Mathematical Methods (Second Year) MT 2009

Problem Set 4: ODEs

1. *Legendre polynomials*

   Position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are such that $r_2 \gg r_1$, where $r_1 = |\mathbf{r}_1|$ and $r_2 = |\mathbf{r}_2|$. Show that

$$\frac{1}{\mid \mathbf{r}_2 - \mathbf{r}_1 \mid} = \frac{1}{r_2}\left\{1 + \left(\frac{r_1}{r_2}\right) P_1(\cos\theta_{12}) + \left(\frac{r_1}{r_2}\right)^2 P_2(\cos\theta_{12}) + ...\right\}$$

   where θ_{12} is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$, and $P_1(\cos\theta) = \cos\theta, P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$.

   An electric quadrupole is formed by charges Q at coordinates $(0, \pm a, 0)$ and charges $-Q$ at coordinates $(\pm a, 0, 0)$. Show that the potential V in the (x, y) plane at a distance r large compared with a is approximately

$$V = \frac{-3Qa^2 \cos 2\theta}{4\pi\varepsilon_0 r^3},$$

   where θ is the angle between $\mathbf{r}$ and the x-axis.

   Derive an expression for the couple exerted on the quadrupole by a positive point charge Q at a position $\mathbf{r}$ in the (x, y) plane, where $r >> a$.

   Deduce the angles θ for which this couple is zero. If the charges of the quadrupole are rigidly connected and free to rotate about the z-axis, determine whether the equilibrium is stable or unstable in each case.

2. *Orthogonal, normalized, eigenfunctions*

   The real functions $u_n(x)(n = 1 \text{ to } \infty)$ are an orthogonal, normalized, set on the interval (a, b). The function $f(x)$ is expressed as a linear combination of the $u_n(x)$'s via

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x).$$

   Show

   (i)

$$a_n = \int_a^b u_n(x) f(x) dx;$$

   (ii)

$$\int_a^b [f(x)]^2 dx = \sum_{n=1}^{\infty} a_n^2.$$

   [Hint for this part: the left hand side is, when written out in long-hand notation,

$$\int_b^a (a_1u_1(x) + a_2u_2(x) + ...)(a_1u_1(x) + a_2u_2(x) + ...)dx = \int_a^b \{a_1^2[u_1(x)]^2 + a_2^2[u_2(x)]^2 + ...$$

$$+2a_1a_2u_1(x)u_2(x) + ...\}dx.$$

   Why do the $\int [u_n(x)]^2 dx$ terms each give 1? Why do the $\int u_n(x).u_m(x)dx$ terms with $n \neq m$ each give 0?]

3. *Hermiticity*

Consider the set of functions $\{f(x)\}$ of the real variable x defined on the interval $-\infty < x < \infty$ that go to zero faster than $1/x$ for $x \to \pm\infty$, i.e.

$$\lim_{x \to \pm\infty} x f(x) = 0. \tag{1}$$

For unit weight function, determine which of the following linear operators is Hermitian when acting upton $\{f(x)\}$: (a) $\frac{d}{dx} + x$ (b) $-i\frac{d}{dx} + x^2$ (c) $ix\frac{d}{dx}$ (d) $i\frac{d^3}{dx^3}$

4. *More Hermiticity*

Recall that an operator $\hat{H}$ is hermitian if

$$\int dx\ v^*(x)\ \hat{H}\ u(x) = \left[\int dx\ u^*(x)\ \hat{H}\ v(x)\right]^* = \int dx\ u(x)\ (\hat{H}\ v(x))^*\ . \tag{2}$$

The action of the hermitian conjugate of an operator A is defined as

$$\int dx\ u^*(x) A^\dagger\ v(x) = \int dx\ (Au(x))^* v(x). \tag{3}$$

(a) Let A be a non-hermitian operator. Show that $A + A^\dagger$ and $i(A - A^\dagger)$ are hermitian operators.

(b) Using the preceding result, show that every non-hermitian operator may be written as a linear combination of two hermitian operators.

5. *Eigenvalues and Eigenfunctions*

By substituting $x = e^t$, find the normalized eigenfunctions $y_n(x)$ and the eigenvalues λ_n of the operator $\hat{\mathcal{L}}$ defined by

$$\hat{\mathcal{L}} y = x^2 y'' + 2xy' + \frac{1}{4} y\ , \qquad 1 \le x \le e, \tag{4}$$

with boundary conditions $y(1) = y(e) = 0$.

6. *Orthogonality*

In problem set 2 you showed that N orthogonal vectors are automatically linearly independent. Let the functions $\Psi_n(x)$ be orthogonal over the interval $[a, b]$ with recpect to the weight function $w(x)$. Show that the functions $\Psi_n(x)$ are linearly independent.

7. *Sturm-Liouville Problem*

The equation

$$\hat{\mathcal{L}} y(x) = \lambda y(x) \tag{5}$$

is a Sturm-Liouville equation for the operator

$$\hat{\mathcal{L}} = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] - q(x)\ , \tag{6}$$

where $p(x)$ and $q(x)$ are real functions. Any two real solutions $y_n(x)$, $y_m(x)$ with distinct eigenvalues λ_n, λ_m satisfy the boundary condition

$$\left[y_m p \frac{dy_n}{dx}\right]\Bigg|_{x=a} = \left[y_m p \frac{dy_n}{dx}\right]\Bigg|_{x=b}\ . \tag{7}$$

Show that for $n \neq m$ $y_n(x)$ and $y_m(x)$ are orthogonal.

8. *Quantum Harmonic Oscillator*

Consider the time-independent Schrödinger equation for the quantum harmonic oscillator

$$\begin{aligned} H\Psi(x) &= E\Psi(x) \ , \\ H &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \ . \end{aligned} \tag{8}$$

(a) Using the substitutions $y = x\sqrt{\frac{m\omega}{\hbar}}$ and $\epsilon = \frac{E}{\hbar\omega}$ reduce the Schrödinger equation to

$$\frac{d^2}{dy^2}\Psi(y) + (2\epsilon - y^2)\Psi(y) = 0. \tag{9}$$

(b) Consider the limit $y \to \infty$ and show that in this limit

$$\Psi(y) \to Ay^m e^{-y^2/2}. \tag{10}$$

Hint: you can neglect ϵ compared to y^2 in this limit.

(c) Separate off the exponential factor and define

$$\Psi(y) = u(y)e^{-y^2/2} \ . \tag{11}$$

Show that $u(y)$ fulfils the ODE

$$u'' - 2yu' + (2\epsilon - 1)u = 0. \tag{12}$$

(d) Show that this differential equation can be converted to Sturm-Liouville form by multiplying both sides of the equation by e^{-y^2}. What is the weight function $w(y)$ of the Sturm-Liouville problem?

(e) Solve (12) by the ansatz

$$u(y) = \sum_{n=0}^{\infty} a_n \ y^n \tag{13}$$

by deriving a recurrence relation for the coefficients a_n. You should get

$$a_{n+2} = a_n \frac{(2n + 1 - 2\epsilon)}{(n + 2)(n + 1)}. \tag{14}$$

(f) We know from (b) that for $y \to \infty$ the function $u(y)$ must go to Ay^m. This means that the recurrence relation must *terminate*, i.e. we must have $a_n = 0$. This quantizes the allowed values of $\epsilon = E/(\hbar\omega)$

$$\epsilon_n = n + \frac{1}{2} \ , \quad n = 0, 1, 2, \ldots \tag{15}$$

Find the polynomial solutions $H_n(x)$ corresponding to these values of ϵ for $n = 0, 1, 2, 3$. These polynomials are called *Hermite polynomials.*

(g) Show that your results for $n = 0, 1, 2, 3$ agree with *Rodrigues' formula*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{16}$$

(h) Show that the H_n can be normalized such that

$$\int_{-\infty}^{\infty} dy \ e^{-y^2} \ H_n(y) \ H_m(y) = \delta_{nm}\sqrt{\pi}2^n n!. \tag{17}$$