Quantum Matter 2, Problem Sheet 1

1 Volume and Area Laws for Bipartite Entanglement Entropy

Consider a tight-binding model of free, spinless fermions with Hamiltonian

$$H = -J\sum_{j=1}^{L} c_j^\dagger c_{j+1} + c_j^\dagger c_{j+1} - \mu\sum_{j=1}^{L} c_j^\dagger c_j\ , \qquad J>0. \tag{1}$$

(a) Construct the ground state $|\mathrm{GS}\rangle$ of the model for $|\mu|<2J$.

(b) Determine the $L\times L$ *correlation matrix* C with entries

$$C_{j,k} = \langle \mathrm{GS}|c_j^\dagger c_k|\mathrm{GS}\rangle. \tag{2}$$

Consider a sub-system made from sites 1,..., N. Show that in the thermodynamic limit $L\to\infty$, keeping N fixed, one obtains (k_F is the Fermi momentum)

$$C_{j,k} = \frac{\sin\big(k_F(j-k)\big)}{\pi(j-k)}\ , \quad j,k\in[1,N]. \tag{3}$$

Importantly, all other two-point functions $\langle \mathrm{GS}|c_j c_k|\mathrm{GS}\rangle$ vanish (we will use this throughout what follows). Why?

(c) As the ground state is a Fock state in momentum space we have a Wick's theorem, so multi-point correlction functions can be reduced to sums of products of two-point functions. *This allows us to determine the reduced density matrix of the sub-system from its correlation matrix!* The correlation matrix is Hermitian and hence can be diagonalized by a unitary transformation

$$UCU^\dagger = D = \mathrm{diag}(\nu_1,\nu_2,\ldots,\nu_N)\ , \tag{4}$$

where D is a diagonal matrix. Let's write this out

$$D_{j,n} = \sum_{k,\ell=1}^{N} \langle \mathrm{GS}|U_{j,k}c_k^\dagger c_\ell (U^\dagger)_{\ell,n}|\mathrm{GS}\rangle\ ,\ j,n=1,\ldots,N. \tag{5}$$

Show that the operators $\alpha_j,\alpha_j^\dagger$ defined by

$$\alpha_j^\dagger = U_{j,k}c_k^\dagger \tag{6}$$

obey canonical anticommutation relations. At this point we have from (5) that

$$\langle \mathrm{GS}|\alpha_j^\dagger\alpha_k|\mathrm{GS}\rangle = \delta_{j,k}\nu_j. \tag{7}$$

Next we use that all expectation values on the subsystem can be obtained from its reduced density matrix as

$$C_{j,k} = \mathrm{Tr}\big[\rho_{[1,N]}\ c_j^\dagger c_k\big]. \tag{8}$$

Using the same unitary transformation conclude that

$$\mathrm{Tr}\big[\rho_{[1,N]}\ \alpha_j^\dagger \alpha_k\big] = \delta_{j,k} - \mathrm{Tr}\big[\rho_{[1,N]}\ \alpha_k \alpha_j^\dagger\big] = \nu_j \delta_{j,k}. \tag{9}$$

Now argue that the reduced density matrix that has this property can be written as

$$\rho_{[1,N]} = \otimes_{j=1}^N \rho_j\ , \tag{10}$$

where the ρ_j are operators that can be written as 2×2 matrices acting on a Hilbert space with basis states

$$|\tilde{0}\rangle_j\ ,\quad |\tilde{1}\rangle_j = \alpha^\dagger |\tilde{0}\rangle_j\ , \tag{11}$$

where $|\tilde{0}\rangle$ is defined as the state annihilated by α_j. In order to obtain (9) we need to take

$$\rho_j = \nu_j |\tilde{1}\rangle_j\ {}_j\langle\tilde{1}| + (1-\nu_j)|\tilde{0}\rangle_j\ {}_j\langle\tilde{0}| \tag{12}$$

Conclude that the 2^N eigenvalues of the reduced density matrix $\rho_{[1,N]}$ are

$$\prod_{j=1}^N \nu_j^{n_j}(1-\nu_j)^{1-n_j}\ ,\quad n_j \in \{0,1\}. \tag{13}$$

(d) Use the results of (c) to write the entanglement entropy in the form

$$S_{[1,N]} = -\sum_{j=1}^N \nu_j \ln(\nu_j) + (1-\nu_j)\ln(1-\nu_j)\ . \tag{14}$$

Use Mathematica or Matlab to determine $S_{[1,N]}$ as a function of the subsystem length N for $J = 1$ and several values of μ (to do this you need to compute the eigenvalues of the $N \times N$ matrix C). You should find that for large N

$$S_{[1,N]} = \frac{1}{3}\ln(N) + \mathrm{const} \tag{15}$$

This is the behaviour characteristic of *conformal field theories* in 1+1 dimensions (instead of a coefficient $1/3$ in a CFT one obtains $c/3$, where c is the central charge of the CFT). This suggests that our tight-binding model is related to a $c = 1$ CFT, which is indeed the case.

(e) Finally repeat your analysis for an energy eigenstate with an energy that corresponds to temperature T. You can calculate its correlation matrix on a sub-system using the equivalence of the micro-canonical and Gibbs ensemble

$$C_{j,k} = \frac{1}{Z_G}\mathrm{Tr}\big(e^{-\beta H} c_j^\dagger c_k\big) = \frac{1}{L}\sum_{p,q} e^{-ipj+iqk}\ \underbrace{\frac{1}{Z_G}\mathrm{Tr}\big(e^{-\beta H} c^\dagger(p) c(q)\big)}_{\frac{\delta_{p,q}}{1+e^{\beta\epsilon(p)}}} \to \int_{-\pi}^{\pi} \frac{dp}{2\pi}\frac{e^{-ip(j-k)}}{1+e^{\beta\epsilon(p)}}. \tag{16}$$

Show that the entanglement entropy fulfils a *volume law.*

2 AKLT spin-1 Chain

(a) Use the MPS representation for periodic boundary conditions

$$|\mathrm{GS}\rangle = \sum_{\sigma_1,\dots,\sigma_L} \mathrm{Tr}\left[A^{\sigma_1}\dots A^{\sigma_L}\right] |\sigma_1,\dots,\sigma_L\rangle\ ,$$

$$A^+ = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}\ ,\quad A^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ,\quad A^- = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}\ , \tag{17}$$

to obtain an expression for the string-order parameter

$$\mathcal{O}^z_{\text{string}} = \lim_{|j-k|\to\infty} \lim_{L\to\infty} \frac{\langle \text{GS}|S^z_j e^{i\pi \sum_{l=j+1}^{k=1} S^z_l} S^z_k|\text{GS}\rangle}{\langle \text{GS}|\text{GS}\rangle}. \tag{18}$$

Hints: Argue that the numerator of (18) corresponds to a diagram of the form (the example is for $L = 8$, $j = 1$, $k = 5$)

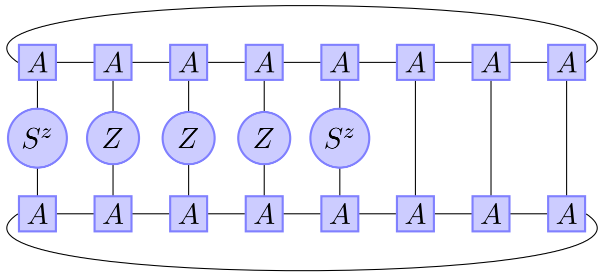


Figure 1: Graphical representation of the numerator in the expression for $\mathcal{O}^z_{\text{string}}$.

Then work out 4×4 matrix representations of $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ shown in Fig. 2. You should find

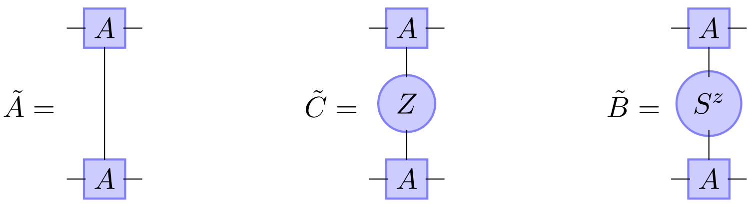


Figure 2: Building blocks $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$.

$$\tilde{C} = \frac{1}{4}\begin{pmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \tilde{B} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{A} = \frac{1}{4}\begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{19}$$

This allows you to express the numerator in (18) in the form

$$\text{Sp}\left[\tilde{B}\tilde{C}^{k-j-1}\tilde{B}\tilde{A}^{L-k+j-1}\right], \tag{20}$$

where Sp denotes the 4×4 matrix trace.

(b) Consider now the AKLT model on a chain with open boundaries, i.e.

$$H = \sum_{j=1}^{L-1} \frac{1}{2}\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{6}\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3} = \sum_{j=1}^{L-1} P^{(2)}_{j,j+1}. \tag{21}$$

Show that there are four linearly independent degenerate ground states, which can be written in MPS form

$$|\text{GS}\rangle = \sum_{\sigma_1,\dots,\sigma_L} \ell^{\sigma_1} \cdot A^{\sigma_2} A^{\sigma_3} \dots A^{\sigma_{L-2}} r^{\sigma_L}. \tag{22}$$

Here A^σ are the 2×2 matrices and ℓ^σ, r^σ are respectively two-dimensional row and column vectors. Hint: impose that $|\mathrm{GS}\rangle$ is annihilated by each $P^{(2)}_{j,j+1}$. You should find that the choices

$$\sum_\sigma \ell^\sigma|\sigma\rangle \in \{(|+\rangle, \frac{1}{\sqrt{2}}|0\rangle)\ ,\ (|\frac{1}{\sqrt{2}}|0\rangle, |-\rangle)\}\ ,\quad r^\sigma|\sigma\rangle \in \{\begin{pmatrix} \frac{1}{2}|0\rangle \\ -\frac{1}{\sqrt{2}}|+\rangle) \end{pmatrix}\ ,\ \begin{pmatrix} \frac{1}{\sqrt{2}}|-\rangle \\ -\frac{1}{2}|0\rangle) \end{pmatrix}\} \tag{23}$$

lead to the ground states.

(c) For each of the ground states obtained by the choices (23), determine the magnetization close to the boundaries in the thermodynamic limit. For the left and right boundaries this amounts to calculating

$$\begin{aligned} \langle S^z_{l,j}\rangle &= \lim_{L\to\infty}\frac{\langle \mathrm{GS}|S^z_j|\mathrm{GS}\rangle}{\langle \mathrm{GS}|\mathrm{GS}\rangle}\ ,\quad j=1,2,\dots\ ,\\ \langle S^z_{r,j}\rangle &= \lim_{L\to\infty}\frac{\langle \mathrm{GS}|S^z_{L+1-j}|\mathrm{GS}\rangle}{\langle \mathrm{GS}|\mathrm{GS}\rangle}\ ,\quad j=1,2,\dots \end{aligned} \tag{24}$$

Hint: For the four choices ($a=1,2,3,4$) write the numerator in the form

$$\langle \mathrm{GS,a}|S^z_j|\mathrm{GS,a}\rangle = \begin{cases} \langle \tilde{L}_a|\tilde{A}^{L-2}|R_a\rangle & \text{if } j=1\\ \langle L_a|\tilde{A}^{j-2}\tilde{B}\tilde{A}^{L-j-1}|R_a\rangle & \text{if } j>1 \end{cases} \tag{25}$$

where $\langle L_a|, \langle\tilde{L}_a|$ and $|R_a\rangle$ are four-dimensional row and column vectors respectively.

Argue that this indicates that we have two spin-1/2 degrees of freedom exponentially localized in the vicinity of the boundaries by considering e.g.

$$\sum_{j=1}^{\infty}\langle S^z_{l,j}\rangle \tag{26}$$

How are these findings compatible with the $SU(2)$ symmetry of the model?

(d) Show that the bipartite entanglement entropy of a segment of ℓ neighbourings spins in the ground state of the periodic AKLT chain in the limit $L\to\infty$ is given by

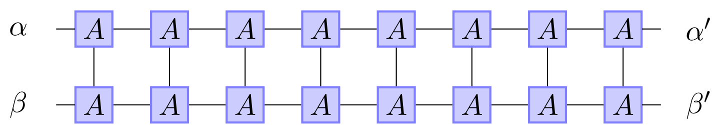
$$\lim_{L\to\infty} S_\ell = 2\ln(2). \tag{27}$$

Give an interpretation of your result.

Hint: As shown in the lectures you can decompose the ground state as

$$|\Psi\rangle = \frac{1}{\mathcal{N}}\sum_{\alpha,\beta=1}^{\chi}\underbrace{\sum_{\sigma_1,\dots,\sigma_\ell}[A^{\sigma_1}\dots A^{\sigma_\ell}]_{\alpha\beta}\,|\sigma_1,\dots,\sigma_\ell\rangle}_{|\phi_{\alpha\beta}\rangle}\otimes\underbrace{\sum_{\sigma_{\ell+1},\dots,\sigma_L}[A^{\sigma_{\ell+1}}\dots A^{\sigma_L}]_{\beta\alpha}\,|\sigma_{\ell+1},\dots,\sigma_L\rangle}_{|\psi_{\alpha\beta}\rangle}, \tag{28}$$

but the states $|\phi_{\alpha\beta}\rangle$ are not orthonormal. In order to orthonormalize them work out the scalar products $\langle\phi_{\alpha\beta}|\phi_{\alpha'\beta'}\rangle$. In the graphical notations these take the form shown in the diagram below i.e.



they correspond to particular matrix elements of $\tilde{A}^\ell$.