 0$, the change of variables to $\mathbf{U}$ and $\mathbf{g}$ leaves

$$I_2^{ee} = -\frac{3}{4\pi^{5/2}}\left(\frac{m_e}{2T_e}\right)^{5/2}\int \mathrm{d}^3U \int \mathrm{d}^3g \, \exp\left(- \frac{(m_3+m_4)U^2}{2T_e} - \frac{m_3 m_4}{m_3+m_4}\frac{g^2}{2T_e}\right)$$
$$\times \mathbf{S}_3(\xi,\mathbf{U},\mathbf{g})\cdot\left(\frac{\mathbf{I}}{g} - \frac{\mathbf{gg}}{g^3}\right)\cdot\mathbf{S}_4(\eta,\mathbf{U},\mathbf{g}). \quad \text{(B 22)}$$

with

$$\mathbf{S}_3(\xi,\mathbf{U},\mathbf{g}) = \hat{\mathbf{e}}_3 - \frac{\xi}{1-\xi}\frac{m_e(\mathbf{U}\cdot\hat{\mathbf{e}}_3)\mathbf{U}}{T_e} - \frac{\xi}{1-\xi}\frac{m_e m_4}{m_3+m_4}\frac{(\mathbf{g}\cdot\hat{\mathbf{e}}_3)\mathbf{U}}{T_e} \quad \text{(B 23)}$$

and

$$\mathbf{S}_4(\eta,\mathbf{U},\mathbf{g}) = \hat{\mathbf{e}}_3 - \frac{\eta}{1-\eta}\frac{m_e(\mathbf{U}\cdot\hat{\mathbf{e}}_3)\mathbf{U}}{T_e} + \frac{\eta}{1-\eta}\frac{m_e m_3}{m_3+m_4}\frac{(\mathbf{g}\cdot\hat{\mathbf{e}}_3)\mathbf{U}}{T_e}. \quad \text{(B 24)}$$

This integral can be done analytically because it is composed of integrals very similar to those given in equations (B 13) and (B 14). The final result is

$$I_2^{ee} = -\frac{m_e(m_3+m_4)}{m_3 m_4}\left(\frac{m_e}{m_3+m_4}\right)^{3/2}\left[1 - \frac{\xi+\eta-2\xi\eta}{(1-\xi)(1-\eta)}\frac{m_e}{m_3+m_4}\right.$$
$$\left. + \frac{3\xi\eta}{(1-\xi)(1-\eta)}\left(\frac{m_e}{m_3+m_4}\right)^2\right]. \quad \text{(B 25)}$$

Using (B 18) and (B 19), we find

$$I_2^{ee} = -\frac{2(1-\xi)^{5/2}(1-\eta)^{5/2}}{(2-\xi-\eta)^{5/2}}(4-2\xi-2\eta+3\xi\eta). \quad \text{(B 26)}$$

Summing I_1^{ee} and I_2^{ee} and using equation (B 4), we finally obtain equation (5.35).

B.2. *Generating functions $G^{ei}(\xi,\eta)$*

To calculate the generating function $G^{ei}(\xi,\eta)$, defined in (5.33), we use the expression

$$-\int \frac{f_e}{f_{Me}}\mathcal{L}_{ei}[h_e]\,\mathrm{d}^3v = \frac{3\sqrt{\pi}}{8}\nu_{ei}\left(\frac{2T_e}{m_e}\right)^{3/2}\int f_{Me}\nabla_v\left(\frac{f_e}{f_{Me}}\right)\cdot\nabla_v\nabla_v v\cdot\nabla_v\left(\frac{h_e}{f_{Me}}\right)\mathrm{d}^3v, \quad \text{(B 27)}$$

where ν_{ei} is defined in (5.31). With this expression and equation (B 1), we obtain

$$G^{ei}(\xi,\eta) = \frac{3m_e}{8\pi T_e}\frac{1}{(1-\xi)^{5/2}(1-\eta)^{5/2}}\int \exp\left(-\frac{1-\xi\eta}{(1-\xi)(1-\eta)}\frac{m_e v^2}{2T_e}\right)\frac{v^2-(\mathbf{v}\cdot\hat{\mathbf{e}}_3)^2}{v^3}\,\mathrm{d}^3v. \quad \text{(B 28)}$$

Integrating this equation, we finally find equation (5.36)