Mathematical Methods (Second Year) MT 2009

Problem Set 2: Linear Algebra II

A. Rotations, Eigenvalues, Eigenvectors

1. Which of these matrices represents a rotation?

$$\begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1/4 & 3/4 & -\sqrt{3/8} \\ 3/4 & 1/4 & \sqrt{3/8} \\ \sqrt{3/8} & -\sqrt{3/8} & -1/2 \end{pmatrix}$$

Find the angle and axis of the rotation. What does the other matrix represent?

2. (a) Multiply together the two matrices $A = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ and $B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and interpret the result in terms of 2D rotations.

(b) Evaluate the product $A^T A$ and interpret the result. Evaluate $A^T B$ and interpret the result.

(c) Find the eigenvalues and eigenvectors of A. (Remember that they do not have to be real. Why not?)

3. (a) Find the eigenvalues of the Pauli matrix $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Normalize the two corresponding eigenvectors, $\vec{u}_1$, $\vec{u}_2$, so that $\vec{u}_1^\dagger \cdot \vec{u}_1 = \vec{u}_2^\dagger \cdot \vec{u}_2 = 1$. Check that $\vec{u}_1^\dagger \cdot \vec{u}_2 = 0$. Form the matrix $U = (\,\vec{u}_1 \quad \vec{u}_2\,)$ and verify that $U^\dagger U = I$. Evaluate $U^\dagger\ \sigma^y\ U$. What have you learned from this calculation?

(b) A general 2-component complex vector $\vec{v} = (c_1, c_2)^T$ is expanded as a linear combination of the eigenvectors $\vec{u}_1$ and $\vec{u}_2$ via

$$\vec{v} = \alpha\,\vec{u}_1 + \beta\,\vec{u}_2, \tag{1}$$

where α and β are complex numbers. Determine α and β in terms of c_1 and c_2 in two ways: (i) By equating corresponding components of (1), (ii) By showing that $\alpha = \vec{u}_1^\dagger \cdot \vec{v}$, $\beta = \vec{u}_2^\dagger \cdot \vec{v}$ and evaluating these products.

4. Verify that the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has eigenvalues -1, 1, 2 and find the associated normalized eigenvectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$. Construct the matrix $R = (\vec{u}_1\, \vec{u}_2\, \vec{u}_3)$ and show that it is orthogonal and that it diagonalizes A.

5. Construct a real symmetric matrix whose eigenvalues are 2, 1 and -2, and whose corresponding normalized eigenvectors are $\frac{1}{\sqrt{2}}(0,1,1)^T$, $\frac{1}{\sqrt{2}}(0,1,-1)^T$ and $(1,0,0)^T$.

6. Find the eigenvalues and eigenvectors of the matrix $F = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$. Hence, proving the validity of the method you use, find the values of the elements of the matrix F^n where n is a positive integer.

7. Write down the matrix R_1 for a three dimensional rotation through $\pi/4$ about the z-axis and the the matrix R_2 for a rotation through $\pi/4$ about the x-axis. Calculate $Q_1 = R_1 R_2$ and $Q_2 = R_2 R_1$; explain geometrically why they are different.

8. By finding the eigenvectors of the Hermitian matrix $H = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}$ construct a unitary matrix U such that $U^\dagger\ H\ U = D$, where D is a real diagonal matrix.

9. By taking the trace of both sides prove that there are no finite dimensional matrix representations of the momentum operator p and the position operator x which satisfy $[p, x] = -i\hbar$. Why does this argument fail if the matrices are infinite-dimensional (as Heisenberg's were)?

10. Which of the following matrices have a complete set of eigenvectors in common? $A = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}$, $C = \begin{pmatrix} -9 & -10 \\ -10 & 5 \end{pmatrix}$, $D = \begin{pmatrix} 14 & 2 \\ 2 & 11 \end{pmatrix}$. Construct the set of common eigenvectors where possible.

11. What are the eigenvalues and eigenvectors of the matrix $\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$? Can σ^+ be diagonalized?

C. Quadratic Forms

12. (i) Show that the quadratic form $4x^2 + 2y^2 + 2z^2 - 2xy + 2yz - 2zx$ can be written as $\vec{x}^T V \vec{x}$ where V is a symmetric matrix. Find the eigenvalues of V. Explain why, by rotating the axes, the quadratic form may be reduced to the simple expression $\lambda x'^2 + \mu y'^2 + \nu z'^2$; what are λ, μ, ν.?

    (ii) The components of the current density vector $\vec{j}$ in a conductor are proportional to the components of the applied electric field $\vec{E}$ in simple (isotropic) cases: $\vec{j} = \sigma \vec{E}$. In crystals, however, the relation may be more complicated, though still linear, namely of the form $j_i = \sum_{j=1}^{3} \sigma_{ij} E_j$, where σ_{ij} form the entries in a real symmetric 3×3 matrix, and i runs from 1 to 3.

    In a particular case, the quantities σ_{ij} are given (in certain units) by $\begin{pmatrix} 4 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$.

    Explain why by a rotation of the axes, the relation between $\vec{j}$ and $\vec{E}$ can be reduced to $j'_1 = \tilde{\sigma}_1 E'_1, j'_2 = \tilde{\sigma}_2 E'_2, j'_3 = \tilde{\sigma}_3 E'_3$ and find $\tilde{\sigma}_1, \tilde{\sigma}_2$ and $\tilde{\sigma}_3$.

D. Determinants

13. Let A be an $n \times n$ matrix of the form

$$A = \begin{pmatrix} \vec{a_1} & \dots & \lambda\vec{b} + \mu\vec{c} & \dots & \vec{a_n} \\ \vdots & \dots & \vdots & \dots & \vdots \end{pmatrix}, \tag{2}$$

i.e. the columns of A are formed by the n-dimensional vectors $\vec{a}_j$, except for the k^{th} column, which is formed by the vector $\lambda\vec{b} + \mu\vec{c}$. Using the general definition of the determinant show that

$$\det(A) = \lambda \det(B) + \mu \det(C) \ , \tag{3}$$

where B and C are the matrices

$$B = \begin{pmatrix} \vec{a_1} & \dots & \vec{b} & \dots & \vec{a_n} \\ \vdots & \dots & \vdots & \dots & \vdots \end{pmatrix}, \qquad C = \begin{pmatrix} \vec{a_1} & \dots & \vec{c} & \dots & \vec{a_n} \\ \vdots & \dots & \vdots & \dots & \vdots \end{pmatrix}, \tag{4}$$

Hint: Start by using that $\det(A) = \det(A^T)$ and then write down the definition of $\det(A^T)$ in terms of a sum over permutations.

14. * Let A be an $n \times n$ matrix.

$$A = \begin{pmatrix} \vec{a_1} & \dots & \vec{a_k} & \dots & \vec{a_n} \\ \vdots & \dots & \vdots & \dots & \vdots \end{pmatrix}, \tag{5}$$

and A'_P the matrix obtained from A by permuting it columns, i.e.

$$A'_P = \begin{pmatrix} \vec{a}_{P(1)} & \dots & \vec{a}_{P(k)} & \dots & \vec{a}_{P(n)} \\ \vdots & \dots & \vdots & \dots & \vdots \end{pmatrix}. \tag{6}$$

Here $P = (P(1), P(2), \ldots, P(n))$ is an arbitrary permutation of $(1, 2, \ldots, n)$. Show that

$$\det(A'_P) = \mathrm{sgn}(P)\det(A), \tag{7}$$

where $\mathrm{sgn}(P)$ is the sign of the permutation P.

Hint: It is sufficient to prove this for a permutation P that differs from the identity only by a single pair exchange (Why?).

15. * Let A and B be $n \times n$ matrices. Prove that

$$\det(AB) = \det(A)\ \det(B). \tag{8}$$

Hint: Express the column vectors of the matrix AB as linear combinations of the column vectors forming the matrix A. Then use the result of problem 13. repeatedly and finally the result of problem 14.

16. * Let A be a **diagonalizable** $n \times n$ matrix. This means that there exists an invertible $n \times n$ matrix S such that

$$SAS^{-1} = D\ , \tag{9}$$

where D is a diagonal matrix. Prove that

(i)

$$\det(A) = \det(D). \tag{10}$$

(ii) For positive integers k

$$\mathrm{Tr}(\mathrm{A^k}) = \mathrm{Tr}(\mathrm{D^k}). \tag{11}$$

(iii) Hence show that

$$\det(I + A) = \exp\left(\mathrm{Tr}(\ln(I + A))\right). \tag{12}$$

Here I is the unit matrix and $\ln(I + A)$ is defined through the power series expansion of the logarithm (assume that this expansion is well-defined)

$$\ln(I + A) = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{A^j}{j}. \tag{13}$$