Quantum Mechanics HT 2019: Problem Sheet 6

The Hydrogen Atom

[6.1] Some things about hydrogen's gross structure that it's important to know (ignore spin throughout, and you may set the reduced mass to be equal to the electron mass):

(a) What quantum numbers characterise stationary states of hydrogen?

(b) What combinations of values of these numbers are permitted?

(c) Give the formula for the energy of a stationary state in terms of the Rydberg $\mathcal{R}$. What is the value of $\mathcal{R}$ in eV?

(d) How many stationary states are there in the first excited level and in the second excited level?

(e) What is the wavefunction of the ground state?

(f) Write down an expression for the mass of the reduced particle.

(g) We can apply hydrogenic formulae to any two charged particles that are electrostatically bound. How does the ground-state energy then scale with (i) the mass of the reduced particle, and (ii) the charge Ze on the nucleus? (iii) How does the radial scale of the system scale with Z?

[6.2] An electron is in the ground state of a hydrogen-like atom with nuclear charge $+Ze$. For simplicity neglect the difference between the reduced mass and the electron mass.

(a) What is its average distance from the nucleus?

(b) At what distance from the nucleus is it most likely to be found?

(c) Show that the expectation value of the potential energy of the electron is the same as that given by the Bohr model, namely $-Ze^2/4\pi\epsilon_0 r_0$ where $r_0 = a_0/Z$.

(d) Show that the expectation value of the kinetic energy is equal to the value given by the Bohr model, namely $Ze^2/8\pi\epsilon_0 r_0$.

(e) Hence verify that the expectation value of the total energy agrees with the Bohr model.

[6.3] Show that the speed of a classical electron in the lowest Bohr orbit is $v = \alpha c$, where $\alpha = e^2/4\pi\epsilon_0\hbar c$ is the fine-structure constant. What is the corresponding speed for a hydrogen-like Fe ion (atomic number $Z = 26$)? Given these results, what fractional errors must we expect in the energies of states that we derive from non-relativistic quantum mechanics.

[6.4] Show that the electric field experienced by an electron in the ground state of hydrogen is of order $5 \times 10^{11}\,\mathrm{V\,m^{-1}}$. Why is it impossible to generate comparable macroscopic fields using charged electrodes. Lasers are available that can generate beam fluxes as big as $10^{22}\,\mathrm{W\,m^{-2}}$. Show that the electric field in such a beam is of comparable magnitude.

[6.5] Positronium consists of an electron e^- and a positron e^+ (both spin-half and of equal mass) in orbit around one another. What are its energy levels? By what factor is a positronium atom bigger than a hydrogen atom?

[6.6] Muonium consists of an electron e^- and a positive muon μ^+ (both spin-half particles but $m_\mu = 206.7 m_\mathrm{e}$) in orbit around one another. What are its energy levels? By what factor is muonium atom bigger than a hydrogen atom?

[6.7] The emission spectrum of the He^+ ion contains the Pickering series of spectral lines that is analogous to the Lyman, Balmer and Paschen series in the spectrum of hydrogen

Balmer $i = 1, 2, \ldots$	0.456806	0.616682	0.690685	0.730884
Pickering $i = 2, 4, \ldots$	0.456987	0.616933	0.690967	0.731183

The table gives the frequencies (in $10^{15}\,\mathrm{Hz}$) of the first four lines of the Balmer series and the first four even-numbered lines of the Pickering series. The frequencies of these lines in the Pickering series are almost coincident with the frequencies of lines of the Balmer series. Explain this finding. Provide a quantitative explanation of the small offset between these nearly coincident lines in terms of the reduced mass of the electron in the two systems. (In 1896 E.C. Pickering identified the odd-numbered lines in his series in the spectrum of the star ζ Puppis. Helium had yet to be discovered and he believed that the lines were being produced by hydrogen. Naturally he confused the even-numbered lines of his series with ordinary Balmer lines.)