Fourier Series

1. Sketch graphs of the following functions in the range $-2\pi < x < 2\pi$ given that all the functions are periodic with period 2π:

$$\begin{array}{ll} \text{(i)} & f(x) = |x|, \quad -\pi < x < \pi; \\ \text{(ii)} & f(x) = x, \quad -\pi < x < \pi; \\ \text{(iii)} & f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}; \\ \text{(iv)} & f(x) = \begin{cases} -e^{-x} & -\pi < x < 0 \\ e^{-x} & 0 < x < \pi \end{cases}. \end{array}$$

2. State whether the following functions are even, odd or neither even nor odd:

(i) $\sin x$ (ii) $\cos x$ (iii) $\sin^2 x$ (iv) $\cos^2 x$
(v) $\sin 2x$ (vi) $\cos 2x$ (vii) $x \cos x$ (viii) $\sin x + \cos x$
(ix) $x + \sin x$ (x) e^x.

3. Find the Fourier cosine series that represents the function

$$f(x) = \begin{cases} x & 0 < x < \frac{1}{2}\pi \\ \pi - x & \frac{1}{2}\pi < x < \pi \end{cases}$$

4. Find (a) the Fourier sine series and (b) the Fourier cosine series for the function

$$f(x) = x \sin x, \qquad 0 < x < \pi.$$

5. Find the Fourier series for the function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}.$$

6. Find (a) the Fourier sine series and (b) the Fourier cosine series for the function $f(x) = x^2$ in the interval $0 < x < 3$.

7. Find (i) the Fourier cosine series, and (ii) the Fourier sine series for the function $f(x) = x(\pi - x)$ in the interval $0 < x < \pi$, and show by a sketch the functions represented by the two series in the intervals $-\pi < x < 0$ and $\pi < x < 2\pi$.

From your results show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \tfrac{1}{6}\pi^2$$

and sum the series

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$$

8. The function $f(x)$ has period 2π and in the interval $-\pi < x < \pi$ is defined by

$$f(x) = \begin{cases} \cos x & -\frac{1}{2}\pi \le x \le \frac{1}{2}\pi \\ 0 & \frac{1}{2}\pi < |x| \le \pi \end{cases}.$$

Prove that its Fourier series is given by

$$\frac{1}{\pi} + \tfrac{1}{2}\cos x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2-1}\cos 2nx,$$

and find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$.

9. Find the Fourier series in the range $-\frac{1}{2}\pi \le x \le \frac{1}{2}\pi$ for the function $f(x) = \cosh x$.
Prove that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2+1} = \frac{\pi}{4}\coth(\tfrac{1}{2}\pi) - \tfrac{1}{2}.$$

10. Find the Fourier cosine series for the range $0 \le x \le \pi$ of the function

$$f(x) = \begin{cases} 1 & 0 \le x < \alpha \\ 0 & \alpha < x \le \pi \end{cases}.$$

Hence show that for $0 < \alpha < \pi$

$$\tfrac{1}{2}(\pi - \alpha) = \sin\alpha + \tfrac{1}{2}\sin 2\alpha + \tfrac{1}{3}\sin 3\alpha + \cdots$$

Prove that this result is also true for $\pi < \alpha < 2\pi$.

11. Show that the solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

that satisfies the boundary conditions $u = 0$ on the lines $y = \pm 1$ for $x > 0$, $u = 1$ on $x = 0$ for $-1 < y < 1$, and $u \to 0$ as $x \to \infty$ is:

$$u = \sum_{m=1}^{\infty} A_m \cos\left(\frac{m\pi y}{2}\right) \mathrm{e}^{-m\pi x/2}$$

where $A_m = \int_0^1 \cos(m\pi y/2)\,\mathrm{d}y - \int_1^2 \cos(m\pi y/2)\,\mathrm{d}y$.

12. The temperature Θ in a rod satisfies the diffusion equation $\partial\Theta/\partial t = K\partial^2\Theta/\partial x^2$. At $t = 0$ the rod is at temperature Θ_0 Celsius and the ends of the rod at $x = 0$ and $x = l$ are kept at zero temperature at all times. Show that for $t > 0$ the temperature is given by

$$\Theta = \sum_{m=1,3,}^{\infty} \frac{4\Theta_0}{m\pi} \sin(m\pi x/l)\,\mathrm{e}^{-(m\pi/l)^2 Kt}.$$

13. Show that the solution Φ of Laplace's equation in two dimensions in the region $r \leq a$, subject to the boundary conditions

$$\Phi = \begin{cases} \Phi_0 & \text{for } 0 < \theta < \pi, \\ -\Phi_0 & \text{for } -\pi < \theta < 0, \end{cases}$$

is

$$\Phi = \sum_{m=1,3,}^{\infty} \frac{4\Phi_0}{m\pi} \left(\frac{r}{a}\right)^m \sin(m\theta).$$