Mathematical Methods (Second Year) MT 2009

Problem Set 1: Linear Algebra I

A. Linear Vector Spaces, Linear Independence, Dimension, Bases

1. Show that the space of 2×2 matrices is a linear vector space. What is its dimension? Give a basis for this space.

2. What is the dimension of the space of $n \times n$ matrices? Give a basis for this space.

3. What is the dimension of the space of $n \times n$ matrices all of whose components are zero except possibly the diagonal components?

4. What is the dimension of the space of symmetric 2×2 matrices, i.e. 2×2 matrices such that $A = A^T$ (recall that the transpose matrix is defined by $(A^T)_{ij} = A_{ji}$)? Exhibit a basis for this space.

5. Consider the vector space of all functions of a variable t. Show that the following pairs of functions are linearly independent. (a) 1, t (b) t, t^2 (c) e^t, t (d) $\sin(t)$, $\cos(t)$

6. What are the coordinates of the function $f(t) = 3\sin(t) + 5\cos(t)$ with respect to the basis $\{\sin(t), \cos(t)\}$?

7. What are the dimensions of the vector spaces spanned by the following sets of vectors (they are given in Cartesian form)?

   (a) $\{(1,1)^T,\ (1,2)^T\}$ (b) $\{(1,0)^T,\ (1,0)^T\}$ (c)$\{(1,1,2)^T,\ (-2,0,1)^T,\ (-1,1,3)^T\}$
   (d) $\{(1,1,1,1)^T,\ (1,-1,1,-1)^T,\ (1,1,-1,-1)^T,\ (1,-1,-1,1)^T\}$
   (e) $\{(1,2,3)^T,\ (1,-2,1)^T,\ (3,0,2)^T,\ (4,5,6)^T\}$

   If the number of vectors is greater than the dimension, choose some of them to form a set of basis vectors and express the remaining vectors as linear combinations of them. Which of the bases are orthogonal?

B. Scalar Product, Orthogonality

8. Prove the **Triangle Inequality**: given a norm $\parallel |\mathbf{a}\rangle \parallel = \sqrt{\langle \mathbf{a}|\mathbf{a}\rangle}$ defined through the scalar product $\langle\ |\ \rangle$ we have for any two vectors $|\mathbf{v}\rangle$ and $|\mathbf{w}\rangle$ in a linear vector space

$$\parallel |\mathbf{v}\rangle + |\mathbf{w}\rangle \parallel \leq \parallel |\mathbf{v}\rangle \parallel + \parallel |\mathbf{w}\rangle \parallel. \tag{1}$$

9. Let $\vec{a}_1, \ldots \vec{a}_n$ be vectors in R^n and assume that they are mutually perpendicular (i.e. any two of them are perpendicular) and none of them is equal to 0. Prove that they are linearly independent.

10. Find real values α and β such that the complex vectors $\mathbf{u} = \alpha \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$ and $\mathbf{v} = \beta \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$ are normalised. What is the value of the scalar product $\mathbf{u}^\dagger.\mathbf{v}$? Prove that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Are there any further linearly independent two-dimensional complex vectors? If so, find the necessary vectors to make an orthogonal basis. Express the vector $\begin{pmatrix} 1 \\ i \end{pmatrix}$ as a linear combination of the basis vectors.

11. Construct a third vector which is orthogonal to the following pairs and normalise all three vectors

    (a) $(1,2,3)^T, \quad (-1,-1,1)^T$ (b) $(1+i\sqrt{3}, 2, 1-i\sqrt{3})^T, \quad (1,-1,1)^T$
    (c)* $(1-i, 1, 3i)^T, \quad (1+2i, 2, 1)^T$

12. Using the Schmidt procedure construct an orthonormal set of vectors from the following:

$$\vec{x}_1 = (0,0,1,1)^T\ ,\ \vec{x}_2 = (1,0,-1,0)^T\ ,\ \vec{x}_3 = (1,2,0,2)^T\ ,\ \vec{x}_4 = (2,1,1,1)^T.$$

13. Consider the vector space of continuous, complex-valued functions on the interval $[-\pi, \pi]$. Show that

$$\langle \mathbf{f} | \mathbf{g} \rangle = \int_{-\pi}^{\pi} dt \; f^*(t) \; g(t) \tag{2}$$

defines a scalar product on this space. Are the following functions orthogonal with respect to this scalar product? (a) $\sin(t), \; \cos(t)$ (b) $\exp(int), \; \exp(ikt)$ n,k, integers (c) $t^2, \; t^4$

14. Let V be the real vector space of all real symmetric $n \times n$ matrices and define the scalar product of two matrices A, B by (Tr (A) denotes the trace of A)

$$\langle A | B \rangle = \text{Tr (AB)}. \tag{3}$$

Show that this indeed fulfils the requirements on a scalar product.

C. Matrices, Rotations

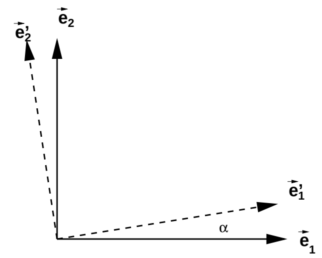
15. Find the rank of the following matrices by reducing their determinants to upper triangular form

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & -1 \\ 5 & -2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & x & y \\ 3x & 2y & 1 \\ x & y & 1 \end{pmatrix}$$

16. By considering the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix}$ show that $AB = 0$ does not imply that either A or B is the zero matrix. Allowing A, B to be any square matrices, show that $AB = 0$ implies that at least one of them is *singular*, i.e. has zero determinant.

17. Consider the three matrices $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Which of the matrices are symmetric, which are hermitian? By calculating the commutators of these matrices show that they can be written as $[\sigma^a, \sigma^b] = 2i\epsilon_{abc}\sigma^c$, where ϵ_{abc} is the epsilon tensor and on the right hand side the summation convention is employed (i.e. the index c is summed over). Write $\exp(i\alpha\sigma^y)$ (α is a real number) as a 2×2 matrix. What does it represent? Show that $\exp(i\alpha\sigma^y)$ is unitary without writing it explicitly as a 2×2 matrix.

18. Consider the vector space V of arrows in the plane. Let A be the linear operator that rotates all vectors by 45 degrees and then reflects them with respect to the horizontal. Let $B_1 = \{\vec{e}_1, \vec{e}_2\}$ be the standard cartesian basis and $B_2 = \{\vec{e}_1{'}, \vec{e}_2{'}\}$ another orthonormal basis of V, which is obtained from $\{\vec{e}_1, \vec{e}_2\}$ by a rotation by an angle α, see Fig. 1. Write down the transformation that takes $\vec{e}_{1,2}$ to $\vec{e}_{1,2}{'}$ in matrix form. What are the coordinate representations of A with respect to the bases B_1 and B_2 respectively? What is the matrix equation that relates these two coordinate representation?



19. (i) Show that $(AB)^T = B^T A^T$.
   (ii) The trace of a matrix A is defined as $\text{Tr A} = \sum_n \text{A}_{\text{nn}}$ *ie* the sum of the diagonal elements. Show that Tr AB = Tr BA for any two matrices B, A and hence that the trace of any product of matrices $AB \ldots Z$ is unchanged by a cyclic permutation of the entries.
   (iii) Show that if R is an orthogonal matrix, then $\text{Tr R}^\text{T}\text{AR} = \text{Tr A}$ *ie* the trace is invariant under change of orthonormal basis